MA134 Geometry and Motion

EXAMPLES SHEET 8

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 7th March, 2019 (week 9).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Compute the Jacobian of the transformation from cylindrical coordinate to Cartesian coordinates. Thus obtain the volume element dV for cylindrical coordinates. Do the same for spherical coordinates.

2. Derive the volume element dV for ellipsoidal coordinates (r, θ, ϕ) defined by $x = ar \sin\phi \cos\theta$, $y = br \sin\phi \sin\theta$, $z = cr \cos\phi$, where x, y, z are Cartesian coordinates.

3. (a) Evaluate

$$\iint_\Omega xy\, dxdy$$

over the square Ω with corners $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$ in two ways: directly and using $x = (u + v)/2$, $y = (u - v)/2$.

(b) Evaluate

$$\iint_\Omega (x + y)e^{x-y}\, dxdy$$

over the triangle Ω with vertices $(0, 0)$, $(-1, 1)$, and $(1, 1)$, in two ways: directly and using $x = (u + v)/2$, $y = (u - v)/2$.

B. Questions for credit

4. Sketch the following vector fields in the region $-5 \le x \le 5$, $-5 \le y \le 5$, including enough vectors that the nature of the vector field is clear.

   (a) $\mathbf{v}(x, y) = (1, \sin y)$

   (b) $\mathbf{v}(x, y) = (\sin y, -\sin x)$

   (c) $\mathbf{v}(x, y)$ is the gradient vector field of $f(x, y) = (x + y)^2$.

5. Evaluate the double integral

$$\int\int_S \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dxdy,$$

where $a > 0$, $b > 0$ and the region S is the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Hint. Change to *elliptical* coordinates (r, ϕ): $x = ar \cos\phi$, $y = br \sin\phi$.

6. Calculate

$$I = \int\int_D e^{a(x+y)^2} dxdy,$$

where $a \in \mathbb{R}$ is a constant and D is a region defined by inequalities

$$x \ge 0, y \ge 0, x + y \le 1.$$

Hint. Introduce new coordinates (u, v) such that $x(u, v) = u - y(u, v)$. Determine $y(u, v)$ from the condition that the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is equal to u and $y(u, 0) = 0$. (Then the integral is easily computable.)
Comment. The integral can also be computed using a linear change of variables.

7. Evaluate

$$\int\int\int_E \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz,$$

where a, b, c are positive constants and E is the interior of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

C. More exercises.
(They should not be handed in for marking.)

8. Evaluate

$$\iint_\Omega \cos\left(\frac{y - x}{y + x}\right) dA$$

where Ω is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$ and $(0, 1)$.

9. Evaluate

$$\iint_\Omega e^{x+y}\, dA$$

where Ω is the region where $|x| + |y| \leq 1$.

10. Sketch the vector field $\mathbf{v}(x, y)$, where $\mathbf{v}(x, y)$ is the gradient vector field of $f(x, y) = \sin\sqrt{x^2 + y^2}$. Note that in polar coordinates $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{\partial f}{r\partial\theta}\hat{\theta}$.

11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as follows:

$$f(\mathbf{r}) = \frac{e^{-ar}}{r},$$

where $a > 0$ is a constant and $r = ||\mathbf{r}||$ is the magnitude of $\mathbf{r}$. Calculate

$$F(\mathbf{k}) = \lim_{R\to\infty} \int\int\int_{B_R} f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} dxdydz,$$

where $\mathbf{k}$ is a fixed vector in $\mathbb{R}^3$, $B_R \subset \mathbb{R}^3$ is a ball of radius R centred at the origin and i is the imaginary unit, $i^2 = -1$.

Hint. Check that $F(O\mathbf{k}) = F(\mathbf{k})$, where O is a rotation matrix. (A three-by-three matrix such that $OO^T = I$ and $det(O) = 1$. Use this freedom to bring $\mathbf{k}$ to the simplest possible form before computing the integral.
Comment. The function f is called the Yukawa potential, it describes interactions between electrons in the presence of screening positive charges. The function F is called the three-dimensional Fourier transform of f.