## Catalan percolation

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### §1. Model and results.

Gravner & Kolesnik '23

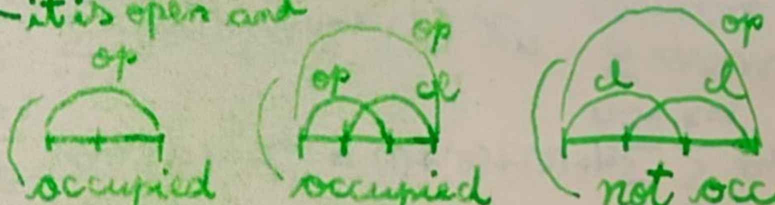
Increasing cellular automaton indexed by edges of a graph. $p \in [0,1]$ $G=(V,E)$ $V=\mathbb{Z}$, $E$ all edges

Initially: independently, each edge of length $\geq 2$ is open with probability $p$, closed $1-p$.

Now: say all edges of length 1 are occupied.

An edge $\{i,j\}$ with $|i-j| \geq 2$ becomes occupied if it is open and $\exists\, i'$: $\{i,i'\}$ and $\{i',j\}$ are both occupied.

Examples




[Can be seen as growth model with immunity]

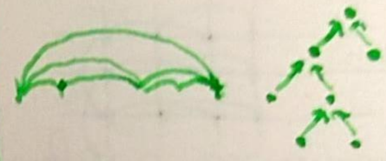
$\theta_n(p) := \mathbb{P}_p(\{0,n\} \text{ becomes occupied}) \in [0,p]$ → monotone in $p$, unclear in $n$.

$p_c := \inf\{p : \liminf_{n\to\infty} \theta_n(p) > 0\}$.

Why "Catalan"? $\{C_n\}_{n\geq 0}$ Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad C_0 = C_1 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

Fix a particular "path" that can cause an edge to become occupied



Path represented as binary tree with $n$ leaves. $C_{n-1}$ such trees.

Gravner & Kolesnik: $\frac{1}{4} \leq p_c \leq 1 - 2^{-32}$

Simple refinement: $\frac{1}{4} \leq p_c \leq p_c^{(o)}$ oriented site percolation $\simeq 0.7$

First inequality: union bound over "paths": $\theta_n(p) \leq \sum_{\gamma \text{ to } \{0,n\}} \mathbb{P}(\gamma \text{ all open})$

(Binary tree with $n$ leaves has $n-1$ internal nodes) $= p^{n-1} \cdot C_{n-1}$

$C_n \simeq 4^n$ as $n \to \infty$

$p < 1/4 \Rightarrow \theta_n(p) \xrightarrow{n\to\infty} 0$

**Theorem.** $\frac{1}{4} < p_c < p_c^{(o)}$. Simulations suggest $p_c \simeq 0.4$

### §2. Lower bound.

$$p_c \geq p_c^{(-)} := \sup\{p : \underbrace{\limsup_{n\to\infty} \theta_n(p)^{1/n}}_{1/\mathrm{rad}(\{\theta_n(p)\})} < 1\}$$

$= \sup\{p : \mathrm{rad}(\{\theta_n(p)\}) > 1\}$.

Strategy: find $\{a_n(p)\}_{n\geq 1}$ such that $a_n(p) \geq \theta_n(p)$ and $\mathrm{rad}(\{a_n(p)\})$ easy to find

This gives $\mathrm{rad}(\{a_n(p)\}) \le \mathrm{rad}(\{\theta_n(p)\})$, so
$p_c^{(-)} \ge \sup\{p : \mathrm{rad}(\{a_n(p)\}) > 1\}$.

Key inequality: $\theta_n(p) \le p \sum_{k=1}^{n-1} \theta_k(p)\, \theta_{n-k}(p)$   Union bound

Take $a_1(p) = \theta_1(p) = 1$
$a_2(p) = \theta_2(p) = p$
$a_3(p) = \theta_3(p)$
$n \ge 3$ $a_n(p) = p \sum_{k=1}^{n-1} a_k(p)\, a_{n-k}(p)$

Get $a_n \ge \theta_n$ by induction

How to find $\mathrm{rad}(\{a_n(p)\})$?

$$C(x) = \sum_{n=1}^{\infty} a_n(p) \cdot x^n = (1)+(2)+(3)+p\sum_{n=4}^{\infty} \sum_{k=1}^{n-1} a_k\, a_{n-k}\, x^n$$

$$C(x)^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} a_k(p)\, a_{n-k}(p) \cdot x^n$$

Expressions coincide for $n \ge 4$

Manipulating this get

$$p\, C(x)^2 - C(x) + x - p^3 x^3 = 0. \quad (*)$$

It turns out that $C(x)$ converges at $x=1$ if and only if there is some value that, when replacing $C(x)$ above, makes the equality right

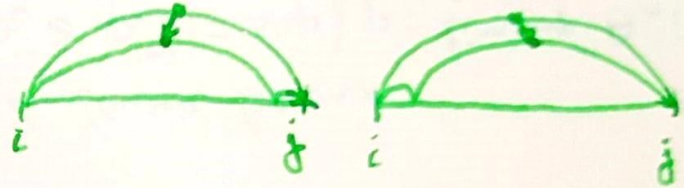
$$pX^2 - X + 1 - p^3 = 0$$

Discriminant $4p^4 - 4p + 1$ positive for $p \in [0, 0.254)$

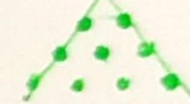
Further iterations seem to accumulate at $\approx 0.28$.

§3. Upper bound. Modify model to make percolation harder.

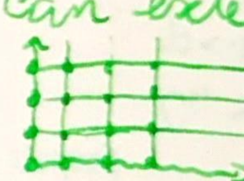
$\{i,j\}$ becomes occupied if either $\{i+1, j\}$ or $\{i, j-1\}$ is occupied.




Get lattice model



Sites open with probability $p$, closed $1-p$, all independent. Can extend lattice to $-\infty$, rotate, get:



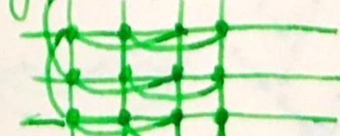
1st quadrant
Can move $\rightarrow$ and $\uparrow$

Site percolation

$p_c^{(s)} = \inf\{p : \mathbb{P}_p((0,0) \to \infty) > 0\}$.

Further modification: $\{i,j\}$ becomes occupied if any of the following holds: $\{i, j-2\}, \{j-2, j\}$
$\{i+1, j\}$ occupied
$\{i, i+2\}, \{i+2, j\}$ "   $\{i, j-1\}$

After extending and rotating, get model:



Sites $p$ (indep)
Length-two bonds (dependent)

One further modification: $\left.\begin{array}{l} p \text{ for sites} \\ q \text{ for edges} \end{array}\right\} p_c(q)$

Theorem. $q > 0 \Rightarrow p_c(q) < p_c(0) = p_c^{or}$

Standard method to prove this kind of thing: essential enhancements (Aizenman & Grimmett). Not applicable here.

Our method – Ingredients

- Classical "edge speed estimates" from IPS literature (Durrett, Griffeath)
- Renormalization for oriented percolation (Durrett)
- RSW theory for critical oriented percolation (Duminil-Copin, Tassion, Teixeira)
- Oriented percolation with infinite-range defects

(Hilário, Sá, Sanchis, Teixeira)