MA134 Geometry and Motion

Examples Sheet 9

IMPORTANT: Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 14th March, 2019 (week 10).

Part A. Easier questions. Not to be handed in for marking.

1. Derive the element of surface area dS for each of the following. A disk (using polar coordinates). The surface of a cylinder of radius R (using cylindrical coordinates). The surface of a sphere of radius R (using spherical coordinates).

2. For each of the following two parametrisations of the upper hemisphere, compute the area of the upper unit hemisphere by integrating surface area:

$$S_+ = \{\mathbf{r} \in \mathbb{R}^3 \mid \mathbf{r} = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), 0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi\}$$

$$S_+ = \{\mathbf{r} \in \mathbb{R}^3 \mid \mathbf{r} = (x, y, \sqrt{R^2 - x^2 - y^2}), 0 \le x^2 + y^2 \le R^2\},$$

where $R > 0$. (For one of these you can use dS from the previous question. For the other you need to work out dS again.)

3. Parametrise the conical surface $z^2 = x^2 + y^2$ for $0 \le z \le 1$. From this find the element of surface area dS. Then by integration find the surface area of a circular cone with height $h = 1$ and radius $R = 1$.

4. Find the flux of $\mathbf{v}(x, y, z) = (0, 0, x^2 + y^2)$ through the disk $z = 0, x^2 + y^2 \le 1$ in the $+\mathbf{k}$ direction.

B. Questions for credit

5. Find the tangent plane and a unit normal vector to the torus

$$\mathbf{r}(u, v) = \big((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u\big), 0 \le u \le 2\pi, 0 \le v \le 2\pi.$$

at the point $(2 + 1/\sqrt{2}, 0, 1/\sqrt{2})$. For the tangent plane give the answer as an equation in three variables.

6. Let T be the torus parameterised by

$$\mathbf{r}(u, v) = \big((A + a\cos u)\cos v, (A + a\cos u)\sin v, a\sin u\big), \quad 0 \le u \le 2\pi, 0 \le v \le 2\pi.$$

Here $A > a > 0$ are constants. A is called the major radius of the torus (distance from the origin to the centre of the 'tube' making up the torus), a is the minor radius (the radius of the tube). See `https://en.wikipedia.org/wiki/Torus` for more details. Find the surface area of T.

7. Find the flux of $\mathbf{v}(x, y, z) = (2x, 2y, 2z)$ in the outward direction through the sphere of radius R centred on the origin.

8. Let $\mathbf{f}(\mathbf{r}) = F(||\mathbf{r}||)\frac{\mathbf{r}}{||\mathbf{r}||}$ be a radial vector field defined on $\mathbb{R}^3 \setminus \mathbf{0}$, where $F : (0, \infty) \to \mathbb{R}$ is an unknown continuous function. Find F from the condition that the flux of $\mathbf{f}$ in the outward direction through the sphere of radius R centred on the origin is independent of R and is equal to some constant $\Phi \in \mathbb{R}$. Depending on the interpretation of Φ, you have discovered the gravitational (or electric) field created by a point mass (or charge) placed at the origin.

C. More exercises.

(They should not be handed in for marking.)

In these questions you should use, as needed, the element of surface area dS obtained in Question 1.

8. Evaluate $\iint_S z dS$ where S is the upper hemisphere of radius 2.

9. Find the area of the portion of the unit sphere that is inside the cone $z \geq \sqrt{x^2+y^2}$.

10. Find the area cut out of the cylinder $x^2+z^2=1$ by the cylinder $x^2+y^2=1$.

11. Let S be the paraboloid parametrised in polar coordinates as

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, r^2), \quad 0 \leqslant r,\ \theta \in [0, 2\pi].$$

Evaluate $\iint_S \frac{1}{(1+4z)^2}\, dS$.

12. Find the flux of $\mathbf{v}(x,y,z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$ through the unit sphere centred on the origin in the outward direction.

13. Find the flux of $\mathbf{v}$ through S in the direction indicated for
   (a) $\mathbf{v}(x,y,z) = (x, -z, y)$. S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant. Direction towards the origin.

   (b) $\mathbf{v}(x,y,z) = (xy, yz, zx)$. S is the part of the paraboloid $z = 4-x^2-y^2$ above the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Upward direction.

14. Write the tangent plane from Q5 as a parametrised plane rather than an equation in 3 variables.