Week 3: Differential Geometry of Curves

Introduction

We now know how to differentiate and integrate along curves. This week we explore some of the geometrical properties of curves that can be addressed using differential calculus. We will be particularly interested in the bending and twisting of curves and how to describe and quantify this bending and twisting.

3.1 Curves in the Plane

Unit tangent vector

Given a regular parametrisation $\mathbf{r}$ of a curve $\mathcal{C}$, we know that $\mathbf{r}'(t) \neq \mathbf{0}$ is tangent to $\mathcal{C}$ at $\mathbf{r}(t)$. However the length of this tangent vector will depend on the particular parametrisation. We define the **unit tangent vector** to be

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|},$$

which is the same for all parametrisations corresponding to a given orientation. If $\mathbf{r}$ is an arc-length parametrisation of $\mathcal{C}$, then $||\mathbf{r}'|| \equiv 1$ and the expression for $\mathbf{T}$ simplifies to

$$\mathbf{T} = \mathbf{r}'.$$

Principal normal vector

Start with a little calculation. We know that by definition $\|\mathbf{T}(t)\| = 1 = const$ for all t. So

$$\|\mathbf{T}\|^2 = \mathbf{T} \cdot \mathbf{T} = const$$

Differentiating gives

$$\frac{d}{dt}\big(\mathbf{T} \cdot \mathbf{T}\big) = \mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2\,\mathbf{T}' \cdot \mathbf{T} = 0,$$

so

$$\mathbf{T}' \cdot \mathbf{T} = 0.$$

Thus either $\mathbf{T}' = 0$ or $\mathbf{T}'$ is perpendicular to $\mathbf{T}$. (Compare this with the Week 2 mechanical example of a particle moving with constant speed.) We define the **principal normal vector** to be the unit vector in the direction of $\mathbf{T}'$:

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}.$$

If $\mathbf{T}' = \mathbf{0}$ the principal normal vector is not defined.

Curvature

Intuitively curves can have different amounts of bending and this is seen in how quickly the unit tangent vector $\mathbf{T}$ changes as one moves along the curve. We know $\mathbf{T}'$ points in the direction of $\mathbf{N}$, but its size quantifies how rapidly the curve is bending with parameter t. Unfortunately, $\mathbf{T}'$ itself will depend on the parametrisation used. To remove this dependence, one defines the **curvature** κ to be the magnitude of instantaneous change in $\mathbf{T}$ with respect to arc length

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

The **radius of curvature** ρ is defined to be

$$\rho = \frac{1}{\kappa}.$$

From an arc-length parametrisation the calculation of curvature and radius of curvature is this easy. However, we will not always have an arc-length parametrisation at our disposal. For a general parametrisation $\mathbf{r}$, one must additionally normalise using $\|\mathbf{r}'\|$

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|}.$$

To check the equivalence to the previous expression for curvature, use the chain rule and recall from Eq. (2) of Week 2, that $\dfrac{ds}{dt} = \|\mathbf{r}'\|$. Hence

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \mathbf{T}'\frac{1}{ds/dt} = \mathbf{T}'\frac{1}{\|\mathbf{r}'\|}.$$

Discussion

Example: Consider a circle of radius R parametrised by $\mathbf{r} = (R\cos t, R\sin t)$. Then $\mathbf{T} = (-\sin t, \cos t)$ and $\mathbf{N} = (-\cos t, -\sin t)$. The curvature is $\kappa = 1/R$ and the radius of curvature is $\rho = R$.

Note any regular parametrisation of the circle would give the same $\mathbf{T}$, within a $\pm$ sign. Flipping the orientation flips $\mathbf{T}$ but not $\mathbf{N}$. You should verify this.

An intuitive way to think about curvature is in terms of the **osculating circle** – the circle which most closely approximates a curve $\mathcal{C}$ near some point P on $\mathcal{C}$. The radius of the osculating circle is the radius of curvature of $\mathcal{C}$ at P. In other words, moving along $\mathcal{C}$ in the vicinity of P is like moving along a circle of radius $\rho = 1/\kappa$. (If the curvature is zero, then the osculating circle becomes a straight line.)

As you will soon learn, the calculation of curvature can get messy. The following is a useful alternative formula for both plane and space curves

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

We will not derive this, but it is not hard to show that this is equivalent to $\|\mathbf{T}'\|/\|\mathbf{r}'\|$.

Note, for a plane curve the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are two dimensional. The cross product is therefore understood to mean taking these vectors to be in $\mathbb{R}^3$, but with zero third component. The cross product will then be a vector with a single component, pointing out of the plane of the curve. It is common in many situations to consider the cross product of vectors in the plane using this interpretation. After a few examples this becomes very easy and natural.

Frenet coordinates

$\mathbf{T}$ and $\mathbf{N}$ are basis vectors for a coordinate system called **Frenet coordinates**. These basis vectors vary along the curve. While the idea of a local system of coordinates varying along a curve might seem odd at first, the concept is well known from your everyday experience. Think of driving in a car and/or giving driving directions. When studying the geometry of a curve, it is very useful to have an intrinsic system of coordinates dictated by a curve itself.

It is easy to quantify how Frenet coordinates change as we move along the curve. Let $\mathbf{r} : [0, L] \to \mathbb{R}^2$ be the arc length parametrisation of curve $\mathcal{C}$. Combining the definitions of principal normal $\mathbf{N}$ and curvature κ, we find that

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s), \ s \in [0, L],$$

which expresses the rate of change of $\mathbf{T}$ in terms of the principal normal vector and the curvature. What about $\mathbf{N}'$? As $\mathbf{N}$ is a unit vector, $\mathbf{N}'$ is perpendicular to it, which means that in two dimensions it must be parallel to $\mathbf{T}$. Therefore, we must have $\mathbf{N}' = b\mathbf{T}$ for some real valued function b. Can we determine b? As $\mathbf{N} \cdot \mathbf{T} \equiv 0$, $(\mathbf{N} \cdot \mathbf{T})' \equiv 0$. Or, $\mathbf{N}' \cdot \mathbf{T} + \mathbf{N} \cdot \mathbf{T}' = 0$. Substituting $\mathbf{T}' = \kappa\mathbf{N}$, $\mathbf{N}' = b\mathbf{T}$ and using that $\mathbf{N} \cdot \mathbf{N} = 1$, $\mathbf{T} \cdot \mathbf{T} = 1$ we find that

$b = -\kappa$. Therefore,

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s),\ \ s \in [0, L].$$

The derived pair of equations relating $\mathbf{N}'$ and $\mathbf{T}'$ to $\mathbf{N}$ and $\mathbf{T}$ is called Frenet-Serret's equations. They have a simple geometric meaning: as we move along the curve, Frenet's basis is rotating with angular velocity equal to the curvature.

If we fix the curvature and treat Frenet-Seret's equations as a system of ODE's for $\mathbf{T}, \mathbf{N}$, we can prove the fundamental theorem of planar curves: given a positive continuous function $\kappa : [a, b] \to \mathbb{R}^2$ there exists a unique regular curve with curvature κ. (The uniqueness is understood up to rigid shifts and rotations of the curve.) Therefore, all the information about the *shape* of a planar curve is contained in a single positive function of one variable!

3.2 Curves in Space

Everything we have discuss thus far applies to curves in space. The unit tangent vector, principal normal vector, curvature and radius of curvature all apply. However, in $\mathbb{R}^3$ we need one more basis vector and also a new concept: torsion.

Binormal vector

The **binormal vector** is defined to be:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, and has unit length since both $\mathbf{T}$ and $\mathbf{N}$ do. Together $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form the Frenet basis in $\mathbb{R}^3$. Since $\mathbf{N}$ is not defined where $\kappa = 0$, neither is $\mathbf{B}$.

Torsion

Just as curvature measures how quickly the unit tangent vector changes direction as one moves along a curve, the torsion measures how quickly the binormal vector changes direction along a curve.

Why is the change in the binormal direction interesting? Consider a curve that is confined to the $x - y$ plane. Think of a bent wire lying flat on a table top. Both $\mathbf{T}$ and $\mathbf{N}$ will be in the plane, and while they vary, $\mathbf{B}$ will always be perpendicular to the plane and thus will not vary. The torsion of such a curve is zero. Now, bend the wire so that it does not lie flat on the table. This will introduce some torsion. The plane defined by $\mathbf{T}$ and $\mathbf{N}$ is not the same everywhere along the curve and so $\mathbf{B}$ must vary. The change in $\mathbf{B}$ along the curve is the relevant measure of how the plane formed by $\mathbf{T}$ and $\mathbf{N}$ varies along the curve.

The magnitude of the torsion is defined as

$$|\text{torsion}| = \left\| \frac{d\mathbf{B}}{ds} \right\|$$

Without going into unnecessary detail, the sign of torsion is significant and the full definition accounts for this sign. The **torsion** τ is defined by:

$$\tau = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}$$

The minus sign is such that a right-handed curve, e.g. a right-handed helix, has positive torsion. Notice that the compatibility of the last two definitions is not obvious - why should the magnitude of $\mathbf{B}'$ be related to the projection of $\mathbf{B}'$ onto $\mathbf{N}$? Fortunately, there is no inconsistency, as $\mathbf{B}'$ is always parallel or anti-parallel to $\mathbf{N}$, see below.

For a general parametrisation $\mathbf{r}(t)$ the definition of torsion is

$$\tau = -\frac{\mathbf{N} \cdot \mathbf{B}'}{\|\mathbf{r}'\|}$$

Discussion

Example: Compute the curvature and torsion of the right-handed helix $\mathbf{r} = (\cos t, \sin t, t)$. Answer: $\kappa = 1/2$, $\tau = 1/2$.

Example: Compute the curvature and torsion of the left-handed helix $\mathbf{r} = (\cos t, \sin t, -t)$. Answer: $\kappa = 1/2$, $\tau = -1/2$.

Example: Compute the torsion of the curve parametrised by

$$\mathbf{r} = (\frac{1}{\sqrt{2}} \cos t, \sin t, -\frac{1}{\sqrt{2}} \cos t)$$

Answer: $\tau = 0$. In fact the curve is a circle and lies in the plane of all points perpendicular to $\mathbf{B} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}$.

3.3 Summary

General regular parametrisation

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} \qquad \mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} \qquad \rho = \frac{1}{\kappa} \qquad \tau = -\frac{\mathbf{N} \cdot \mathbf{B}'}{\|\mathbf{r}'\|}$$

Alternative formula for curvature:

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}$$

Arc-length parametrisation

For an arc-length or natural parametrisation $\mathbf{r}_{\text{arc}}(s)$

$$\mathbf{T} = \mathbf{r}'_{\text{arc}} \qquad \mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\kappa = \|\mathbf{T}'\| \qquad \rho = \frac{1}{\kappa} \qquad \tau = -\mathbf{N} \cdot \mathbf{B}'$$

$$\kappa = \|\mathbf{r}'_{\text{arc}} \times \mathbf{r}''_{\text{arc}}\|$$

where $'$ is understood to be $\frac{d}{ds}$.

Beads on a wire

The differential geometry we have considered can be viewed with the following physical picture using discrete approximations to derivatives. Two closely spaced beads on a bent wire (curve) can be used to approximation the tangent to the wire. (Just draw a unit vector through the beads). With three closely spaced beads one can approximate the tangent at two points nearby points. If the three beads to not line on a straight line then the wire has non-zero curvature. The three beads define a plane that approximates the **T**-**N** plane at the middle bead. (This plane is call the osculating plane and if one were to draw the osculating circle, it would lie in this plane.) Now put 4 beads close together on the wire. If the 4 beads do not lie in a plane, then the wire has non-zero torsion. Essentially, the plane defined by beads 1, 2 and 3 is different from the plane defined by beads 2, 3 and 4.

Small final aside

In case you were wondering, the expression for curvature can be written analogously to the expression for torsion. Consider an arc-length parametrisation $\mathbf{r}(s)$. (A general parametrisation is similar with the requisite factors of $\|\mathbf{r}'\|$.) We know that $\mathbf{T}'$ points in the direction of $\mathbf{N}$, and its magnitude is the curvature κ. Thus we could have written $\kappa\mathbf{N} = \mathbf{T}'$ for the definition of curvature rather than $\kappa = \|\mathbf{T}'\|$. Similarly, it is straightforward to show that $\mathbf{B}'$ points in the direction of $\mathbf{N}$, and its magnitude is $|\tau|$.

One can then write the expressions defining the curvature and torsion in a nearly symmetric fashion

$$\mathbf{T}' = \kappa\mathbf{N} \qquad \mathbf{B}' = -\tau\mathbf{N}$$

The interpretation is that the derivative of the tangent vector has a magnitude κ and direction $\mathbf{N}$. κ cannot be negative. The derivative of the binormal has magnitude $|\tau|$. τ itself may be positive or negative. By convention, τ is positive when $\mathbf{B}'$ points in the opposite direction of $\mathbf{N}$. By analogy with the two dimensional case one can show that

$$\mathbf{N}' = \tau\mathbf{B} - \kappa\mathbf{T}.$$

We get a triple of Frenet-Serret's equations which can be applied to the study of geometric properties of curves in $\mathbb{R}^3$ using analytical methods. In particular, one can show that the *shape* of a regular curve in $\mathbb{R}^3$ is completely determined by its curvature and torsion.

Additional Material

Cross product

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ be vectors in $\mathbb{R}^3$. The **cross product** or **vector product** of $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$ and is the vector

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \tag{3}$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \tag{4}$$

or symbolically

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$

where bars, $|\,|$, mean determinant.

The above definition is in terms of the components of vectors $\mathbf{a}$ and $\mathbf{b}$. There is an alternative definition of the cross product not involving the components explicitly:

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta\, \hat{\mathbf{n}}, \tag{6}$$

where θ is the smaller angle between $\mathbf{a}$ and $\mathbf{b}$ ($0 \le \theta \le 180°$), and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose direction is given by the *right-hand rule.* (See Wikipedia or other sources for the right-hand rule.)

Properties:

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, i.e. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.
- The magnitude of $\mathbf{a} \times \mathbf{b}$ follows directly from the second definition above: $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$. Of particular importance is that $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram formed from $\mathbf{a}$ and $\mathbf{b}$.
- If $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector functions, $\dfrac{d}{dt}\big(\mathbf{r}(t) \times \mathbf{s}(t)\big) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$

There are many uses for the cross product. We have already used it to define the binormal vector as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. From the properties of cross product we know that $\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, and the its length is $\|\mathbf{T}\|\|\mathbf{N}\| \sin 90° = 1$. You should be able to work out that $\mathbf{B}'$ is perpendicular to both $\mathbf{T}$ and $\mathbf{B}$.

Triple product

Important later is the triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Writing $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ and using the above definition of cross product (together with properties of the dot product):

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The importance of the triple product is that it gives the volume of the parallelepiped formed from the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Specifically, let P be the parallelepiped with three adjacent sides given by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Then the volume of P is given by the absolute value of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Moreover, if the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are order correctly (using the right-hand rule so that $\mathbf{c}$ is "in the direction" of $\mathbf{a} \times \mathbf{b}$) then the absolute value is not needed.

Finally, note that:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Planes

You should know that given a non-zero vector $\mathbf{n}$ in $\mathbb{R}^3$, the set of points

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, \mathbf{r} \cdot \mathbf{n} = 0\}$$

is a plane passing through the origin. The vector $\mathbf{n}$ is said to be **normal** to the plane.

More generally, if the plane with normal vector $\mathbf{n}$ does not pass through the origin, but is known to pass through come point $\mathbf{r}_0$, then it is given by the set of points

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0\}$$

Using $\mathbf{r} = (x, y, z)$, $\mathbf{r}_0 = (x_0, y_0, z_0)$, and $\mathbf{n} = (A, B, C)$, the equation for the plane can be written in standard form as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

One can instead describe the set of points on plane as follows. Given two non-zero, non-colinear vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$. These vectors generate a plane passing through the origin

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, \mathbf{r} = u\mathbf{a} + v\mathbf{b}, (u, v) \in R^2\}$$

or a plane passing through a point $\mathbf{r}_0$

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, \mathbf{r} = u\mathbf{a} + v\mathbf{b} + \mathbf{r}_0, (u, v) \in R^2\}$$

By the properties of the cross product, a vector normal to this plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b}$.