Week 10: Line Integrals

Introduction

In this final week we return to parametrised curves and consider integration along such curves. We already saw this in Week 2 when we integrated along a curve to find its length. Here we generalise this with particular emphasis on integrating over vector fields.

10.1 Line Integrals

The basic line integral can be motivated as follows. Given an interval $[a, b]$ and a function $f(x)$ which is positive over the interval, $\displaystyle\int_a^b f(x)dx$ is the area under the graph $y = f(x)$. Intuitively one understands that $f(x)\,dx$ is the area of a tall skinny rectangle of height $f(x)$ and width dx and $\displaystyle\int_a^b$ means "add these up" for x's in the interval $[a, b]$.

Why restrict ourselves to just integrating along straight lines? We known how to work with curves so let us generalise and consider a curve in the plane and a function $f(x, y)$ that is positive in some region containing the curve. A surface is formed by f over the curve. Think of a curtain hanging down from f to the curve. We want to compute the area of this curtain by integration.

The formula for the integral is easy once one recalls the formulas from Week 2. Recall the length of a curve $\mathcal{C}$ is given by

$$\ell(\mathcal{C}) = \int_{\mathcal{C}} ds$$

where ds is the infinitesimal arc length, or distance, along the curve.

Thus to find the area of the curtain formed from f over $\mathcal{C}$, we simply multiply the height f times the infinitesimal arc length ds and integrate over the curve

$$\int_{\mathcal{C}} f\,ds$$

In practice, such an integral is evaluated by parametrising the curve. Given a parametrisation of the curve $\mathbf{r}(t)$, $t \in [a, b]$, the infinitesimal arc length

ds can be expressed in terms of the infinitesimal change dt via

$$ds = \|\mathbf{r}'(t)\|dt$$

so that

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(\mathbf{r}(t)) \left\|\mathbf{r}'(t)\right\|\, dt$$

It is not necessary to restrict to positive functions, nor does the method depend on dimension. The relationship $ds = \|\mathbf{r}'(t)\|dt$ holds in any dimension. Thus we can go directly to the general formula.

Given $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{r} : [a, b] \to \mathbb{R}^n$ a parametrisation of a curve $\mathcal{C}$ lying in U, the **line integral** of f along a curve $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(\mathbf{r}(t)) \left\|\mathbf{r}'(t)\right\|\, dt$$

The comments from Week 2 apply here: the curve may be piecewise regular and one must parametrise the curve in a sensible way.

Note that in our definition of the line integral f is a function defined on a region of $\mathbb{R}^n$. It also happens that one may have f defined only on the curve. For example, f might represent the linear density (mass per unit length) of a wire (the curve). Therefore f only has meaning on the curve.

10.2 Line Integrals for Vector Fields

Given a vector field $\mathbf{F}$, it frequently occurs that one wants to compute a line integral where the function f is

$$f = \mathbf{F} \cdot \mathbf{T}$$

where $\mathbf{T}$ is the unit tangent vector to the curve $\mathcal{C}$. Examples of this type of integration are work and circulation discussed below.

Hence we need to evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

To derive a useful formula for such an integral we recall (Week 3) that

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$$

Thus we can write

$$\mathbf{F}\cdot\mathbf{T}\,ds = \mathbf{F}\cdot\frac{\mathbf{r}'}{\|\mathbf{r}'\|}\,\|\mathbf{r}'(t)\|\,dt = \mathbf{F}\cdot\mathbf{r}'\,dt$$

The right-most expression is what we will use in practice to evaluate this type of line integral. However, to stress the independence of the line integral of the parametrisation corresponding to a chosen orientation, it is common to write $\mathbf{r}'\,dt$ as $d\mathbf{r}$.

Let $\mathbf{F}$ be a vector field defined in some region of $\mathbb{R}^n$, and let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be a parametrisation of a curve $\mathcal{C}$ in this region,

the **line integral of F along** $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

One important feature of line integrals of vector fields is that they are *not independent of the orientation of the curve*. The reason is that is that if one reverses the orientation of a curve, then the tangent vector changes sign. Denoting $-\mathcal{C}$ as the curve $\mathcal{C}$ with the opposite orientation, then

$$\int_{-\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds$$

10.3 Fundamental Theorem of Line Integrals

When we introduced vector fields it was noted that an important class of vector fields was that obtained as the gradient of a function of several variables: $\mathbf{F} = \nabla f$. Such vector fields are called **conservative vector fields**. They are important because they arise in practice and because the following holds

Fundamental Theorem of Line Integrals (FTLI). Let $\mathcal{C}$ be a regular curve parametrised by $\mathbf{r} : [a, b] \to \mathbb{R}^n$ and let f be a differentiable function whose gradient vector is continuous on $\mathcal{C}$, then

$$\int_{\mathcal{C}} \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note the analogy with the Fundamental Theorem of Calculus (FTC)

$$\int_a^b F'(t)dt = F(b) - F(a)$$

(See Week 2 notes).

Proving the FTLI is not difficult as it primarily relies on the Chain Rule (Week 4) and the FTC. The manipulations are

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$
$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

The FTLI tells use that if we know our vector field $\mathbf{F}$ is a conservative vector field, and hence given by gradient of some function f, then we can evaluate any line integral of $\mathbf{F}$ over $\mathcal{C}$ simply by evaluating f at the end points of $\mathcal{C}$. Call these points $\mathbf{r}_a$ and $\mathbf{r}_b$. The importance is not just that it simplifies our calculations, but the fact that since the integral depends only on the end points, it in fact must be the same for any curve that starts $\mathbf{r}_a$ and ends at $\mathbf{r}_b$. That is, if $\mathbf{F} = \nabla f$, then

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r},$$

for any two $\mathcal{C}_1$ and $\mathcal{C}_2$ that start at $\mathbf{r}_a$ and end at $\mathbf{r}_b$. The line integral is said to by **path independent**. Note in particular that if $\mathbf{F} = \nabla f$ then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

for any closed curve $\mathcal{C}$ because $\mathbf{r}_a = \mathbf{r}_b$ for a closed curve.

With some mild conditions, it can be shown that if all line integrals of a vector field $\mathbf{F}$ are path independent, or equivalently if the line integral around all closed curves is zero, then $\mathbf{F}$ is a conservative vector field and there is a function f such that a $\mathbf{F} = \nabla f$.

The converse is generally easier, although perhaps less important. If the line integral of $\mathbf{F}$ around a close path is not zero, then $\mathbf{F}$ is definitely not a conservative vector field and it cannot be expressed as the gradient of a function.

10.4 Work and Potential Energy

Work is an important physical concept that you can learn all about in a mechanics module. It is a classic example of a case where one needs to do line integrals of a vector field.

If a force $\mathbf{F}(\mathbf{r})$ acts on a point particle and the particle moves from position $\mathbf{r}_a$ to $\mathbf{r}_b$ along a curve

$\mathcal{C}$, then the work W_{ab} done by the force on the particle is

$$W_{ab} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

This definition is independent of whether or not the force $\mathbf{F}(\mathbf{r})$ is a conservative vector field.

In many situations (gravitational fields for example), but not all, the force $\mathbf{F}(\mathbf{r})$ is a conservative vector field. It can be thus be written as a gradient of a function. One typically defines the function so that $\mathbf{F} = -\nabla V$ where V is a **potential**, and more specifically in this case, V is **potential energy**.

The work done in moving from $\mathbf{r}_a$ to $\mathbf{r}_b$ is given in terms of the potential at the end points

$$W_{ab} = (-V(\mathbf{r}_b)) - (-V(\mathbf{r}_a)) = V(\mathbf{r}_a) - V(\mathbf{r}_b)$$

independently of how the particle moved from $\mathbf{r}_a$ to $\mathbf{r}_b$.

You should visualise the potential V as the height of a hill, or more general landscape. Assume that $V(\mathbf{r}_a) > V(\mathbf{r}_b)$. This means the particle starts out a some high point and moves to some lower point. The work done on the particle is $W_{ab} = V(\mathbf{r}_a) - V(\mathbf{r}_b) > 0$, independently of the path followed from $\mathbf{r}_a$ to $\mathbf{r}_b$. W_{ab} is the energy that can be extracted from the particle as it moves downhill from $\mathbf{r}_a$ to $\mathbf{r}_b$.

Contrarily, if a particle starts at $\mathbf{r}_b$ then one must expend energy to push it uphill to $\mathbf{r}_a$. We must input energy equal to W_{ab}, All work (or energy) differences are encoded in the potential V and are independent of the path taken by the particle. Informally, the force conserves mechanical energy by converting work to potential energy, and back. Gravitational and Coulomb forces are two examples of conservative forces that are frequently described in terms of potentials.

10.5 Circulation

For many vector fields, line integrals around closed curves have physical significance. In fluid dynamics, for example, such integrals give what is known as the **circulation** of the fluid around the curve. In electricity and magnetism, such integrals appear in the integral statement of Maxwell's equations and correspond to a circulation of electric or magnetic fields.

We will focus of the fluids case. Let $\mathbf{v}$ be a vector field corresponding to the velocity of fluid in some region of space (or could be confined to a plane). Then the circulation Γ of $\mathbf{v}$ over a closed curve $\mathcal{C}$ is

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$$

Intuitively this integral corresponds to the net amount the fluid is circulating around the curve.

Knowing the circulation around a body such as a wing or a spinning ball, one can calculate the **lift force** on the body. In the case of a wing, the lift force is what holds the aeroplane up. In the case of a spinning ball, the lift force gives rise to a deflection, or bending, of its path through the air.

10.6 Relationship between Various Line and Surface Integrals

First consider integration of scalar functions. Line integration of scalar functions and surface integration of scalar function are similar concepts. Surface integration can be thought of as the extension of line integration, just as double integration (Week 6) is an extension of single variable integration.

On the other hand, *line integrals of vector functions and flux integrals through a surface correspond to very different concepts.* For line integrals of vector functions, the integrand is the dot product between the vector field $\mathbf{F}(\mathbf{r})$ an the *tangent* $\mathbf{T}$ to the curve. The integral measures total component of $\mathbf{F}(\mathbf{r})$ in direction of the curve.

Such integrals make perfectly good sense in any dimension larger than one. (Dimension $n = 1$ can even be included where vectors just become scalars.) A curve through $\mathbb{R}^n$ makes sense as does a vector field in $\mathbb{R}^n$, for any n. The physical reasons for computing the component of $\mathbf{F}$ along the curve do not depend on dimension. In any dimension

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

gives the net amount by which $\mathbf{F}$ points in the direction of the tangent to the curve.

For flux integrals, the integrand is the dot product between the vector field an the unit *normal* $\mathbf{n}$ to the surface. The integral measures total component of $\mathbf{F}$ crossing the surface. The relevant physical idea is sometime traversing or passing through the surface (whether anything actually crosses the surface of not). The surface must be of dimension one less than the dimension of the space, i.e. we consider a two-dimensional surface in $\mathbb{R}^3$. In fact, it is perfectly sensible to define flux integrals for two dimensional vectors field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$. In this case the flux is through a curve (a "one-dimensional surface"in $\mathbb{R}^2$) and we have

$$\pm \int_C (\mathbf{F} \cdot \mathbf{n}) ds$$

where $\mathbf{n}$ is the principal normal to C. The sign will be determined by context.

Finally, we end with a look ahead to Vector Analysis. In that module you will take this further and learn that there are deep relationships between certain line, surface and volume integrals. For example,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$$

and

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_\Omega \text{div } \mathbf{F} \, dV$$

to be explained ...