Week 9: Surfaces and Surface Integrals

Introduction

We have seen surfaces a few times already, as graphs of functions of two variables and as level sets of functions of three variables. We now approach surfaces through parametrisations - extending the ideas from parametrised curves. You will see many similarities both to parametrised curves from Part I and to coordinate transformation from last week.

9.1 Parametric Surfaces

Recall that a curve in $\mathbb{R}^3$ is parameterised by a continuous map from an interval in $\mathbb{R}$ into $\mathbb{R}^3$.

$$\mathbf{r}\colon [a, b] \to \mathbb{R}^3$$

By extension, a surface S in $\mathbb{R}^3$ is parameterised by a continuous map from a region in Ω in $\mathbb{R}^2$ into $\mathbb{R}^3$, that is

$$\mathbf{r}\colon \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

We think of the region Ω getting mapped to the surface in $\mathbb{R}^3$.

In terms of components, we write

$$\begin{aligned} \mathbf{r}(u, v) = &(x(u, v), y(u, v), z(u, v)) \\ &\text{with} \quad (u, v) \in \Omega \end{aligned}$$

We say that

$$S = \{\mathbf{r}(u, v) | (u, v) \in \Omega\}$$

is a **parametric surface** and is given parametrically by parameters (u, v) in the region (or domain) Ω.

Comments

- One must put conditions on $\mathbf{r}$, or its component functions $x(u, v)$, $y(u, v)$, and $z(u, v)$, to guarantee niceness of the surface. We will not focus on this, but implicitly assume that the component functions are differentiable, except possibly at the boundaries of Ω. Self intersections are ruled out by requiring $\mathbf{r}$ to be injective, except at the boundaries of Ω.

- One often draws and analyses surfaces in terms of individual parameterised curves corresponding to the parameters u and v separately. For example, one fixes v to some value v_0 and varies u. $\mathbf{r}(\cdot, v_0)$ is a parameterised curve in S. We call this a **u-curve**. There is a family of these since the value v_0 can be varied. Similarly there is a family of **v-curves**. These curves are sometimes collectively called **grid curves**.

- It is not sufficient to specify the mapping alone, one must specify the parameter domain Ω.

  Similarly to what we encountered with multiple integration, in the simplest case Ω will be rectangle $[a, b] \times [c, d] \in \mathbb{R}^2$. More complex domains, for example of the form,

  $$\Omega = \{(u, v) | a \leq u \leq b, \quad g_1(u) \leq v \leq g_2(u)\}$$

  can easily be dealt with.

- Just as with curves in multiple segments, some surfaces of interest (see examples) will be most naturally specified as several distinct pieces, each with their own domain and parametrisation.

- The parametric representation of a surface S is not unique.

- Finally, be aware that many of the derivations and formulas that follow are similar to those encountered last week with coordinate transformations. There are important differences, however. Coordinate transformations are mappings between spaces of the same dimensions, $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$. Parametrised surfaces are mappings between spaces of different dimension, $\mathbb{R}^2 \to \mathbb{R}^3$.

Examples

As with parametrised curves, one best learns how to parametrise surfaces by working through many examples.

Unit sphere We obtain the parametrisation for the unit sphere by putting $r = 1$ in the transformation into spherical coordinates:

$$\mathbf{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)\,.$$

Here $\Omega = [0, 2\pi] \times [0, \pi]$. We could of course have labelled the parameters as (θ, ϕ) rather than (u, v).

$$\mathbf{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)\,.$$

This is more natural and generally preferred when a parametrisation is closely connected with particular coordinate system. If we restrict ϕ to take values in $[0, \pi/2]$, we obtain a parametrisation of the upper unit hemisphere.

Another parametrisation of this hemisphere, not based on spherical coordinates, is given by:

$$\mathbf{r}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}),\ u^2 + v^2 \leqslant 1\,.$$

Cylinder of radius R This is obtained by using cylindrical coordinates. We put $r = R$ and let z vary over any range, $[0, h]$ for example, obtaining

$$\mathbf{r}(\theta, t) = (R\cos\theta, R\sin\theta, t) \quad (\theta, t) \in [0, 2\pi] \times [0, h]$$

Cylinder of radius R **with end caps** Frequently one wants to consider surfaces that are closed (fully enclose some volume). For example, a cylinder with end caps. In this case we would specify the surface in 3 pieces, S_1, S_2 S_3, where S_1 would be the above cylinder and S_2 and S_3 would be disks:

$$\mathbf{r}_2(r, \theta) = (r\cos\theta, r\sin\theta, 0) \quad (r, \theta) \in [0, R] \times [0, 2\pi]$$

$$\mathbf{r}_3(r, \theta) = (r\cos\theta, r\sin\theta, h) \quad (r, \theta) \in [0, R] \times [0, 2\pi]$$

Surfaces of revolution Consider a function $f : [a, b] \to \mathbb{R}^+$. The graph of f is a curve $\{(x, y) |\, x \in [a, b], y = f(x) > 0\}$. Now extend to $\mathbb{R}^3$ and generate a surface by rotating the graph of f about the x axis. This surface is given parametrically by

$$\mathbf{r}(x, \theta) = \big(x\,,\ f(x)\cos\theta\,,\ f(x)\sin\theta\big), \qquad (x, \theta) \in [a, b] \times [0, 2\pi]$$

Torus We obtain our parametrisation by taking a circle $C = \{(0, a + b\cos u, b\sin u) \mid 0 \leqslant u \leqslant 2\pi\}$ of radius b and centred at $(0, a, 0)$ in the y-z plane, $0 < b < a$, and rotating it about the z-axis:

$$\mathbf{r}(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big)$$

where $v \in [0, 2\pi]$ is the angle in the x-y plane measured from the x-axis.

Graph The graph of $f\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a surface parametrised by:

$$\mathbf{r}(u,v) = \big(u, v, f(u,v)\big).$$

The hemisphere $f(u,v) = \sqrt{1-u^2-v^2}$ above was such an example.

9.2 Tangent Plane and Normal to a Surface

We have already touched on this when discussing partial derivatives. In the case of a parametrised surface we have grid curves from which we can easily obtain the tangent plane and normal vector at a given point on a surface.

Tangent plane

Consider a particular u-curve $\mathbf{r}(\cdot, v_0)$ on a surface S. We know from our study of parametrised curves that differentiating this with respect to u will give a vector tangent to the curve. Here the derivative will be a partial derivative. Thus $\dfrac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$ is tangent to the u-curve. Since the curve is in S, this vector will be tangent to S at point $\mathbf{r}(u_0, v_0)$. Likewise $\dfrac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$ will be another vector tangent to S at $\mathbf{r}(u_0, v_0)$.

These two vectors form a basis for the tangent plane to S at $\mathbf{r}(u_0, v_0)$. That is any point in the plane tangent to S at $\mathbf{r}(u_0, v_0)$ can be represented by a linear combination of these two vectors, plus the point $\mathbf{r}(u_0, v_0)$ itself. We can express the position vector of every point of the tangent plane at $\mathbf{r}(u_0, v_0)$ as

$$\mathbf{p}(h,k) = \mathbf{r}(u_0, v_0) + h\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) + k\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\,,$$

where $(h,k) \in \mathbb{R}^2$. Note, the plane is itself a parametrised surface: $\mathbf{p}$ is a mapping $\mathbb{R}^2 \to \mathbb{R}^3$ here written with parameters (h,k).

Normal vector

Frequently one does not express as plane via a parametrisation as above, but in terms of the normal vector. In any case the normal vector to a surface plays an important in the calculus of surfaces.

From our previous calculations for the tangent plane we know that the vector

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

is perpendicular to both $\dfrac{\partial \mathbf{r}}{\partial u}$ and $\dfrac{\partial \mathbf{r}}{\partial v}$. This vector is thus **normal to** S at $\mathbf{r}(u, v)$. It is perpendicular to any vector in the tangent plane. Of course there are infinitely many normals because we can multiply the above vector by any constant and obtain another normal. Normalising the normal in the above calculation gives a unit normal vector. In fact there are two possible unit normal vectors to a surface at a point

The two **unit normal vectors** to a surface at a point are

$$\mathbf{n} = \pm \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|}$$

In practice, either it will not be important which of these two vectors to use, or else it will be clear which one to choose. For a closed surface it is common to choose the outward-pointing normal.

From the normal vector at a point $\mathbf{r}_0$ on the surface, one can write an expression for the tangent plane as

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}$$

9.3 Surface Area

Given a parameterised surface S, we would like to find the area of the surface. That is we want to compute something like

$$\iint_S dS$$

where dS is an infinitesimal element of surface area.

The approach follows closely our previous treatment double integrals in generalised coordinates. Let $\mathbf{r}(u, v), (u, v) \in \Omega$ be a parametrisation of S. To integrate over the surface, we perform a double integral over Ω.

$$\iint_S \to \iint_\Omega$$

What we need to work out what is infinitesimal element of surface area dS, in terms of an infinitesimal area $du\, dv$ in Ω.

Proceed as we did for double integrals in generalised coordinates. Consider a small rectangle in Ω of width Δu and height Δv. This gets mapped by $\mathbf{r}$ to a "curved parallelogram", which is well approximated by the parallelogram based at $\mathbf{r}(u_0, v_0)$ and with sides

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\Delta u$$
$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\Delta v.$$

The area ΔS of the surface is given approximately by the area of the small parallelogram. Then, using the fact that the magnitude of the cross product of two vectors is the area of the parallelogram formed by them, we have

$$\Delta S \approx \left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u \Delta v$$

or since $\Delta u \, \Delta v$ is the area ΔA in Ω,

$$\Delta S = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A$$

Thus giving the differential of surface area

$$dS = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

We have dropped the explicit dependence on the point in question, (u_0, v_0), but you should understand that in general the partial derivatives on the right-hand-side are functions of (u, v), and hence so is dS.

Do not confuse this with similar expressions from generalised coordinates. Here $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ is each a three-component vector. In our treatment of two-dimensional integration in generalised coordinates, we had two two-component vectors.

Using this expression for dS we have

$$\begin{aligned} \text{Area of S} = A(S) &= \iint_S dS \\ &= \iint_\Omega \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \end{aligned}$$

where in practice the double integral over Ω is carried out by usual iterated integration over u and v is some suitable order.

One can show that the area of S is independent of the parametrisation of S. The proof is simply an exercise in the use of the change of variables formula.

9.4 Surface Integrals

Given a continuous scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ and a surface S in $\mathbb{R}^3$, we can evaluate f at a each point on S. Thus we have a real value associated to each point on S and we can integrate this over the surface. Rather than again going through all the arguments of summing up pieces and taking the limit as the number of pieces goes to infinity, we simply state the unsurprising result

Given a parametrisation $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ of a surface S the surface integral of f over S is

$$\iint_S f\, dS = \iint_\Omega f\big(\mathbf{r}(u,v)\big) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

Remarks:

- When f takes the value 1 everywhere, (or equivalently is just left out), the integral $\iint_S f\, dS = \iint_S dS$ and so reduces to the previous formula for the area of S.
- We have given the definition assuming that f was defined in a region of $\mathbb{R}^3$ containing the surface. In practice, all that is required is that f be defined on the surface. Such situations arise naturally in practice.
- In the case where S is a closed surface, one often writes
$$\oiint_S f dS$$
to indicate specifically that the surface is closed.

9.5 Flux Integrals

Given a three-dimensional vector field $\mathbf{v}\colon \mathbb{R}^3 \to \mathbb{R}^3$ and a surface S in $\mathbb{R}^3$, one commonly wishes to calculate the *flux of* $\mathbf{v}$ *though* S . This is defined to be the surface integral

$$\text{Flux of } \mathbf{v} \text{ through S} = \iint_S (\mathbf{v} \cdot \mathbf{n})\, dS$$

where $\mathbf{n}$ is a unit normal vector to S whose direction will be set by the context.

Flux integrals are particular surface integrals in which $f = \mathbf{v}\cdot\mathbf{n}$. Notice that this f is an example of a function that is only defined on the surface; it has no meaning except in conjunction with the surface. Since $\mathbf{v} \cdot \mathbf{n}$ is just a scalar, flux integrals are in principle just a particular case of surface integrals.

However, the algebra works out in a particular way to simplify the calculations.

As always, we will evaluate flux integrals by resorting to some parametrisation of S.

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\,dS = \iint_\Omega (\mathbf{v}\cdot\mathbf{n}) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

Recalling our expression of the unit normal vector $\mathbf{n}$ this can be written

$$\iint_\Omega \left(\mathbf{v}\cdot \frac{\pm\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|} \right) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$
$$= \pm \iint_\Omega \mathbf{v}\cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

Hence to find the flux of $\mathbf{v}$ through a surface S we evaluate:

The flux of $\mathbf{v}$ through a surface S is obtain via

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\,dS = \pm \iint_\Omega \mathbf{v}\cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

The direction of $\mathbf{n}$*, and hence the choice of* $\pm$*, will be specified for any given problem.*

Orientation

In our definition of the flux integral we implicitly assumed that we had a well-defined normal vector at each point of S. This sometimes fails. One way it can fail is that S has corners. For example, along the edges of a cube. The tangent plane and normal vector are not defined at corners. In most practical cases it is clear how to divide the surface into finite number of pieces where each piece has a well defined normal and the normals in the different pieces have a sensible relationship with one another, e.g. the normals on all faces of a cube can be taken in the outward direction.

A more serious problem arise for certain surfaces that cannot be oriented. You are probably familiar with a Möbius strip. Such a surface has only one side. This means that if one picks one of the two possible unit normal vectors at a point and then tries to vary that vector continuously over the surface, the choice will fail even though the surface is perfectly smooth. We can only consider orientable surfaces in the computation of flux integrals. Precise definitions of these concepts will appear in later modules.