Week 2: Elementary Calculus of Vector Functions and Curves

2.1 Derivative of a Vector Function

The differential calculus of vector functions follows mostly from the simple fact that taking the limit of a vector function means taking the limit of each component function

$$\begin{aligned}\lim_{t\to a}\mathbf{r}(t) &= \lim_{t\to a}(x_1(t),\cdots,x_n(t))\\ &= \left(\lim_{t\to a}x_1(t),\cdots,\lim_{t\to a}x_n(t)\right)\end{aligned}$$

Just as for real-valued functions, a vector function is continuous at a point a if

$$\mathbf{r}(a) = \lim_{t\to a}\mathbf{r}(t)$$

Component by component this reads

$$x_1(a) = \lim_{t\to a}x_1(t),\ \cdots,\ x_n(a) = \lim_{t\to a}x_n(t)$$

Hence a vector function is continuous if each of its component function is continuous. (Now you have the formal meaning of continuous that we used last week in defining a curve.)

The derivative of the vector function $\mathbf{r}$ is defined exactly as you would expect based on real-valued functions

$$\frac{d\mathbf{r}}{dt}(t) = \lim_{\triangle t\to 0}\frac{\mathbf{r}(t+\triangle t)-\mathbf{r}(t)}{\triangle t}$$

Again, component by component

$$\begin{aligned}&\lim_{\triangle t\to 0}\frac{\mathbf{r}(t+\triangle t)-\mathbf{r}(t)}{\triangle t}\\ &= \lim_{\triangle t\to 0}\left(\frac{x_1(t+\triangle t)-x_1(t)}{\triangle t},\cdots,\frac{x_n(t+\triangle t)-x_n(t)}{\triangle t}\right)\\ &= \left(\lim_{\triangle t\to 0}\frac{x_1(t+\triangle t)-x_1(t)}{\triangle t},\cdots,\lim_{\triangle t\to 0}\frac{x_n(t+\triangle t)-x_n(t)}{\triangle t}\right)\\ &= \left(\frac{dx_1}{dt}(t),\frac{dx_2}{dt}(t),\cdots,\frac{dx_n}{dt}(t)\right)\end{aligned}$$

Thus the derivative of a vector function is itself a vector and is computed by differentiating component by component. We often denote the derivative with a $'$, e.g. $\mathbf{r}'(t)$, so

$$\boxed{\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}(t) = \left(\frac{dx_1}{dt}(t),\frac{dx_2}{dt}(t),\cdots,\frac{dx_n}{dt}(t)\right)}$$

For example, in space we can write the derivative as

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

If $\mathbf{r}$ is a parametrisation of a curve $\mathcal{C}$ and if the derivative exists at t and is nonzero ($\mathbf{r}'(t) \neq \mathbf{0}$), then the vector $\mathbf{r}'(t)$ is called tangent to $\mathcal{C}$ at the point $\mathbf{r}(t)$. This *definition* makes sense: a line through $\mathbf{r}(t)$ parallel to $\mathbf{r}'(t)$ is the best *linear* approximation to the curve near $\mathbf{r}(t)$. Moreover, if $\mathbf{r}$ and $\bar{\mathbf{r}}$ are two parametrisations of the curve such that $\mathbf{r}(t) = \bar{\mathbf{r}}(\tau)$, then $\mathbf{r}'(t)$ is parallel to $\bar{\mathbf{r}}'(\tau)$.

Recall that last week you learned that a curve is regular if there is a parametrisation such that $\mathbf{r}'$ is defined and nonzero at all points. In other words, regular curves have a tangent vector at each point. Regular curves are nice in that they do not have corners or cusps and these are mainly the ones we want to do calculus on. We will commonly assume, without explicitly stating so, that curves we consider are regular, or at least piecewise regular (see below).

2.2 Differentiation Rules

Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector functions, and $f(t)$ is a differentiable real function, and a is a real number. Then the following hold:

- $\dfrac{d}{dt}a\mathbf{r}(t) = a\mathbf{r}'(t)$
- $\dfrac{d}{dt}\big(\mathbf{r}(t) + \mathbf{s}(t)\big) = \mathbf{r}'(t) + \mathbf{s}'(t)$
- $\dfrac{d}{dt}f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
- $\dfrac{d}{dt}\big(\mathbf{r}(t) \cdot \mathbf{s}(t)\big) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
- $\dfrac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

These are obvious extensions of what you know for functions from $\mathbb{R} \to \mathbb{R}$ which can be proved by expressing vector functions in terms of components. You should learn and be able to verify each of these.

2.3 A Little Mechanics

When $\mathbf{r}(t)$ is the path of a particle and t is time, then the following terminology is used:

- $\mathbf{r}'(t)$ is the **velocity**. It is a vector quantity.

$$\text{velocity} = \mathbf{v}(t) = \mathbf{r}'(t)$$

- The magnitude of the velocity is the **speed**. It is a scalar quantity.

$$\text{speed} = c(t) = \|\mathbf{v}(t)\| = \left\| \frac{d\mathbf{r}}{dt}(t) \right\|$$

- The derivative of velocity is the **acceleration**. It is a vector quantity. It is not, in general, tangent to $\mathcal{C}$.

$$\text{acceleration} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt}(t)$$

Example. Consider a particle moving with a constant *speed*, $c^2(t) = \mathbf{v}(t) \cdot \mathbf{v}(t) = const$. Then $0 = \frac{d}{dt}(\mathbf{v}(t)\mathbf{v}(t)) = 2\mathbf{v}(t) \cdot \mathbf{a}(t)$, where we used the rule for differentiating scalar products stated above. We conclude that the acceleration is perpendicular to the tangent at every point of the particle's path.

2.4 Arc Length

We are going to want to perform integration along curves, now and later in this course. To begin, let us ask the question: "given a parametrised curve $\mathcal{C}$, what is its length?" The answer will require integration along $\mathcal{C}$.

Our method for deriving the integral will be to take $\mathcal{C}$ and divide it up into "small" pieces, or arcs, and sum up the lengths of these pieces. We then take the limit of the sum as the size of the pieces goes to zero while the number of pieces going to infinity.

Let $\mathbf{r} : I = [a, b] \to \mathbb{R}^n$ be a parametrisation of $\mathcal{C}$. To divide $\mathcal{C}$ into small arcs, all we need to do is divide I into small segments of size $\triangle t$. Let

$$t_0 = a, t_1 = a + \triangle t, \cdots, t_j = a + j\triangle t, \cdots, t_N = b.$$

So there are N segments of length $\triangle t = (b - a)/N$.

Now, the j^{th} segment $[t_j, t_{j+1}]$ will get mapped by $\mathbf{r}$ to a small arc from $\mathbf{r}(t_j)$ to $\mathbf{r}(t_{j+1})$ of the curve $\mathcal{C}$. The length of the arc, $\triangle s_j$, will be approximately the length of the chord from $\mathbf{r}(t_j)$ to $\mathbf{r}(t_{j+1})$

$$\triangle s_j \approx \|\mathbf{r}(t_{j+1}) - \mathbf{r}(t_j)\|$$

or, multiplying and dividing by $\triangle t$

$$\triangle s_j \approx \left\| \frac{\mathbf{r}(t_j + \triangle t) - \mathbf{r}(t_j)}{\triangle t} \right\| \triangle t \qquad (1)$$

The total length s is then approximately

$$s = \sum_{j=0}^{N-1} \triangle s_j \approx \sum_{j=0}^{N-1} \left\| \frac{\mathbf{r}(t_j + \triangle t) - \mathbf{r}(t_j)}{\triangle t} \right\| \triangle t$$

Taking the limit $N \to \infty$ with $\triangle t \to 0$ we obtain the following

> The length s of a curve $\mathcal{C}$, denoted $\ell(\mathcal{C})$, is given by
>
> $$s = \ell(\mathcal{C}) = \int_a^b \left\| \frac{d\mathbf{r}}{dt}(t) \right\| \, dt$$
>
> where $\mathbf{r}(t)$, $t \in [a, b]$ is a parametrisation of $\mathcal{C}$.

Strictly speaking, the above formula should be regarded as the definition of arc length and the heuristic discussion leading to it - as a motivation.
Example. If $\mathcal{C}$ is the graph of $f : [a, b] \to \mathbb{R}$, $\mathcal{C} = \{(x, f(x)) \mid x \in [a, b]\}$, then $\ell(\mathcal{C}) = \int_a^b dx \sqrt{1 + f'(x)^2}$.

Discussion

- We have implicitly assumed that the parametrisation $r(t)$ does not repeat points of $\mathcal{C}$. In practice this is always obvious. For example, to compute the length of a circle (i.e. its circumference) one would not use $r(t) = (R \cos t, R \sin t)$, with $t \in [0, 4\pi]$.
- Independence of parametrisation. While we defined the length of a curve based on a parametrisation, the length of a curve is independent of the parametrisation. If $\mathbf{r}_1 \colon [a, b] \to \mathbb{R}^n$ and $\mathbf{r}_2 \colon [c, d] \to \mathbb{R}^n$ are two parametrisations of the same curve $\mathcal{C}$, the lengths will be the same. **Exercise.** Let $f : [c, d] \to [a, b]$ be a smooth bijection: $f' > 0$. Check that the arc lengths computed using the parametrisations $\mathbf{r} : [a, b] \to \mathbb{R}^n$ and $\mathbf{r} \circ f : [c, d] \to \mathbb{R}^n$ are equal to each other.
- Arc length is never negative. The limits of integration are *always* from the left endpoint (smallest value) of I to the right endpoint (largest value) of I.
- An infinitesimal increment dt in the parameter t corresponds to an infinitesimal increment of arc length ds along the curve $\mathcal{C}$. These are related by

  $$ds = \|\mathbf{r}'(t)\| dt$$

  This is the infinitesimal version of Eq. (1). It can be equivalently written as

  $$\frac{ds}{dt} = \|\mathbf{r}'(t)\| \qquad (2)$$

- Eq. (2) and the formula for length have simple physical interpretations when $\mathbf{r}(t)$ is a particle path. Recall that we defined $\|\mathbf{r}'(t)\|$ to be the speed $c(t)$ of a particle. Eq. (2) states simply that the change in distance per change in time, i.e. ds/dt, is the speed of the particle. Likewise, the equation for arc length becomes

$$\text{Distance} = \int_a^b c(t)dt$$

i.e. distance travelled is just the integral of speed over time a interval.

- We learned that to differentiate a vector-valued function one simply differentiates component by component, e.g. given

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

the derivative is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

You might have therefore expected to see a similar start to the integration of vector-valued functions, e.g.

$$\int_a^b \mathbf{r}(t)dt = \int_a^b x(t)dt\,\mathbf{i} + \int_a^b y(t)dt\,\mathbf{j} + \int_a^b z(t)dt\,\mathbf{k}$$

This formula is correct and it is the correct meaning of $\int_a^b \mathbf{r}(t)dt$. The reason we did not start with this is that such integration does not often arise in the course, and in any case it is easy. Integration along a curve, such as in computing arc length, is more important and we will see it again later in this course when we do line integrals. Be sure you understand the difference between

$$\int_a^b \mathbf{r}(t)dt \quad \text{and} \quad \int_a^b \|\mathbf{r}'(t)\|dt.$$

Example. If $\mathcal{C} = \{(R\cos(t), R\sin(t)) \mid t \in [0, 2\pi]\}$, a circle of radius R centred on the origin, then $\int_0^{2\pi} \mathbf{r}'(t)dt = 0$, $\int_0^{2\pi} ||\mathbf{r}'(t)||dt = 2\pi R$.

2.5 Curves in Multiple Segments

Last week we already noted that we often want to work with curves composed of multiple segments. Let $\mathcal{C}$ be the union of k segments which meet end to end

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k,$$

Assuming each segment $\mathcal{C}_j$ is regular, then the curve $\mathcal{C}$ is **piecewise regular**. Such curves may fail to be regular where the curves join. For example if $\mathcal{C}$ is square then it is not regular because it has 4 corners. However it can clearly be composed of 4 regular curves and hence $\mathcal{C}$ is piecewise regular.

The length of $\mathcal{C}$ is given by

$$\ell(\mathcal{C}) = \ell(\mathcal{C}_1) + \ell(\mathcal{C}_2) + \cdots + \ell(\mathcal{C}_k)$$

In practice one would typically use whatever parametrisation is most convenient for each segment to compute the length of that segment, and then add up the lengths to obtain the length of $\mathcal{C}$.

2.6 Parametrisation by Arc Length

Imagine that the interval I is a straight piece of wire and that the mapping $\mathbf{r}$ corresponds to taking the wire and bending in it around into a curve $\mathcal{C}$ *without any stretching or compression*. Then there is an exact correspondence between distances along the interval I and arc length along $\mathcal{C}$. For example, let $I = [0, 10]$ (think of a 10 centimetre long straight wire). Each subinterval $[0, 1], [2, 3], \ldots, [9, 10]$ of I gets mapped by $\mathbf{r}$ to a segment of $\mathcal{C}$ with arc length one. (Each centimetre of straight wire gets bent around, but maintains its length of one centimetre.)

Such a parametrisation of a curve $\mathcal{C}$ is called an **arc-length parametrisation** and the curve is said to be parametrised by arc length. This is also call the **natural parametrisation**. Denote such are parametrisation by $\mathbf{r}_{\text{arc}}$. It has the property that the arc length between $\mathbf{r}_{\text{arc}}(t_1)$ and $\mathbf{r}_{\text{arc}}(t_2)$ is precisely $|t_2 - t_1|$, the distance between t_1 and t_2.

We shall simplify our discussion by assuming $\mathcal{C}$ has finite length $\ell(\mathcal{C}) < \infty$ and when parametrised by arc-length the interval I is $I = [0, \ell(\mathcal{C})]$. (One can in fact parametrise infinitely long curves by arc-length.)

Arc-length parametrisations are important for two reasons. The first is conceptual. We know that there are infinitely many ways to parametrise a curve, yet these all give the same arc-length. By choosing arc length along the curve *as the parameter* we select a parametrisation that is intrinsic to the curve rather than arbitrary. Apart from orientation and shifts of the starting point, the parametrisation by arc length is unique.

The second reason is more important in practice. Let $\mathbf{r}_{\text{arc}}(t)$ be an arc-length parametrisation of $\mathcal{C}$. Then by definition

$$\int_0^s \|\mathbf{r}'_{\text{arc}}(t)\| \, dt = s$$

for all $s \in [0, \ell(\mathcal{C})]$. Differentiating this with respect to s gives (see Additional Material),

$$\frac{d}{ds}\int_0^s \|\mathbf{r}'_{\mathrm{arc}}(t)\|\, dt = \frac{d}{ds}\, s$$
$$\|\mathbf{r}'_{\mathrm{arc}}(s)\| = 1$$

Hence the derivative $\mathbf{r}'_{\mathrm{arc}}$ *has unit length for an arc-length parametrisation.* One often says that an arc-length parametrisation has speed one, whether or not one is thinking of particle paths. Many calculations for curves, such as those we will see next week, greatly simplify for arc-length parametrisations.

To obtain an arc-length parametrisation of a curve $\mathcal{C}$, start with some parametrisation $\mathbf{r}(t) : [a, b] \to \mathbb{R}^n$ and compute the arc length from $\mathbf{r}(a)$ to $\mathbf{r}(t)$ for $t \in [a, b]$

$$s = \ell(t) = \int_a^t \|\mathbf{r}'(\tau)\| d\tau$$

Inverting this relationship, $t = \ell^{-1}(s)$, gives us the parameter value t corresponding to arc length s from the end point $\mathbf{r}(a)$. Using this to eliminate t in favour of s, we can in principle obtain an arc-length parametrisation

$$\mathbf{r}_{\mathrm{arc}}(s) = \mathbf{r}(\ell^{-1}(s))$$

Be warned, however, even though this parametrisation exists in principle, obtaining an explicit arc-length parametrisation is generally impossible because no formula exists for $\ell^{-1}(s)$.

It is common practice to use s rather than t for the independent variable in an arc-length parametrisation, and so to write $\mathbf{r}_{\mathrm{arc}}(s)$ as appose to $\mathbf{r}_{\mathrm{arc}}(t)$. We will sometimes do this if we want to emphasise arc-length parametrisation.

Additional Material

Partitions

In our derivation of the formula for arc length we partitioned the interval I into N equal segments of equal size $\triangle t = t_{j+1} - t_j$. It is not necessary that the segments be of equal size. All that matters is that in the limit all the segments of our partition shrink to zero. However, regular partitions are the most intuitive and they suit our purposes well. Hence we use regular partitions now and throughout this course when we derive integration formulas. In later analysis modules this will be treated more rigorously.

You probably are familiar with partitions from the integral calculus you already know. Have a look at the Wikipedia article *Riemann sum* for a nice discussion of different approximating sums converging to the the same limit. (While this does not fully address the issue of irregular partitions, it gives you some idea why in the end different approximations will converge to the same resulting integral.)

Differentiating limits of integrals

In the treatment of arc-length parametrisation we used the following result:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

In other words, if you differentiate a definite integral with respect to the upper limit of integration, the result is the integrand.

This is called the **Fundamental Theorem of Calculus (FTC)** or sometimes the **First Fundamental Theorem of Calculus**. You should be familiar with this result, but if not you can see that it follows immediately from the following method you use to evaluate definite integrals:

$$\int_a^b f(t)dt = F(b) - F(a), \quad \text{where} \quad F'(x) = f(x)$$

This is true for any b, so let b vary and call it x instead

$$\int_a^x f(t)dt = F(x) - F(a)$$

Differentiating with respect to x

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \frac{d}{dx}\left(F(x) - F(a)\right) = F'(x) - 0 = f(x)$$

In most treatments, one starts with the FTC and from this derives as a corollary the method for evaluating definite integrals. If you are not familiar with this then you should read the Wikipedia article *Fundamental theorem of calculus.*

Later in the course we will expand on this and consider cases such as $\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right)$ and more complicated expressions.

Hyperbolic trigonometric functions

It is time you starting recalling or learning about hyperbolic trig function. The basic functions cosh and sinh are defined by

$$\cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}, \quad t \in \mathbb{R}.$$

You should know what the graphs $\cosh t$ and $\sinh t$ and also of $\tanh t \equiv \frac{\sinh t}{\cosh t}$ look like. From the definitions you can verify the following properties:

(i) $\cosh^2 t - \sinh^2 t = 1$,

(ii) $\frac{d}{dt}\cosh t = \sinh t, \quad \frac{d}{dt}\sinh t = \cosh t$,

(iii) $\cosh 2t = \cosh^2 t + \sinh^2 t, \quad \sinh 2t = 2\cosh t \sinh t$,

(iv) $\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sqrt{t^2-1}}, \quad \frac{d}{dt}\sinh^{-1} t = \frac{1}{\sqrt{t^2+1}}$,

(v) $\sinh^{-1} t = \log(t + \sqrt{t^2+1})$.

Hint: Formulas (iv) and (v) are easily verified by letting $t = \cosh y$ or $t = \sinh y$ as needed. In (iv) it is understood that $\cosh^{-1}$ means the positive branch of the inverse of cosh.

The reason we care particularly about the hyperbolic trig functions now is that they occur frequently in arc-length integrals. In particular, the following integral is common.

$$I = \int \sqrt{t^2+1}\, dt$$

This can be integrated by parts to give

$$I = t\sqrt{t^2+1} - \int \frac{t^2}{\sqrt{t^2+1}}\, dt = t\sqrt{t^2+1} - \int \frac{t^2+1}{\sqrt{t^2+1}}\, dt + \int \frac{1}{\sqrt{t^2+1}}\, dt = t\sqrt{t^2+1} - I + \int \frac{1}{\sqrt{t^2+1}}\, dt$$

Therefore

$$2I = t\sqrt{t^2+1} + \int \frac{1}{\sqrt{t^2+1}}\, dt$$

So

$$\int \sqrt{t^2+1}\, dt = \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\sinh^{-1} t = \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log(t + \sqrt{t^2+1})$$

You should be able to redo this calculation for the more general case $\int \sqrt{t^2+a^2}\, dt$

Using integral tables

As you have no doubt noticed, integration is more tricky than differentiation and historically this has been dealt with through *Integral Tables* - collections of integrals organised by category. The most famous is by Gradshteyn and Ryzhik which has over 1000 pages of integrals.

You may use an Integral Tables/web for completing the assignment sheets in this module and I will provide a link to a useful one. However, you should known how to derive the integrals you use and you should practice your substitutions regularly or you will get rusty.