Week 7: Integration: Special Coordinates

Introduction

Many problems naturally involve symmetry. One should exploit it where possible and this often means using coordinate systems other than Cartesian coordinates.

7.1 Overview

We start with a quick reminder of our derivation of multiple integration in Cartesian coordinates. We initially considered a rectangular domain R and partitioned it into sub-rectangles R_{ij} based on a partitioning each of the coordinates x and y separately. Essentially we divided up R using an equally spaced grid with increments $\triangle x$ and $\triangle y$. We then constructed the sum $f(x_i, y_i)\triangle A$, where $\triangle A$ is the area of R_{ij}. In this case $\triangle A = \triangle x \triangle y$. Taking the limit gives the relationship between differential of area dA and the differentials dx and dy of the Cartesian coordinates.

$$\triangle A = \triangle x \triangle y \longrightarrow dA = dx\, dy$$

Similarly for integration in 3D,

$$\triangle V = \triangle x \triangle y \triangle z \longrightarrow dV = dx\, dy\, dz$$

The way you should interpret these relationships is that if one makes a small (infinitesimal) change dx in the x coordinate, then the area or volume element will be proportional to that change, and similarly for y and z.

We are going to consider important non-Cartesian coordinate systems and address the question: what are dA and dV in terms of coordinate increments in these coordinate systems? Next week will will take a more general and systematic approach. However, the intuition in the current approach is correct and valuable.

7.2 Polar Coordinates

You know the relationship between polar coordinates (r, θ) and Cartesian coordinates (x, y), which we reproduce here for consistency with subsequent coordinate systems.

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

It is useful to view the polar coordinate system in terms of a polar grid consisting of curves of constant r-coordinate – circles centred on the origin, and curves of constant θ-coordinate – radial lines.

Now consider the equivalent of a rectangular region R in polar coordinates.

$$R = \{(r, \theta) |\, a \le r \le b, \alpha \le \theta \le \beta\}$$

This is referred to as a **polar rectangle**. Of course this is not a rectangle in the plane, but typically a wedge (although these include disks and annuli). If we want to integrate a function of polar coordinates $f(r, \theta)$ over this region, we partition R using an equally spaced grid in polar coordinates, with spacing $\triangle r$ and $\triangle\theta$

$$a = r_0, \cdots, r_i = a + i\triangle r \cdots, r_N = b,$$
$$\alpha = \theta_0, \cdots, \theta_j = c + j\triangle\theta \cdots, \theta_M = \beta.$$

This gives sub-regions R_{ij} in the form of small wedges

$$R_{ij} = \{(r, \theta) |\, r_i \le r \le r_i + \triangle r, \theta_j \le \theta \le \theta_j + \triangle\theta\}$$

The double integral of f over this region is obtained by sampling the function somewhere within each wedge R_{ij}, multiply by the area of R_{ij}, summing over all sub-regions, and taking the limit $M, N \to \infty$

$$\iint_R f\, dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(r_i^*, \theta_j^*) \triangle A_{ij}$$

where (r_i^*, θ_j^*) is some point in R_{ij} and $\triangle A_{ij}$ is the area of R_{ij}.

To derive the relationship between the double integral of f and its expression as an iterated integral over r and θ, it will be convenient to sample the function at the the mid-point of each R_{ij},

$$r_i^* = \frac{r_i + r_{i+1}}{2} \quad \theta_j^* = \frac{\theta_j + \theta_{j+1}}{2}$$

In the limit $M, N \to \infty$it does not matter where we choose to sample f, but with this we can write the area of $\triangle A_{ij}$ compactly as

$$\triangle A_{ij} = \triangle r \left(\frac{r_i + r_{i+1}}{2} \triangle\theta \right) = \triangle r\, (r_i^* \triangle\theta) = r_i^* \triangle r \triangle\theta$$

Then the expression for double integral gives

$$\begin{aligned}
\iint_R f\,dA &= \lim_{M\to\infty}\lim_{N\to\infty}\sum_{j=0}^{M-1}\sum_{i=0}^{N-1} f(r_i^*,\theta_j^*)\,r_i^*\triangle r\triangle\theta \\
&= \lim_{M\to\infty}\sum_{j=0}^{M-1}\lim_{N\to\infty}\sum_{i=0}^{N-1} f(r_i^*,\theta_j^*)\,r_i^*\triangle r\triangle\theta \\
&= \int_\alpha^\beta\int_a^b f(r,\theta)\,r\,dr\,d\theta
\end{aligned}$$

Hence we have the desired relationship between the double integral and iterated integrals

In polar coordinates:

$$\iint_R f\,dA = \int_\alpha^\beta\int_a^b f(r,\theta)\,r\,dr\,d\theta$$

where

$$R = \{(r,\theta)|\, a\le r\le b, \alpha\le\theta\le\beta\}$$

The key feature that distinguishes integration in polar coordinates from integration in Cartesian coordinates is that now the area element is no longer independent of position. It depends on coordinate r because, for given increments $\triangle r$ and $\triangle\theta$, the area of the little wedges R_{ij} depend on their distance from the origin.

Key fact, in polar coordinates:

$$dA = r\,dr\,d\theta$$

This is the relationship between the infinitesimal area in polar coordinates and the infinitesimal changes in the coordinates dr and $d\theta$. You should understand this as follows. If one makes a small (infinitesimal) change dr in the r coordinate, then the area element will be proportional to that change, while if one makes an small (infinitesimal) change $d\theta$ in the θ coordinate, then the area element will be proportional $rd\theta$.

We will continue with integration in polar coordinates after we treat the other two special coordinate systems.

7.3 Iterated Integration in Cylindrical Coordinates

Cylindrical coordinates (r,θ,z) are a three dimensional coordinate systems composed of polar coordinates (r,θ) in the plane and a Cartesian coordinate z in the third direction, generally thought of as the vertical direction. The relationship between cylindrical and Cartesian coordinates is

$$x = r\cos\theta \qquad r = \left(x^2 + y^2\right)^{1/2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = z \qquad z = z$$

One could give a different symbol ξ to the vertical coordinate in cylindrical coordinates and write relationship between this cylindrical coordinate and the Cartesian coordinate is $\xi = z$, but this is silly.

It is useful to view the cylindrical coordinate system in terms of a grid consisting of surfaces of constant r-coordinate – cylinders centred on the z-axis, surfaces of constant θ-coordinate – radial half-planes, and surfaces of constant z-coordinate – horizontal planes.

We suppose have a function of cylindrical coordinates $f(r, \theta, z)$ defined on a cylindrical wedge given by

$$\Omega = \{(r, \theta, z) \,|\, a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

and we want to compute the triple of f over Ω.

For integration in cylindrical coordinates we partition Ω with increments $\triangle r$, $\triangle\theta$, and $\triangle z$ equally spaced in each of the coordinate directions. The volume of an element of the partition is

$$\triangle V_{ijk} = (\triangle r)(r_i^* \triangle\theta)(\triangle z) = r_i^* \triangle r \triangle\theta \triangle z$$

where r_i^* is again midway between r_i and r_{i+1}. Note that $\triangle V = \triangle A \triangle z$ where $\triangle A$ is the corresponding area in the polar coordinates. The remainder of the derivation is analogous to previous section and so we skip the details and just state the result

In cylindrical coordinates:

$$\iiint_\Omega f\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(r, \theta, z)\, r\, dr\, d\theta\, dz$$

where

$$\Omega = \{(r, \theta, z) \,|\, a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

Key point is the expression of the infinitesimal volume element dV in terms of infinitesimal changes in the coordinates.

Key fact, in cylindrical coordinates:

$$dV = r dr d\theta dz$$

7.4 Iterated Integration in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are a three dimensional coordinate systems where r is the distance from the origin (three-dimensional radial coordinate), θ is same angle as in cylindrical coordinates (corresponds to longitude where the x-axis is zero longitude), and ϕ is the angle from the vertical (angle from the north pole or co-latitude). The relationship between spherical and Cartesian coordinates is

$$x = r \sin\phi \cos\theta \qquad r = \left(x^2 + y^2 + z^2\right)^{1/2}$$

$$y = r \sin\phi \sin\theta \qquad \tan\phi = \frac{(x^2 + y^2)^{1/2}}{z}$$

$$z = r \cos\phi \qquad \tan\theta = \frac{y}{x}$$

Some authors give a different symbol ρ to the radial coordinate in spherical coordinates but we will not do this. (*Be warned, many authors use θ and ϕ in exactly the reverse of the roles here.*) The ranges of spherical coordinates are

$$r \geq 0 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

It is useful to view spherical coordinate system in terms of a grid consisting of surfaces of constant r-coordinate – spheres centred on the origin, surfaces of constant θ-coordinate – radial planes, and surfaces of constant ϕ-coordinate – cones.

We suppose have a function of spherical coordinates $f(r, \theta, \phi)$ defined on a spherical wedge

$$\Omega = \{(r, \theta, \phi) \,|\, a \leq r \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \delta\}.$$

and we want to compute the triple of f over Ω.

For integration in spherical coordinates we partition Ω with increments $\triangle r$, $\triangle\theta$, and $\triangle\phi$ equally spaced in each of the coordinate directions. The key is again the relationship between the volume of an element of the partition and the increments $\triangle r$, $\triangle\theta$, and $\triangle\phi$. Here the relationship is slightly more complicated. At a point (r_i, θ_j, ϕ_k) we have the approximation

$$\triangle V_{ijk} \approx (\triangle r)(r_i \triangle\phi)(r_i \sin\phi_k \triangle\theta) = r_i^2 \sin\phi_k \triangle r \triangle\theta \triangle\phi$$

In can be shown (using the Mean Value Theorem) that there is a point $(r_i^*, \theta_j^*, \phi_k^*)$ within the element such that

$$\triangle V_{ijk} = (\triangle r)(r_i^* \triangle\phi)(r_i^* \sin\phi_k^* \triangle\theta) = r_i^{*2} \sin\phi^* \triangle r \triangle\theta \triangle\phi$$

The remainder of the derivation is analogous to previous sections and so we skip the details and just state the result

In spherical coordinates:

$$\iiint_\Omega f\,dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(r,\theta,\phi)\,r^2\,\sin\phi\,dr\,d\theta\,d\phi$$

where

$$\Omega = \{(r,\theta,\phi)\,|\,a \le r \le b, \alpha \le \theta \le \beta, \gamma \le \phi \le \delta\}.$$

Key point is the expression of the infinitesimal volume element dV in terms of infinitesimal changes in the coordinates.

Key fact, in spherical coordinates:

$$dV = r^2\,\sin\phi\,dr\,d\theta\,d\phi$$

Discussion

- Sometimes we use special coordinates because of geometry, that is the region of integration suggest special coordinates, but equally often we use special coordinates because the integrand simplifies in these coordinates. Another way to say this is that in many cases the function we wish to integrate arises because a problem depends only on distance from the origin (polar coordinates in 2D and spherical coordinates in 3D) or because a problem depends only on distance from an axis (cylindrical coordinates).
- In problems with symmetry, the integral separates in the corresponding coordinate, or coordinates, and the integrals simplify.
- In our derivation of formulas for iterated integration we took the regions to be simple and hence the limits of integration were constants. However, just as with integration in Cartesian coordinates, the regions need not be this simple. Recall Type I and Type II regions for double integral for example. In general, in iterated integration the limits of any inner integrals may depend on coordinates of any of the outer integrals.

7.5 Applications

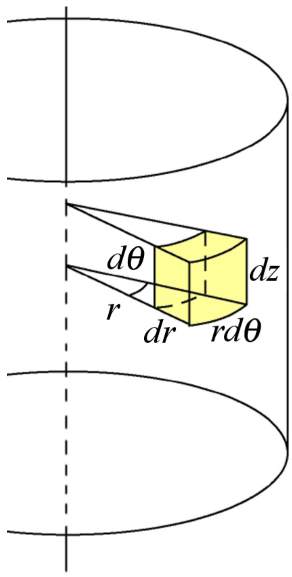
Some interesting application areas of integration in special coordinates are:

- Spherical geometries. Here a good example would be the earth's atmosphere.

- Interaction kernels or interaction potentials. Here good examples would be flocking and herding of animals or the spread of infectious diseases.

- Probability densities. An important example here is computing the Gaussian integral to obtain the normal distribution.

- Central forces. A good example here is electron orbitals about nuclei.



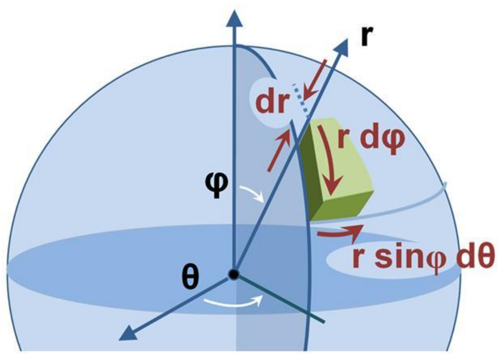


Figure 6: Volume elements in cylindrical and spherical coordinate systems. (Cylindrical case taken from MIT Physics 8.01 course notes. Spherical case from Wikipedia.)