Week 6: Integration: Cartesian Coordinates

Introduction

Given a continuous function $f(x) > 0$, you know that the integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $f(x)$ from $x = a$ to $x = b$. This interpretation as area under the curve is intimately connected with the definition of integration. Since the graph of a function of two variables $f(x, y) > 0$ is a surface above the xy-plane, it is natural to ask about the volume under the sheet. This volume will be expressed as a multiple integral.

What is fun about integration in two and three dimensions is the new freedom in geometry. There are two aspects. The first is that, for a given function $f(x, y)$ one can ask about many uniquely different volumes by considering different geometrical regions in the x-y plane. For example: What is the volume of $f(x, y)$ above the square $\{(x, y)|\ |x| + |y| \leq 1\}$? What is the volume of $f(x, y)$ above the annulus $\{(x, y)|\ 1 \leq x^2 + y^2 \leq 2\}$?, etc. The second, sometimes related aspect, is that coordinate systems and symmetries come into the picture.

6.1 Multiple Integration

This will be a quick, heuristic development of multiple integration. Consider a function f of two variables which is defined, and is positive, on the closed rectangle, i.e. $f : R \subset \mathbb{R}^2 \to \mathbb{R}^+$, where

$$R = [a, b] \times [c, d] = \{(x, y)|\, a \leq x \leq b, c \leq y \leq d\}$$

Our goal is to compute the volume V of the three dimensional solid Ω generated by the graph of f above this rectangle

$$\Omega = \{(x, y, z)|\, 0 \leq z \leq f(x, y), (x, y) \in R\}$$

To compute the volume we partition the rectangle R into subrectangles by partitioning separately each of the intervals $[a, b]$ and $[c, d]$ as follows. First partition $[a, b]$ into N equal subintervals by letting $x_0 = a, x_1 = a + \triangle x, \cdots, x_i = a + i\triangle x \cdots, x_N = b$. This gives subintervals $[x_i, x_{i+1}]$ each with length $\triangle x = (a - b)/N$.

Similarly partition $[c, d]$ into M equal subintervals $[y_j, y_{j+1}]$ of length $\triangle y$ by letting $y_0 = c, y_1 =$

$c + \triangle y, \cdots, y_j = c + j\triangle y \cdots, y_M = d$. Then form subrectangles R_{ij} of R by taking the Cartesian product of the x and y subintervals:

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

So each subrectangle has area $\triangle A = \triangle x \triangle y$.

We can now approximate the volume of the solid over each subrectangle as $f(x_i, y_j)\triangle A$. Summing these we get an approximation for the volume of Ω

$$V \approx \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

For N and M large, the size of each subrectangle is small and f does not vary much over the subrectangle. Each term in the sum then corresponds to a subvolume that is very tall compared with its base dimensions. Now take the limit as the number of elements N and M goes to infinity, with the sizes $\triangle x$ and $\triangle y$ going to zero. We thus get the exact volume

$$V = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

The volume under a function of two variables is just one of numerous quantities one wants to compute in this way. In general there is no reason to restrict functions taking on positive values, so we now drop this restriction and define

The **double integral** of f over the rectangle R is

$$\iint_R f\, dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

Comments

- We have assumed here that the limit exists. If the limit exists, f is said to be **integrable** and in later courses you will learn that f is continuous (in the mulivariable sense which we have not defined), then it is integrable.
- It is unnecessary that R be partitioned into equal rectangles, all with the same area. In fact it is unnecessary that the partition of R use rectangles at all, we did this here for convenience.
- We chose to approximate the subvolumes using f evaluated at (x_i, y_j), but we could have used any sample point within R_{ij}. It does not matter where we evaluate f within each element since in the limit their size goes to zero.

6.2 Iterated Integration over a Rectangle

One does not generally compute integrals by forming the sums and taking the limits in the above definition. Rather one relies on iterated or repeated one-variable integration as we now explain.

It is instructive to again think in terms of volume under a surface. We know that the volume is given by a double integral which itself is expressed as a limits of a double sum

$$V = \iint_R f dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle A.$$

Now we separate and reorder the limits and sums as follows

$$\begin{aligned} V = \iint_R f dA &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \triangle y \\ &= \lim_{M\to\infty} \sum_{j=0}^{M-1} \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \triangle y \\ &= \lim_{M\to\infty} \sum_{j=0}^{M-1} \left[\lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \right] \triangle y \end{aligned}$$

The expression in brackets is just the usual definite integral of $f(x, y)$ over x with y fixed at y_j. Such an integral is call **partial integration** – integration of a function of several variables with respect to one variable while treating any others as constants. You will recognise the analogy with partial differentiation. Let us denote this definite integral, which will depend on y, by $A(y)$,

$$A(y) = \int_a^b f(x, y)\, dx$$

A for area since $A(y)$ is the area under the curve $f(x, y)$ from $x = a$ to $x = b$ with fixed y. Then our multiple integral becomes

$$V = \iint_R f\, dA = \lim_{M\to\infty} \sum_{j=0}^{M-1} A(y_j) \triangle y$$

The expression on the right is just the usual definite integral of $A(y)$, so

$$V = \iint_R f\, dA = \int_c^d A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

It is important to understand the interpretation of this. The inner integral gives the area under a curve at constant y. The outer integral integrates these areas over the range of y to give the volume.

We did not have to re-order such that the x-integral appeared on the inside. One could have equally moved the y sum and limit to the inside. (You should satisfy yourself that this is true).

We now drop the specific interpretation of volume under a function and arrive at the following relationship between double integrals and iterated integrals.

The double integral of f over the rectangle $R = [a,b] \times [c,d]$ is given by iterated, also known as repeated, integration

$$\iint_R f dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Discussion

- *Notation:* You will notice that we dropped the square brackets in boxed expression above. This is common. All of the following are used to denoted iterated integration

$$\int_c^d \int_a^b f(x,y)dxdy, \; \int_c^d \left[\int_a^b f(x,y)dx \right] dy,$$
$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy, \; \int_{y=c}^{d} dy \int_{x=a}^{b} dx f(x,y)$$

  and other combinations of these forms.

- The boxed expression is known as *Fubini's Theorem*, or would be if we stated is as a theorem. Equality is guaranteed if f is a continuous function, and even under weaker conditions on f.

- Understanding the following is crucial to iterated integration: Integrals are nested. (Now just 2 levels, but in higher dimensions there will be more nesting.) Integrals are done from inside out. You work with one variable at a time treating any variables outside the current level as constants. Make sure you understand this now. As we generalise to other integration domains this will be key to your success. If you are ever confused by an iterated integral, explicitly include the square brackets showing the nesting.

- Sometimes the function $f(x,y)$ separates into the product of a function of x only and a function of

y only. Letting $f(x,y) = g(x)h(y)$,

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b g(x)h(y)dx\right] dy$$

but $h(y)$ is a constant as far as the inner x integration is concerned, so it can be pulled out

$$\int_c^d h(y) \left[\int_a^b g(x)dx\right] dy$$

and now the whole x integral is a constant as far as the y integration is concerned to it can be pulled out

$$\left[\int_a^b g(x)dx\right] \int_c^d h(y)dy$$

Thus the double integral is said to **separate**

$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b g(x)dx \int_c^d h(y)dy$$

6.3 Multiple Integration in Three or More Variables

The extension of the above to functions of more than two variables is straightforward. We will only explicitly consider functions of three variables, but functions of four, five, etc variables are essentially the same.

The following treatment is a natural extension of the above. Please fill in the missing details. Rather than consider volume under a surface, a useful physical example would be the total mass of a solid whose density (mass per unit volume) is given by $f(x,y,z)$. f is necessarily positive in this example. Assume the solid is in the shape of a rectangular box. We then have $f : B \subset \mathbb{R}^3 \to \mathbb{R}^+$, where B is the rectangular box

$$\begin{aligned} B &= [a,b] \times [c,d] \times [r,s] \\ &= \{(x,y,z) |\, a \le x \le b, c \le y \le d, r \le z \le s\}. \end{aligned}$$

By partitioning each of the x, y, and z directions, B can be partitioned into sub-boxes

$$B_{ijk} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$$

each with volume $\triangle V = \triangle x \triangle y \triangle z$. The mass of each sub-box B_{ijk} will be approximately the density f evaluate at (x_i, y_j, z_k) multiplied by the box volume: $f(x_i, y_j, z_k) \triangle V$. Then the approximate total mass will be the sum of all the sub-boxes. Taking the limit as the partition becomes finer we have the total mass of the solid given as a triple integral

$$\begin{aligned}\text{Total Mass} &= \iiint_B f\,dV \\ &= \lim_{K,M,N\to\infty} \sum_{k=0}^{K-1}\sum_{j=0}^{M-1}\sum_{i=0}^{N-1} f(x_i, y_j, z_k)\triangle V\end{aligned}$$

Now forgetting the particular physical example, and the requirement that f be positive, we define

The **triple integral** of f over the rectangular box B is

$$\iiint_B f\,dV = \lim_{K,M,N\to\infty} \sum_{k=0}^{K-1}\sum_{j=0}^{M-1}\sum_{i=0}^{N-1} f(x_i, y_j, z_k)\triangle V$$

More important than the definition in terms of limits is evaluations by iterated integrals. Analogously to the two-variable case

The triple integral of f over the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ is given by iterated integration

$$\iiint_B f\,dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\,dxdydz$$

or other orderings of the x, y, and z integrals.

The same comments from Sec. 6.1 and 6.2 apply here as well.

6.4 Iterated Integration over General Domains

Now the fun starts. We are going to consider integration of functions over more general domains, first in 2D then in 3D. Doing integrals is not going to be the issue – getting the geometry right is.

Two variables

Let Ω denote the region or domain in $\mathbb{R}^2$ on which f is defined and over which we wish to integrate.

A **Type I** region can be expressed

$$\Omega = \{(x, y) \in \mathbb{R}^2 |\, a \le x \le b,\, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous functions on $[a, b]$. We evaluate the double integral of f over Ω as an iterated integral in the following way

$$\iint_{\Omega} f dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

Let us elaborate by examining the nesting from inside to outside. Within the brackets, x is treated as a constant. Its actual value is not known, but everywhere it appears it acts like a constant. Hence the limits of the y-integral are the "constants" $g_1(x)$ and $g_2(x)$. Assuming one can partially integrate $f(x, y)$ with respect to y, then one can evaluate the inner brackets resulting in a function of x only. Then the outer integral is a standard integral over the interval $[a, b]$.

A **Type II** region or domain is one that can be expressed

$$\Omega = \{(x, y) \in \mathbb{R}^2 |\, h_1(y) \leq x \leq h_2(y),\, c \leq y \leq d\}$$

where h_1 and h_2 are continuous functions on $[c, d]$. In this case we evaluate the double integral of f over Ω as iterated integral in the following way

$$\iint_{\Omega} f dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$$

Comments

- There are two aspects to iterated integrals over these types of domains. First setting up the integration correctly and second carrying out the integration correctly. One simply needs to practice a lot.
- Our advice is, given some specification of the domain Ω, first try writing it explicitly as a finite union of non-intersecting domains of Type I or Type II. Then your integral will be expressed as a sum of iterated integrals due to the property of additivity, see below.
- Many domains can equally be written in Type I or Type II format. You should choose the one which simplifies the calculations, but the right choice is not always obvious when you start. Again, practice helps.

Example. Let $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \pi; x \leq y \leq \pi\}$. Then

$$\int \int_D \frac{\sin(y)}{y} dA = \int_0^{\pi} dx \int_x^{\pi} dy \frac{\sin(y)}{y}.$$

The inner integral cannot be expressed in terms of elementary functions and we seem to be stuck. However, the domain of integration D can be re-written as Type-II domain: $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \pi; 0 \leq x \leq y\}$. Therefore,

$$\int\int_D \frac{\sin(y)}{y} dA = \int_0^\pi dy \int_0^y dx \frac{\sin(y)}{y}$$
$$= \int_0^\pi dy y \frac{\sin(y)}{y} = -\cos(y) \mid_0^\pi = 2.$$

- Multiple integrals obey linearity

$$\iint_\Omega (cf + g) dA = c \iint_\Omega f dA + \iint_\Omega g dA$$

for function f and g and constant c.

- The following fact is useful: If $\Omega = \Omega_1 \cup \Omega_2$, and Ω_1 and Ω_2 do not overlap except possibly on their boundaries, then

$$\iint_\Omega f dA = \iint_{\Omega_1} f dA + \iint_{\Omega_2} f dA$$

This not only helpful for computing the integral over Ω from integrals over Ω_1 and Ω_1, but it is also helpful for computing the integral over Ω_1 from the integrals over Ω and Ω_2.

Three variables

Now let Ω be a region or domain in $\mathbb{R}^3$ on which f is continuous and over which we wish to integrate. In analogy with the 2D case we could define 6 types of regions. We won't do this, but instead look at a one case and let you deduce the others.

Suppose the set Ω can be written

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 | \, a \leq x \leq b, \, g_1(x) \leq y \leq g_2(x), \\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

Then

$$\iiint_\Omega f dV$$
$$= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz \right] dy \right] dx$$

The inner most integral is a partial integral over z. Here x and y, and hence integration limits $u_1(x, y)$ and $u_2(x, y)$, are viewed as constants. The result will be a function of (x, y). The middle integral is a partial integral over y with x treated as constant, followed by the outer definite integral over x with limits a and b.

6.5 Applications

There are numerous applications of multiple integration. The point here is not to give a long list but rather to state a few general concepts from applications which you are expected to learn and know. Specifically, you should know the formulas below that appear in boxes.

Area and Volume

This is not so much an application as some special cases you should be aware of. Suppose you want to compute the area of a region Ω in the plane. Using the same formalism we used to derived double integrals, you can derive the formula for area as a double integral. Basically, just set the function $f \equiv 1$. That is, rather than summing up $f(x_i, y_j)\triangle A$ to get the volume of f over Ω, just sum up $\triangle A$ to get the area of Ω. For rectangular region R

$$\text{Area of R} = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} \triangle A = \iint_R dA$$

This extendes to general regions Ω in the plane. Similarly one can obtain the volume of a region Ω in space as a triple integral – just set $f(x, y, z) \equiv 1$ in triple integration.

Denoting the area of a planar region Ω by $A(\Omega)$ and the volume of a space region by $V(\Omega)$, we have

Area of a region Ω in the plane

$$A(\Omega) = \iint_\Omega dA$$

Volume of a region Ω in space

$$V(\Omega) = \iiint_\Omega dV$$

Density and centre of mass

Let the shape of a solid be represented by the domain Ω. Let the density (mass per unit volume) at point (x, y, z) now be denoted $\rho(x, y, z)$, (ρ is a common symbol for density in three dimensions). Physically $\rho(x, y, z) \geq 0$ everywhere on Ω.

The mass of the solid is

$$M = \iiint_\Omega \rho \, dV$$

There are a number of interesting quantities besides the total mass. One is the centre of mass, a point in space, i.e. a vector. Coordinates of the centre of mass of solid Ω with density ρ is

The centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M} \iiint_\Omega x\rho \, dV \quad \bar{y} = \frac{1}{M} \iiint_\Omega y\rho \, dV$$

$$\bar{z} = \frac{1}{M} \iiint_\Omega z\rho \, dV$$

There are many other types of densities, charge density, energy density, etc. and many other quantities that can be computed. The concept of density applies in dimensions other than three. For example in the two-dimensional setting on can have a population density or a charge density on a surface, in which case the density will be in units of something per unit area.

A particularly important example of density is probability density. In this case the analog of total mass is total probability which will be 1 by definition. The analog of centre of mass will be expectation values (or mean values) of the random variables described by the probability density.