Stochastic differential equations and higher order derivatives of heat semigroups.

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Based on : [1] *Elworthy, K. David, Higher order derivatives of heat semigroups on spheres and Riemannian symmetric spaces, in Geometry and invariance in stochastic dynamics, Springer Proc. Math. Stat., volume 378, 113–136, 2021*

Taken from: [2] *Generalised Weitzenböck formulae for differential operators in Hörmander form II: Natural bundles and higher order derivative formulae* D El. In preparation.

Approach derived from: [3] K. David Elworthy, Yves Le Jan, and Xue-Mei Li. *The geometry of filtering*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2010.

Baxendale integrability lemma for groups of random transformations

Consider a Wiener process $\{g_t\}_{t \geq 0}$ on a polish group $\mathcal{G}$, ie sample cts, $g_0 = Id$, increments $g_t g_s^{-1}$ independent and time homogeneous.

Theorem (Baxendale, Comp. Math. 1984)

Let $\mathcal{G} \times B \to B$ be a continuous group action on a Banach space B by linear transformations. Then $\exists c, d \in \mathbf{R}$ with

$$||\mathbf{E}\{g_t b\}|| \leq c e^{dt} ||b|| \qquad b \in B, t \geq 0.$$

What are c, d?

First example

M compact Riemannian. $\mathcal{G} := C^r\,\mathrm{Diff}(M)$. $B = C^0(M; R)$. Take SDE for BM on M, solution flow $\{\xi_t\}_t$ is Wiener process on $\mathcal{G}$.

$$P_t(f) := \mathbf{E}\{f \circ \xi_t\}$$

What are c, d ? Not interesting, take $f = \mathit{constant}$.
More interesting if take $B = C^r(M; \mathbf{R})$, cf Kifer.
This shows: $f \in C^r(M; \mathbf{R}) \Rightarrow P_t f \in C^r(M; \mathbf{R})$

Special cases; quantitative version, after Elworthy-LeJan-Li

Suppose $\mathcal{G}$ is a smooth Hilbert manifold and a polish group and $\{g_t\}_t$ is an $\mathcal{L}_t$-diffusion for a smooth non-autonomous right invariant nuclear diffusion operator $\{\mathcal{L}_t\}_{t\geq 0}$, with $g_0 = Id$. Let $\rho : \mathcal{G} \to GL(V)$ be a smooth representation for finite dimensional (or Hilbert) V. Then

$$\frac{d}{dt}\mathbf{E}\rho(g_t) = \lambda_t \circ \mathbf{E}\rho(g_t)$$

for $\lambda_t \in \mathbf{L}(V; V)$ given by

$$\lambda_t = \mathrm{Comp}(\rho_* \otimes \rho_*)\sigma_{Id}^{\mathcal{L}_t} + \rho_* \delta^{\mathcal{L}_t}(\varpi)(Id).$$

Note the two terms.

Nuclear diffusions: the symbol

Symbol $\sigma_g^{\mathcal{L}_t} : T_g^*\mathcal{G} \to T_g\mathcal{G}$.
If trace class we say $\mathcal{L}_t$ is nuclear and can consider

$$\sigma_{Id}^{\mathcal{L}_t} \in \mathfrak{g} \otimes_\pi \mathfrak{g}.$$

Assume cts in t.

Aside:

$$\begin{aligned} \sigma_g^{\mathcal{L}_t}(df, dh) : &= dh\left(\sigma_g^{\mathcal{L}_t}(df)\right) \\ &= \Gamma(f, h). \end{aligned}$$

The first term

Since the derivative ρ_* of ρ at identity is

$$\rho_* : \mathfrak{g} \to \mathbf{L}(V; V)$$

We have

$$(\rho_* \otimes \rho_*)\sigma_{Id}^{\mathcal{L}_t} \in \mathbf{L}(V; V) \otimes \mathbf{L}(V; V)$$

and

$$\mathrm{Comp}(\rho_* \otimes \rho_*)\sigma_{Id}^{\mathcal{L}_t} \in \mathbf{L}(V; V).$$

The second term: generalised divergence

$\delta^{\mathcal{L}_t}$ maps smooth one forms to functions on $\mathcal{G}$ s.t.

- $\mathcal{L}_t(f) = \delta^{\mathcal{L}_t}(df)$
- $\delta^{\mathcal{L}_t}(f\phi)(g) = f(g)\delta^{\mathcal{L}_t}(\phi) + \sigma_g^{\mathcal{L}_t}(df, \phi)$.

The vector valued form ϖ is $\varpi_g := TR_g^{-1} : T_g\mathcal{G} \to \mathfrak{g}$, Maurer-Cartan. Then

$$\delta^{\mathcal{L}_t}(\varpi)(g) \in \mathfrak{g}.$$

Proof of formula

To prove $\frac{d}{dt}\mathbf{E}\rho(g_t) = \lambda_t \circ \mathbf{E}\rho(g_t)$
for $\lambda_t \in \mathbf{L}(V;V)$ given by

$$\lambda_t = \mathrm{Comp}(\rho_* \otimes \rho_*)\sigma^{\mathcal{L}_t}_{Id} + \rho_* \delta^{\mathcal{L}_t}(\varpi)(Id).$$

We have

$$\rho(g_t) = Id_V + M^{d\rho}_t + \int_0^t \mathcal{L}_s(\rho)(g_s)\, ds.$$

By right invariance:

$$\begin{aligned}
\mathcal{L}_s(\rho)(g_s) = \mathcal{L}_s(\rho \circ R_{g_s})(Id) &= \mathcal{L}_s(\rho(-)\rho(g_s))(Id) \\
&= \mathcal{L}_s(\rho)(Id).\rho(g_s)) \\
&= \delta^{\mathcal{L}_s}(d\rho)(Id).\rho(g_s).
\end{aligned}$$

But $d\rho_a(-) = \rho(a)(\rho_* \circ \varpi_a)$.
Q.E.D.

Special case of BM on compact G

For Brownian motion on a compact Lie group G

$$\lambda_t = \mathrm{Comp}(\rho_* \otimes \rho_*)\sigma_{Id}^{\mathcal{L}_t} + \rho_* \delta^{\mathcal{L}_t}(\varpi)(Id).$$

reduces to

$$\lambda = \frac{1}{2}\mathrm{Comp}(\rho_*(\alpha^j) \otimes \rho_*(\alpha^j))$$

for $\{\alpha^j\}_j$ an o.n. base for $\mathfrak{g}$.

For ρ reducible

If
$V = V^1 \bigoplus V^2$ with $\rho(g) = \rho^1(g) \oplus \rho^2(g)$
then

$$\lambda_t(v^1, v^2) = (\lambda_t^1(v^1), \lambda_t^2(v^2)).$$

Application to higher order derivative estimates. $M = S^n$

Fix $x_0 \in S^n$.
Consider $SO(n+1) = isom(S^n)$, with $SO(n) = SO(n+1)_{x_0}$.
Define
$p : SO(n+1) \to S^n$ by Note $SO(n)$ acts on the right
$p(k) = k.x_0$. on $SO(n+1)$ preserving fibres.

Take a BM $\{k_t\}_t$ on $K = SO(n+1)$ with $k_0 = 1$.
Then $\xi_t(y) := k_t.y$ is a flow of BM's on S^n.
Set

$$x_t = k_t.x_0 = p(k_t).$$

decomposition

By El-LeJan-Li 2004, El-Kendall '86, the BM on $K = SO(n+1)$ has skew product decomposition:

$$k_t = \tilde{x}_t . g_t^{\tilde{x}_\cdot}$$

for $\tilde{x}_t \in K$ "horizontal lift" of $x_\cdot$ and $g_t^\sigma \in SO(n)$ independent of $\{x_t\}_t$.

The derivative $T\tilde{x}_t : T_{x_0}S^n \to T_{x_t}S^n$ of $y \mapsto \tilde{x}_t . y$ is just Levi-Civita parallel translation $/\!/_t$ along $\{x_t\}_t$. Thus

$$T\xi_t = /\!/_t \rho^1(g_t^{\tilde{x}_\cdot}) : T_{x_0}S^n \to T_{x_t}S^n$$

for $\rho^1 : SO(n) \to O(T_{x_0}S^n)$ given by

$$\rho^1(g) = T_{x_0}L_g$$

Hessians

For $f : \mathbf{R}^n \to \mathbf{R}$,

$$Hess(f)_x : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} \text{ with } Hess(f)_x(u,v) = D^2 f(x)(u,v).$$

For $f : M \to \mathbf{R}$

$$Hess(f)_x : T_x M \times T_x M \to \mathbf{R} \text{ with } Hess(f)_x(u,v) = \nabla^2 f(x)(u,v).$$

It is bilinear, symmetric (using Levi-Civita), so consider linear

$$Hess(f)_x : T_x M \odot T_x M \to \mathbf{R}.$$

To look at $Hess(P_t f)$

$$P_t f(x_0) = \mathbf{E}\{f(\xi_t(x_0))\} \qquad f : M \to \mathbf{R}$$

$$dP_t f_{x_0}(v_0) = \mathbf{E}\{df_{x_t} \circ T\xi_t(v_0)\} \qquad v_0 \in T_{x_0}M$$

$$\begin{aligned} Hess(P_t f)(u_0 \odot v_0) &= \mathbf{E}\{df_{x_t} \circ \nabla T\xi_t(u_0 \odot v_0) \\ &\quad + Hess(f)_{x_t}\left(T_{x_0}\xi_t(u_0) \odot T_{x_0}\xi_t(v_0)\right)\} \end{aligned}$$

Helped via isometries

Now for us:

$$
\begin{aligned}
& Hess(P_t f)_{x_0}(u_0 \odot v_0) \\
= \quad & \mathbf{E}\{df_{x_t} \circ \nabla T\xi_t(u_0 \odot v_0) + Hess(f)_{x_t}\,(T_{x_0}\xi_t(u_0) \odot T_{x_0}\xi_t(v_0))\} \\
= \quad & \mathbf{E}\{Hess(f)_{x_t}\,(T_{x_0}\xi_t(u_0) \odot T_{x_0}\xi_t(v_0))\}
\end{aligned}
$$

since $\nabla(T\xi_t) = 0$ because ξ_t is an isometry.

Use of decomposition $\xi_t = k_t = \tilde{x}_t . g_t^{\tilde{x}.}$, redundant noise as El-LeJan-Li

$$
\begin{aligned}
Hess(P_t f)(u_0 \odot v_0) &= \mathbf{E}\{Hess(f)_{x_t}\left(T_{x_0}\xi_t(u_0) \odot T_{x_0}\xi_t(v_0)\right)\} \\
&= \mathbf{E}\{Hess(f)_{x_t}(T_{x_0}\tilde{x}_t \odot T_{x_0}\tilde{x}_t)\rho(g_t^{\tilde{x}.})(u_0 \odot v_0)\} \\
&= \mathbf{E}\{Hess(f)_{x_t}(/\!/_t \odot /\!/_t)\rho(g_t^{\tilde{x}.})(u_0 \odot v_0)\}
\end{aligned}
$$

where $\rho : \mathcal{G} = SO(n) \to GL(\mathbf{R}^n \odot \mathbf{R}^n) \approx GL(T_{x_0}M \odot T_{x_0}M)$ is

$$\rho(g)(u \odot v) = T_{x_0}g(u) \odot T_{x_0}g(v).$$

Using independence replace $(/\!/_t \odot /\!/_t)\rho(g_t^{\tilde{x}.})$ by $W_t^{\odot} : \odot^2 T_{x_0}M \to \odot^2 T_{x_t}M$ where

$$\frac{D}{dt}W_t^{\odot} = \Lambda_{x_t} W_t^{\odot}.$$

$$\Lambda_{x_t} := (/\!/_t \odot /\!/_t)\lambda_t(/\!/_t \odot /\!/_t)^{-1} : \odot^2 T_{x_t}M \to \odot^2 T_{x_t}M$$

The splitting

ALSO $T_{x_0}M \odot T_{x_0}M$ splits under the othogonal group into

$$\mathbf{R}\sum(e_i \odot e_i) \bigoplus \{\ker\langle -, -\rangle : T_{x_0}M \odot T_{x_0}M \to \mathbf{R}\}$$

Set

$$\Xi_{x_0} = \sum e_j \odot e_j$$

.
Then

$$u_0 \odot v_0 = \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0} + \{u_0 \odot v_0 - \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0}\}.$$

Preserved by $/\!/_t \odot /\!/_t$.

Splitting of Λ

Set

$$\Xi_{x_t} = \sum /\!/_t e_j \odot /\!/_t e_j \qquad t \geq 0.$$

Easy computation gives:

$$\begin{aligned}\Lambda_{x_t}(\Xi_{x_t}) &= 0\\ \Lambda_{x_0}\left(u_0 \odot v_0 - \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0}\right) &= -n\left(u_0 \odot v_0 - \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0}\right).\end{aligned}$$

$\therefore$ from

$$u_0 \odot v_0 = \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0} + \{u_0 \odot v_0 - \frac{1}{n}\langle u_0, v_0\rangle \Xi_{x_0}\},$$

we get

$$W_t^{\odot}(u_0 \odot v_0) = \frac{1}{n}(1 - e^{-nt})\langle u_0, v_0\rangle \Xi_{x_t} + e^{-nt} /\!/_t u_0 \odot /\!/_t v_0.$$

The result for S^n

We saw

$$W_t^{\odot}(u_0 \odot v_0) = \frac{1}{n}(1 - e^{-nt})\langle u_0, v_0\rangle \Xi_{x_t} + e^{-nt} /\!/_t u_0 \odot /\!/_t v_0.$$

$\therefore$

$$\begin{aligned} Hess(P_t f)(u_0 \odot v_0) &= \mathbf{E}\{Hess(f)_{x_t}(W_t^{\odot}(u_0 \odot v_0))\} \\ &= \mathbf{E}\{Hess(f)_{x_t}(\frac{1}{n}(1 - e^{-nt})\langle u_0, v_0\rangle \Xi_{x_t} \\ &\qquad + e^{-nt} /\!/_t u_0 \odot /\!/_t v_0)\} \\ &= \frac{1}{n}(1 - e^{-nt})\langle u_0, v_0\rangle P_t(\Delta f) \\ &\qquad + e^{-nt}\mathbf{E}\{Hess(f)_{x_t}(/\!/_t u_0 \odot /\!/_t v_0)\}. \end{aligned}$$

In particular if u_0, v_0 are perpendicular,

$$\nabla^2 P_t f(u_0, v_0) = e^{-nt}\mathbf{E}\{Hess(f)_{x_t}(/\!/_t u_0, /\!/_t v_0)\}.$$

Higher order

$$\odot^q(\mathbf{R}^n) = \odot_0^q(\mathbf{R}^n) + \odot_0^{q-2}(\mathbf{R}^n) \odot \Xi + \ldots$$

For $q = 3$

$$\begin{aligned} u \odot v \odot w &= \frac{1}{n+2}\left(\langle u, v\rangle w + \ldots\right) \odot \Xi \\ &+ u \odot v \odot w - \frac{1}{n+2}\left(\langle u, v\rangle w + \ldots\right) \odot \Xi \end{aligned}$$

3rd order for spheres

We get $\nabla^3(P_t f)(u \odot v \odot w)$ given by

$$\begin{aligned} &= \frac{1}{n+2} e^{-(n-1)t}(1 - e^{-(n+2)t})\left(\langle u, v\rangle \mathbf{E}\{d\Delta f(/\!/_t w) + ...\}\right) \\ &+ e^{-3nt}\mathbf{E}\{\nabla^3 f_{x_t}(/\!/_t u \odot /\!/_t v \odot /\!/_t w)\} \\ &= \frac{1}{n+2} e^{-t}(1 - e^{-(n+2)t})\left(\langle u, v\rangle \{P_t^1(\Delta^1(df))/\!/_t w + ...\}\right) \\ &+ e^{-3nt}\mathbf{E}\{\nabla^3 f_{x_t}(/\!/_t u \odot /\!/_t v \odot /\!/_t w)\}. \end{aligned}$$

for the Kodaira-Hodge semigroup on 1-forms.

Further

Written for higher order, symmetrised, for S^n in [1].

Should be do-able for Riemannian symmetric spaces, compact, eg oriented Grassmannians $\frac{SO(p+q)}{SO(p)xSO(q)}$, compact Lie groups. But complicated: it will depend on the representation theory for tensor products of representations of a given compact Lie group.

A more algebraic approach is mentioned in [1].

Non-compact symmetric spaces seem a challenge; for example spaces of positive definite symmetric matrices, (or hyperbolic spaces $\frac{O(1,n)}{SO(n)}$??).

For general compact M, with cohesive diffusion generator $\mathcal{A}$: higher order derivatives

Need to consider r- jets ie Taylor expansions order r , $d^{(r)}f_x$, and their duals, higher order tangent vectors.
Higher order tangent space: $T_x^{(r)}M$

$$d^{(r)}f_x : T_x^{(r)}M \to \mathbf{R}.$$

For general compact M, with cohesive diffusion generator $\mathcal{A}$: Step 1

Choose SDE for $\mathcal{A}$. Equivalently put $\mathcal{A}$ in Hörmander form.

Get stochastic flow $\{\xi_t\}_t$. Wiener process in $\mathrm{Diff}^s(M)$, any $s \geq 0$.

For general compact M, with cohesive diffusion generator $\mathcal{A}$

Replace $p : SO(n+1) \to S^n$ by

$$p : \mathrm{Diff}^s \to M \qquad p(\theta) = \theta(x_0)$$

and $SO(n)$ by $\mathcal{G} = \mathrm{Diff}^s_{x_0}$. By El-LeJan-Li, flow is a skew product

$$\xi_t = \tilde{x}_t \circ g_t^{\tilde{x}_\cdot}.$$

Set

$$\hat{/\!/}_t = T^{(r)} \tilde{x}_t : T^{(r)}_{x_0} M \to T^{(r)}_{x_0} M$$

parallel translation for a certain connection on $T^{(r)}M$. Take

$$\rho : \mathrm{Diff}^s_{x_0} \to GL(T^{(r)}_{x_0} M) \qquad g \mapsto T^{(r)} g.$$

general formula

We get

$$
\begin{aligned}
d^{(r)}(P_t(f))_{x_0} &= \mathbf{E}\{d^{(r)}f \circ T^{(r)}_{x_0}\xi_t\} \\
&= \mathbf{E}\{d^{(r)}f \circ \hat{/\!/}_t \rho(g^{\tilde{x}_\cdot})\} \\
&= \mathbf{E}\{d^{(r)}f \circ W^{T^{(r)}}_t\}
\end{aligned}
$$

for

$$
\frac{D}{dt} W^{T^{(r)}}_t = \Lambda^{\rho}_{x_t} W^{T^{(r)}}_t.
$$

Main problem: choose flow/SDE/ Hörmander form for $\mathcal{A}$ so that $\hat{/\!/}_t$ preserves metric.

Aside on connections and SDE

Consider SDE on M for elliptic diffusion generator $\mathcal{A}$

$$dx_t = A(x_t)dt + X(x_t) \circ dB_t$$

with $X : Mx\mathbf{R}^m \to TM$ such that $X(x)- = X(x,-) : \mathbf{R}^m \to T_xM$ onto.
It induces Riemannian metric on M, essentially $\sigma^{\mathcal{A}}$
Get pseudo-inverse $Y_x : T_xM \to \mathbf{R}^m$ each $x \in M$.
Determines a metric connection on M : For vector field V and $u \in T_xM$ define

$$\breve{\nabla}_u V = X(x)(d\{y \to Y_y(V(y))\}u) \in T_xM$$

The other connection

For SDE as above, if $u \in T_x M$, define vector field

$$Z^u(-) = X(-)Y_x(u) \in T_- M$$

Get another connection by Lie differentiation

$$\hat{\nabla}_u V = \mathbf{L}_{Z^u}(V)(x)$$

This has parallel translation $\hat{/\!/}_t$ used above.
It may or may not be metric.
It gives a connection on jet bundles etc.

gradient SDE

For isometric immersion $j : M \to \mathbf{R}^m$ let

$$X(x)(e) = \nabla\{y \to \langle j(y), e\rangle\}(x)$$

Then

$$Y_x = d_y j$$

Both connections are the Levi-Civita connection on M.

gradient SDE inducing connections on jet bundles

Can use an isometric immersion and extend X to give
$X : MxJ^q\mathbf{R}^m \to J^q(M)$ with a right inverse Y
so that the corresponding $\check{\nabla}$ agrees with $\hat{\nabla}$.
But the Y is not a pseudo-inverse so this does not prove they are metric.

Can't expect flows of isometries

An SDE for BM on M corresponds to a a “virtual immersion” in sense of Mendes & Radeschi (2019) if it induces the Levi-Civita connection on TM.

For M compact they show that it induces a flow of isometries iff M is a symmetric space and it is the SDE we used.

Can expect volume preserving flows

Baxendale (unpublished, early 1980's) showed that there are virtual immersions with volume preserving flows for all Riemannian manifolds.

Different SDE give "different" answers

E.G. Treat S^3 as a Lie group, $SU(2)$. Take left invariant SDE and solution $\{g_t\}_t$ from identity, so flow is

$$k \mapsto k.g_t = R_{g_t} k.$$

$$\therefore \nabla^2 P_t f(u_0, v_0) = \mathbf{E}\{\nabla^2 f_{g_t.x_o}(TR_{g_t} u_0, TR_{g_t} v_0)\}.$$

Do different symmetric space structures give different answers?

E.G. Identify S^3 as $SU(2)$ with symmetric space structure from

$$p : SU(2)xSU(2) \rightarrow SU(2)$$

$$p(h, k) = hk^{-1}$$

so we treat S^3 as $\frac{SU(2)xSU(2)}{SU(2)}$ with $SU(2)$ identified with the diagonal in $SU(2)xSU(2)$.

References for derivatives of heat semigroups and kernels

See the bibliography of *Second order Bismut Formulae and Applications to Neumann Semigroups on manifolds* Li-Juan Cheng, Anton Thalmaier, Feng-Yu Wang, Arxiv 2022 but these are mostly for Bismut type formulae.
See also the discussion in Xue-Mei Li.*Hessian formulas and estimates for parabolic Schrodinger operators.* J. Stoch. Anal.,2(3): Art.7, 53, 2021
Earlier work by Norris, Driver, Sheu, Krylov, Stroock & Turetsky.

THAT'S IT!

THANK YOU!