Week 5: Geometry and Applications

Introduction

Now that we have some tools from differentiation, we can study geometry, motion, and few other issues associated with functions of several variables.

Much of the emphasis will be on the new tool at our disposal – level sets of functions of several variables. Not only are level sets valuable for visualising and understanding functions, they also provide a powerful technique in applications, particularly where one is interested in geometrical structures that evolve over time.

You may want to look now at the *Recap of Curves and Surfaces* at the end of the chapter and return to it again after you have finished the chapter.

5.1 Linear Approximation and Tangent to a Graph

Recall from functions of one variable $f : \mathbb{R} \to \mathbb{R}$ that the derivative provides a linear approximation to a function near a point (assuming the derivative f' exists)

$$f(x) \approx f(a) + f'(a)(x - a) \tag{9}$$

In this expression a is the fixed (but arbitrary) point at which the derivative is evaluated.

The graphical view is that the graph $y = f(x)$, (generally not linear), is approximated by the tangent line to the graph at any point where f is differentiable. Equivalently, if one zooms in on the graph of any function around a point where it is differentiable, eventually the graph looks linear.

From the point of view of approximations, one can use the derivative to approximate values of a function near a known value, e.g.

$$\sin(x) \approx x, \quad \frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x,$$

for small x. This is known as the **linear approximation**.

<u>Remark</u>: Linear approximation is not primarily valuable because it lets us compute numbers, such as $\sin(0.1) \approx 0.1$. The main value is in replacing a non-linear function with an approximate linear one.

You know this from Differential Equations. The ODE for a pendulum without friction is

$$\ddot{\theta} = -k \sin\theta$$

Have fun solving this. However, if the motion is small about the bottom, then we can use $\sin\theta \approx \theta$ to obtain the easily solved equation

$$\ddot{\theta} = -k\theta$$

We will extend linear approximation to several variables. We focus on functions of two variables, for which the geometry can be visualised, but the expressions are similar in any number of variables. The two-variable generalisation of Eq. (9) is

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \tag{10}$$

where f is assumed differentiable at (a,b).

The graphical view of this expression is that the graph $z = f(x,y)$ is in general a "curved" surface, i.e. not a plane. The graph of the linear approximation on the right hand side is a plane, called the **tangent plane** to the graph of f at the point $(x, y, z = f(x,y))$.

Tangent lines to curves laying the graph of f and passing through $(x, y, f(x,y))$ belong to the tangent plane through $x, y, f(x,y))$. This is particularly easy to see for curves formed by cutting the graph by planes $x = const$ and $y = const$. See figures below. Fixing $y = b$, Eq. (10) reduces to a one-variable equation

$$\underbrace{f(x,b)}_{\text{curve: } z=g(x)} \approx \underbrace{f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a)}_{\text{tangent: } z=g(a)+g'(a)(x-a)}$$

where we have indicated what the corresponding graphs of the two sides would be, using the notation from Week 4, $g(x) = f(x,b)$. Similarly if we can cut at $x = a$

$$\underbrace{f(a,y)}_{\text{curve: } z=h(y)} \approx \underbrace{f(a,b) + \frac{\partial f}{\partial y}(a,b)(y-b)}_{\text{tangent: } z=h(b)+h'(b)(y-b)}$$

where $h(y) = f(a,y)$.

The two tangent lines lie in the tangent plane. There is nothing special about cuts at constant x and constant y, any similar cut will intersect the graph of f in a curve and intersect the tangent plane in a corresponding tangent line.

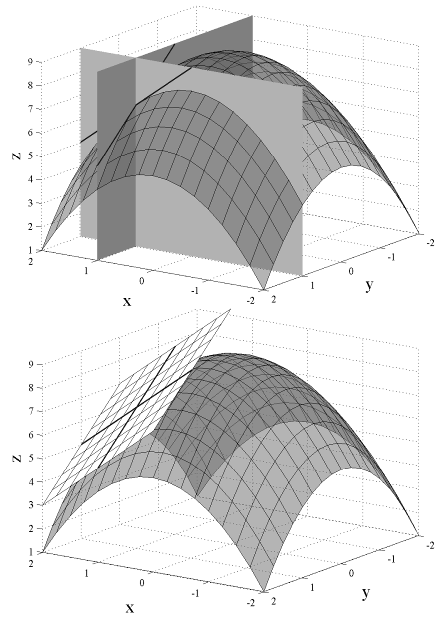

Figure 5: Both the x and y partial derivatives (top) and tangent plane to the graph (bottom) of the function $f(x, y) = 9 - x^2 - y^2$.

Thus, assuming the function is differentiable, the surface $z = f(x, y)$ can be approximated by a plane tangent to the surface. If one zooms in on the graph of the function, eventually the graph looks like a plane.

You can easily show that the equation for the tangent plane can be written in standard form as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

where

$$A = \frac{\partial f}{\partial x}(a, b), \quad B = \frac{\partial f}{\partial y}(a, b), \quad C = -1,$$

$$x_0 = a, \quad y_0 = b, \quad z_0 = f(a, b)$$

From the point of view of approximations, one can use equation (10) to obtain the linear approximation to a function near a known value, e.g.

$$(x + y)e^{xy} \approx x + y, \quad \sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$$

for small x and y. This is sometimes referred to as the tangent plane approximation.

The generalisation of Eq. (10) to any number of variables is obvious. For this it is best to use vector notation and let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Then

If $f(\mathbf{x})$ is differentiable at a point $\mathbf{x}_0$, the linear approximation of $f(\mathbf{x})$ about $\mathbf{x}_0$ is

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

5.2 Differentials

We begin by re-arranging our tangent plane approximation into the following form

$$f(\mathbf{x}) - f(\mathbf{x}_0) \approx \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

Then define $\triangle f = f(\mathbf{x}) - f(\mathbf{x}_0)$ and $\triangle \mathbf{x} = (\mathbf{x} - \mathbf{x}_0)$ so we can write

$$\triangle f \approx \nabla f(\mathbf{x}_0) \cdot \triangle \mathbf{x}$$

Here, as throughout the preceding, we have "$\approx$" because the actual function difference are only approximately given by the linear, tangent plane approximation. The idea of differentials (also call infinitesimals), is to define new variables which are exactly related by the linear relation. We could call these variables anything, such as $thing_1$ and $thing_2$ and write

$$thing_2 = \nabla f(\mathbf{x}_0) \cdot thing_1$$

but is it more usual to just put a d in front of the variable name and write

$$df = \nabla f(\mathbf{x}_0) \cdot \mathbf{dx}$$

df and $\mathbf{dx}$ are **differentials**. You will recognise this as a generalisation of

$$df = f'(x_0)dx$$

The way one understands this informally is as follows. We think of differentials as being infinitely small. So small that the tangent approximation is not as approximation for them, but exact.

More formally, differentials should be thought of as elements of certain vector spaces: for any $\mathbf{y} \in \mathbb{R}^n$, the vector $(\mathbf{y}, \nabla f(\mathbf{x}_0) \cdot \mathbf{y}) \in \mathbb{R}^{n+1}$ belongs to the tangent plane to the graph of f at the point $\mathbf{x}_0$. Therefore, the gradient of f gives rise to a map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ involving tangent spaces, a fruitful idea which will be developed much further in more advances courses.

5.3 Level Sets and the Gradient Vector

Contours

We begin with the planar case and consider a contour $\mathcal{L}_k$ in the plane. Assume it can be written in the form of a parametrised curve $\mathbf{r}(t)$. (We may need several curves to capture the full level set.) Since $\mathbf{r}(t)$ is a contour we have

$$f(\mathbf{r}(t)) = k$$

Differentiating with respect to t using the Chain Rule

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{d}{dt}k = 0$$

so

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$$

Hence the gradient vector is perpendicular to the tangent to the contour. We say that *the gradient vector is perpendicular to the contour.* Since $\nabla f(\mathbf{r}(t))$ is perpendicular to $\mathbf{r}'(t)$, $\nabla f(\mathbf{r}(t))$ is perpendicular to the tangent line to the contour at the point $\mathbf{r}(t)$. The generalisation of these observation to higher dimensions will be important to us, so please understand the planar case now.

We have implicitly assumed that neither the vector ∇f nor $\mathbf{r}'$ is zero. The points where $\nabla f = \mathbf{0}$ are critical points of the function f. You have seen critical points in Differential Equations and we will review them later. At critical points contours may consist of isolated points or they may be complicated. For now ignore them and assume $\nabla f \neq \mathbf{0}$.

Steepest ascent/descent

Now consider a closely related, but more general calculation. Fix a point $\mathbf{x}_0$ in the domain of f and compute the directional derivative of f in any possible direction (unit vector) $\mathbf{u}$

$$\begin{aligned} D_{\mathbf{u}}f(\mathbf{x}_0) &= \nabla f(\mathbf{x}_0) \cdot \mathbf{u} \\ &= \|\nabla f(\mathbf{x}_0)\|\|\mathbf{u}\| \cos\theta = \|\nabla f(\mathbf{x}_0)\| \cos\theta \end{aligned}$$

where θ is the angle between $\nabla f(\mathbf{x}_0)$ and $\mathbf{u}$. Maximising $D_{\mathbf{u}}f(\mathbf{x}_0)$ over all directions $\mathbf{u}$ gives $D_{\mathbf{u}}f(\mathbf{x}_0)$ $= \|\nabla f(\mathbf{x}_0)\|$ when $\theta = 0$, or $\mathbf{u}$ in the direction of $\nabla f(\mathbf{x}_0)$. *Hence ∇f points in the direction of the steepest increase, or steepest ascent, of the function f and $\|\nabla f\|$ is the derivative of f in this direction.*

One can further read off from the above equation that the most negative value of $D_{\mathbf{u}}f(\mathbf{x}_0)$ occurs for $\theta = \pi$. *Hence $-\nabla f$ points in the direction of the steepest decrease, or steepest descent, of the function f.* For physical reasons the direction of steepest descent, $-\nabla f$, is often more important than steepest ascent, basically because things want to "roll down hill", not "roll up hill".

Finally, $D_{\mathbf{u}}f(\mathbf{x}_0) = 0$ when $\mathbf{u}$ is perpendicular to $\nabla f(\mathbf{x}_0)$. This is consistent with the fact that the contour through $\mathbf{x}_0$ is perpendicular to $\nabla f(\mathbf{x}_0)$ and the function values do not change along a contour. The equation for tangent line to the contour at point $\mathbf{x}_0$ can be expressed as $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.

Extension to functions of three variables

The basic calculation we just did is independent of dimension. Fixing a point $\mathbf{x}_0$ and computing $D_{\mathbf{u}}f(\mathbf{x}_0)$ in any possible direction $\mathbf{u}$ gives

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \|\nabla f(\mathbf{x}_0)\| \cos\theta$$

Thus $D_{\mathbf{u}}f(\mathbf{x}_0)$ will assume its maximum of $\|\nabla f(\mathbf{x}_0)\|$ for $\theta = 0$ and its minimum of $-\|\nabla f(\mathbf{x}_0)\|$ for $\theta = \pi$

The fact is that ∇f is the direction of steepest ascent ($-\nabla f$ is the direction of steepest descent) is independent of dimension.

In what directions $\mathbf{u}$ is $D_{\mathbf{u}}f(\mathbf{x}_0) = 0$? The answer is still any direction where $\mathbf{u}$ is perpendicular to $\nabla f(\mathbf{x}_0)$ and this does depend on dimension. For a function of three variables these vectors all lie on a plane with $\nabla f(\mathbf{x}_0)$ as its normal. (The vectors for which $\mathbf{u}$ is $D_{\mathbf{u}}f(\mathbf{x}_0) = 0$ form a circle in this plane.)

The situation is that at each point $\mathbf{x}_0$ there is a plane passing through $\mathbf{x}_0$ with $\nabla f(\mathbf{x}_0)$ as its normal. The plane is tangent to the isosurface passing through $\mathbf{x}_0$. The gradient vector is said to be normal to the isosurface at each point $\mathbf{x}_0$. The equation for the tangent plane at $\mathbf{x}_0$ is

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

with

$$A = \frac{\partial f}{\partial x}(\mathbf{x}_0), \quad B = \frac{\partial f}{\partial y}(\mathbf{x}_0), \quad C = \frac{\partial f}{\partial z}(\mathbf{x}_0)$$

One can also examine the tangent plane to the isosurface by considering parametrised curve $\mathbf{r}(t)$

on the surface passing through $\mathbf{x}_0$ at $t = 0$. Exactly as we argued for contours, since $f(\mathbf{r}(t)) = k$, differentiating gives

$$\nabla f(\mathbf{x}_0) \cdot \mathbf{r}'(0) = 0$$

Thus $\nabla f(\mathbf{x}_0)$ is perpendicular to the tangent vector of every curve on the surface passing through $\mathbf{x}_0$, hence $\nabla f(\mathbf{x}_0)$ is normal to the surface, or equivalently, every tangent vector is in the tangent plane with $\nabla f(\mathbf{x}_0)$ as its normal.

Our previous results for tangent planes to graphs of functions is a particular case of the above. To see this notice that the graph

$$G_f = \{(x, y, f(x, y))|(x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

of a function of two variables f can be thought of as the zeroth level set for the following function of three variables: $F_f : \mathbb{R}^3 \to \mathbb{R}$, $F_f(x, y, z) = z - f(x, y)$.

One of the reasons to care about normal vectors and tangent planes is that they are used extensively in computer graphics. Because of the way light reflects from objects depends on the direction of incoming light relative to the normal vector, knowing the normal vectors is essential for realistic rendering of surfaces.

5.4 Motion

Partial derivatives come up in many applications involving motion. We will consider briefly two interesting and important examples.

Paths

The first example is particle paths where particles want to move "down hill", that is in the direction of minus some gradient vector. A good example of this might be particles in a fluid. Particles will feel a force in the direction of $-\nabla p$ where p is the fluid pressure and as a result particles will move in the direction of $-\nabla p$. (In practice particles will move around in whatever direction the fluid is moving in addition to responding to the pressure, think of dust in the air or snowflakes blown around by the wind. Nevertheless, the response of particles to a pressure is a real effect and we will focus just on this motion.)

We imagine that we are given the fluid pressure in some region of space, $p(\mathbf{x})$ where $\mathbf{x}$ might be in

$\mathbb{R}^2$ or in $\mathbb{R}^3$. We want to find a parametrised path $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = -\nabla p(\mathbf{r}(t))$ starting from some given initial position $\mathbf{r}(0)$ say. It is important that t might not correspond to physical time here.

There are other physical situations in which, to a good approximation, particles move along paths that are everywhere down the gradient of some function, known as a potential. Generically the potential function would be denoted V and we would look for paths $\mathbf{r}'(t) = -\nabla V(\mathbf{r}(t))$. We will see this again when we consider vector fields later in the module.

PDEs

A second example is partial differential equations (PDEs) involving time. While there are many, many example here, we want to focus on one particularly interesting class of problems which relates to the level-set geometry considered earlier in this chapter.

Consider $u(\mathbf{x}, t)$ where $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$. We have written the function this way to distinguish the coordinates $\mathbf{x}$ which will correspond to space, from the coordinate t which will correspond to time. The interest will be the level sets of u, let us say the particular level set $k = 1$. At each time t, $u(\mathbf{x}, t) = 1$ will be a level set in (x, y) or (x, y, z) depending on the space dimension. As t evolves, the $k = 1$ level set will evolve. Thus in two space dimensions we will have a moving curve (the contour $u = 1$) while in three space dimensions we will have a moving surface (the isosurface $u = 1$). The $u(\mathbf{x}, t)$ may itself obey a PDE.

5.5 Critical Points

The final topic is critical points for functions of two variables. This topic can be viewed both from a geometrical perspective or from a dynamical perspective familiar from ODEs. The geometrical perspective concerns the geometry of graphs or equivalently contours near critical points. The dynamical perspective comes from thinking of $f(x, y)$ as a potential and considering the system of ODEs

$$\dot{x} = f_1(x, y) = -\frac{\partial f}{\partial x}(x, y) \tag{11}$$

$$\dot{y} = f_2(x, y) = -\frac{\partial f}{\partial y}(x, y) \tag{12}$$

First some definitions

A point (a, b) is called a **stationary point** of $f(x, y)$ if

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0$$

A point (a, b) is called a **critical point** of $f(x, y)$ if it is a stationary point or if one of these partial derivatives does not exist.

Stationary points of f thus give fixed points of ODEs (11)-(12). We shall only concern ourselves with the case of stationary points, rather than more general critical points.

We are interested in three generic stationary points for functions $f(x, y)$ and it worth discussing these using prototypical functions.

- $f(x, y) = x^2+y^2$ is a canonical example of a function with a **local minimum** at the stationary point $(a, b) = (0, 0)$. The graph of this function is a circular paraboloid. The contours are circles for $k > 0$ and a single point for $k = 0$. The ODEs are

  $$\dot{x} = -2x \quad \dot{y} = -2y$$

  so the fixed point at $(0, 0)$ is a sink (because the system evolves down to the bottom of the function f).
- $f(x, y) = -x^2 - y^2$ is a canonical example of a function with a **local maximum** at the stationary point $(a, b) = (0, 0)$. The graph of this function is again a circular paraboloid. The contours are circles for $k < 0$ and a single point for $k = 0$. The ODEs are

  $$\dot{x} = 2x \quad \dot{y} = 2y$$

  so the fixed point at $(0, 0)$ is a source (because the system evolves away from the peak at $(0, 0)$).
- $f(x, y) = x^2-y^2$ is a canonical example of a function with neither a local maximum nor a local minimum at the stationary point $(a, b) = (0, 0)$. Such a point is called a **saddle point**. The graph of this function is a hyperbolic paraboloid. The contours are hyperbolas for $k \neq 0$ and crossed lines for $k = 0$. The ODEs are

  $$\dot{x} = -2x \quad \dot{y} = 2y$$

  The fixed point at $(0, 0)$ is a called a saddle fixed point.

Continuing with the definitions, $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ for all (x, y) near (a, b). The value $f(a, b)$ is called the **local maximum value**. $f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \ge f(a, b)$ for all (x, y) near (a, b). The value $f(a, b)$ is called the **local minimum value**. Here "(x, y) near (a, b)" means that $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ for some $\delta > 0$. We do not care how small δ is as long as it is positive. It is easily shown, using arguments from functions of one variable, that if $f(x, y)$ has a local maximum or a local minimum at (a, b), then (a, b) is a stationary point of f.

It is frequently possible to decide whether a stationary point is a local maximum, local minimum, or a saddle point using a test based on the second-order partial derivatives.

We first state the test and then discuss it.

Second Derivative Test: Suppose $f(x, y)$ has a stationary point at (a, b). Let

$$D = \frac{\partial^2 f}{\partial x^2}(a, b)\frac{\partial^2 f}{\partial y^2}(a, b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a, b)\right]^2$$

then

- if $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- if $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- if $D < 0$, then $f(a, b)$ is a saddle point.
- if $D = 0$, then the test in inconclusive.

The tests assumes that the second-order partial derivatives exist, and moreover they must be continuous in the vicinity of (a, b).

We are not going to derive this test. It is not difficult but requires Taylor's Theorem for functions of several variables, which we have not yet covered. You should verify that the test works for the three prototype examples above.

You should also recognise that in terms of the ODEs

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial f_1}{\partial x}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{\partial f_2}{\partial y},$$
$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{\partial f_2}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{\partial f_1}{\partial y}$$

Hence D in the test is just the determinant of the Jacobian matrix at the fixed point, and hence D is the product of the eigenvalues at the fixed point. This should give you some intuition into the test.

Finally, one can construct examples for which the test fails. For example,

$$f(x,y) = x^4 + y^4, \quad f(x,y) = -x^4 - y^4$$

$$f(x,y) = x^4 - y^4$$

showing that when the tests fails the function may have a local minimum, local maximum, or a saddle point.

Additional Material

Recap of curves and surfaces

This may be helpful in orienting you on the various approaches we have considered, and will consider, in the description of curves and now surfaces. You may also want return and re-read it after Week 9.

Curves

If you think about it, you now know three potential ways to describe a curve in the plane.

- Graph of a function of one variable $f : I \subseteq \mathbb{R} \to \mathbb{R}$,

$$G_f = \{(x, y)|\, x \in I, y = f(x)\}$$

- Parametrised curve $\mathbf{r} \colon I \to \mathbb{R}^2$,

$$\mathcal{C} = \{\mathbf{r}(t)|t \in I\}$$

- The level set, or contour, of a function of two variables $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\mathcal{L}_k = \{(x, y) \in U|\, f(x, y) = k\}$$

A level set is itself equivalent to the set of points satisfying an equation in two variables, $f(x, y) = k$.

As you know, not all curves can be described by graphs. When a curve can be represented as a graph $y = f(x)$, it is easy to also write it as a parametrised curve or a level set. A simple parametrisation is $\mathbf{r}(t) = (t, f(t))$. For a level set, take the zero set, $\mathcal{L}_0$, of the function $F(x, y) = f(x) - y$. It is useful to keep in mind this interconnection between the various concepts.

As we emphasised in our study of parametrised curves, tangents and normals to curves are important. You know how to compute the tangent to the graph of a function from the derivative f'. You may not have computed the normal to a graph, but you could without much trouble. We have yet to address tangents and normals to level sets, but will this week.

Surfaces

The same three approaches exist for describing surfaces in space.

- Graph of a function of two variables $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, i.e.

$$G_f = \{(x, y, z)|\, (x, y) \in U, z = f(x, y)\}$$

- Parametrised surface $\mathbf{r} \colon U \in \mathbb{R}^2 \to \mathbb{R}^3$, which we treat in Week 9.
- The level set, or isosurface, of a function of three variables $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$

$$\mathcal{L}_k = \{(x, y, z) \in U|\, f(x, y, z) = k\}$$

As in the case of curves, not all surfaces can be represented as graphs, but if the surface can be represented as a graph $z = f(x, y)$, then it can also be parametrised and obtained as a level set. The parametrised surface can wait, but the simple level set description is again the zero level set of $F(x, y, z) = f(x, y) - z$. Again, it will help you if you keep in mind the interconnection between these approaches.

As with curves, tangents and normals to surfaces are important. In this chapter we describe the tangent plane to a surface from two points of view, the tangent plane to a graph of $f(x, y)$ and the tangent plane to a level set of a function of three variables $F(x, y, z)$. These will be the same surface if we take the zero level set of $F(x, y, z) = f(x, y) - z$. We know from the section on isosurfaces that the gradient vector ∇F is normal to the tangent plane to the isosurface. At a point $(a, b, f(a, b))$ on the surface, the normal vector is

$$\nabla F\big(a, b, f(a, b)\big) = \big(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\big)$$

You should satisfy yourself that this gives exactly the same plane and as tangent plane to the graph of $f(x, y)$.

Discussion

There is room for confusion. Functions $\mathbb{R} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}^2$, or $\mathbb{R}^2 \to \mathbb{R}$, could all potentially be used in describing a curve in the plane. Functions $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R}^2 \to \mathbb{R}^3$, or $\mathbb{R}^3 \to \mathbb{R}$, could all potentially be used in describing a surface in the space. Note in particular that $\mathbb{R}^2 \to \mathbb{R}$ appears is both lists: its levels sets are curves in the plane and its graph is a surface in space. Yes, much room for confusion.

Generally speaking, graphs are a tool for understanding and visualising functions, as opposed to describing the geometry of curves or surfaces. In most cases, parametrisations are the best approach to study the geometry of curves and surfaces.

Levels sets play multiple roles. On the one hand they are very useful is visualising functions. A notable example, in this module will be contour maps for functions of two variables (known as potential landscapes) that govern motion. On the other hand, an individual level sets can at times be the ideal way to characterise a particular curve or surface. This is particularly the case if the curve or surface is simply described by an equation. See for example the quadric surfaces at the end of Week 4. Again, set of points satisfying equations are nothing other than level sets of a particular function.