A Quantitative Central Limit Theorem for Simple Symmetric Exclusion Process

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Joint Kyiv-Warwick Stochastic Analysis seminar

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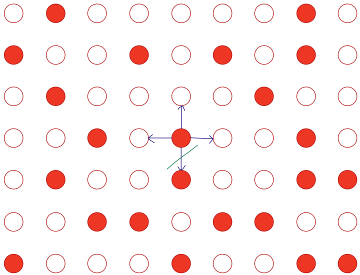
Simple symmetric exclusion process

On the d-dim discrete torus

$$\mathbb{T}_n^d := \left\{ \frac{k}{n} : \; k \in \mathbb{Z}_n^d := \{-m, \ldots, m\}^d \right\} \subset \mathbb{T}^d = (\mathbb{R}/\{\mathbb{Z} - 1/2\})^d$$

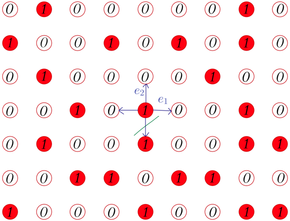
for $n = 2m + 1$ we consider a **Simple Symmetric Exclusion Process** (SSEP)



State space and generator

Particle configuration $\eta \in \{0,1\}^{\mathbb{T}_n^d}$:

$$\eta(x) = 0 \Leftrightarrow \text{side } x \text{ is empty}$$
$$\eta(x) = 1 \Leftrightarrow \text{side } x \text{ is occupied}$$



$$\eta^{x\leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases}$$

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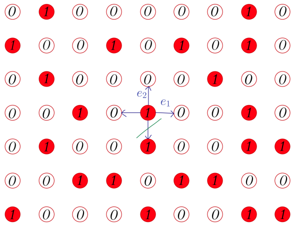

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$$\mathcal{G}_n^{EP} F(\eta) := \frac{n^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x \leftrightarrow x+e_j}) - F(\eta) \right]$$ [Kipnis, Landim '99]

SSEP is already parabolically rescaled: space $\sim \frac{1}{n}$ time $\sim n^2$!

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Equilibrium SSEP:

For fixed $\rho_0 \in [0,1]$, the measure $B(\rho_0)^{\otimes \mathbb{T}_n^d}$ is invariant for SSEP η_t^n:

If $\eta_0^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent then $\eta_t^n(x) \sim B(\rho_0)$, $x \in \mathbb{T}_n^d$, are independent.

$\rightsquigarrow \mathbb{E}\eta_t^n(x) = \rho_0$, $\mathrm{Var}\, \eta_t^n = \rho_0(1-\rho_0)$.

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What is $\mathbb{E}\eta_t^n(x)$?

Mean behavior

Set $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$.

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Then

$$d\rho_t^n(x) = \mathbb{E}\mathcal{G}_n^{EP}\eta_t^n(x)dt = \mathbb{E}\frac{n^2}{2}\sum_{j=1}^{d}\sum_{y\in\mathbb{T}_n}\left[\eta^{y\leftrightarrow y+e_j}(x) - \eta(x)\right] dt$$

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$$= \frac{n^2}{2}\sum_{j=1}^{d}\mathbb{E}\left(\eta_t^n(x+e_j) + \eta_t^n(x-e_j) - 2\eta(x)\right) dt$$

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In particular, the empirical distribution

$$\tilde{\rho}_t^n := \frac{1}{n^d}\sum_{x\in\mathbb{T}_n^d}\rho_t(x)\delta_x,$$

solves

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$$d\langle\varphi, \tilde{\rho}_t^n\rangle = \frac{1}{2}\langle\Delta_n\varphi, \tilde{\rho}_t^n\rangle dt \quad \overset{n\to\infty}{\to} \quad d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt, \quad \rho_0^\infty = \rho_0$$

Law of large numbers

Theorem **[see e.g. in Kipnis, Landim '99]**

Let $\rho_0 : \mathbb{T}^d \to [0,1]$ be an initial density profile and $\eta_0^n(x) \sim B(\rho_0(x))$ be independent. Then

$$\tilde{\eta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \eta_t(x)\delta_x$$

converges in probability to $\rho_t^\infty(x)dx$, where $\rho_t^\infty := P_t^{HE}\rho_0$ solves

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$$= \frac{1}{2} f'(\langle \varphi, \tilde{\eta} \rangle) \langle \Delta_n \varphi, \tilde{\eta} \rangle + \frac{1}{4n^2} f''(\langle \varphi, \tilde{\eta} \rangle) \left\langle \left| \partial_{n,j} \varphi \right|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle + \ldots,$$

where $\tau_j \eta(x) := \eta(x + e_j)$.

Density fluctuation field and CLT

We now consider the fluctuations of the SSEP around its mean:

$$\zeta_t^n(x) := n^{\frac{d}{2}} \left(\eta_t^n(x) - \rho_t^n(x)\right).$$

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can be expanded as follows

$$\mathcal{G}_n^{FF} f\left(\langle \varphi, \tilde{\zeta} \rangle\right) = \frac{1}{2} f'\left(\langle \varphi, \tilde{\zeta} \rangle\right) \langle \Delta_n \varphi, \tilde{\zeta} \rangle + \frac{n^d}{4n^d} f''\left(\langle \varphi, \tilde{\zeta} \rangle\right) \left\langle \left|\partial_{n,j}\varphi\right|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle$$
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$$d\langle \varphi, \tilde{\zeta}_t^n \rangle = \frac{1}{2} \langle \Delta_n \varphi, \tilde{\zeta}_t^n \rangle dt + \text{mart.}$$
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$$d\langle \varphi, \tilde{\zeta}_t^n \rangle = \frac{1}{2} \langle \Delta_n \varphi, \tilde{\zeta}_t^n \rangle dt + \text{mart.} \to d\langle \varphi, \zeta_t^\infty \rangle = \frac{1}{2} \langle \Delta \varphi, \zeta_t^\infty \rangle dt + \text{mart.}$$

$$d\langle \text{mart.} \rangle_t = \frac{1}{2} \left\langle |\partial_{n,j}\varphi|^2, \tau_j \tilde{\eta}_t^n + \tilde{\eta}_t^n - 2\widetilde{\eta_t^n \tau_j \eta_t^n} \right\rangle dt \to d\langle \text{mart.} \rangle = \langle \Delta \varphi, \rho_t^\infty - \rho_t^\infty \rho_t^\infty \rangle dt$$

Central limit theorem

Theorem 2 [Galves, Kipnis, Spohn; Ravishankar '90]

Let the initial density profile ρ_0 be smooth. Then the density fluctuation field

$$\tilde{\zeta}_t^n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta_t(x) \delta_x$$

converges in $D\left([0, T], \mathcal{D}'\right)$ to the generalized Ornstein-Uhlenbeck process that solves the linear SPDE

$$d\zeta_t^\infty = \frac{1}{2} \Delta \zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right)$$

with the centered Gaussian initial condition such that

$$\mathbb{E}\left[\langle \zeta_0^\infty, \varphi \rangle^2 \right] = \langle \rho_0 (1 - \rho_0) \varphi, \varphi \rangle$$

Idea of proof

- Consider the semimartingale $\langle \varphi, \tilde{\zeta}_t \rangle$ and its quadratic variation.

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$$\eta_t^n \tau_j \eta_t^n = \rho_t^n \tau_j \rho_t^n + \frac{1}{n^{d/2}} \left(\rho_t^n \tau_j \zeta_t^n + \zeta_t^n \tau_j \rho_t^n \right) + \frac{1}{n^d} \zeta_t^n \tau_j \zeta_t^n,$$

where $\eta_t^n = \rho_t^n + \frac{1}{(2n+1)^{d/2}} \zeta_t^n$

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- Control of $\mathbb{E}\left[\frac{1}{n^d} \zeta_t^n(x) \tau_j \zeta_t^n(x) \right] = \mathbb{E}\left[\left(\eta_t^n(x) - \rho_t^n(x) \right) \left(\eta_t^n(x+e_j) - \rho_t^n(x+e_j) \right) \right] \to 0$
  [Ferrari et al. '91]

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- Use of tightness argument and the Holley-Stroock theory [Holley, Stroock '79]

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$$\tilde{\zeta}_t^n - \zeta_t^\infty = O(?)$$

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- [Kolokoltsov '10] Central limit theorem for the Smoluchovski coagulation model (mean field limit, non-local Smoluchowski's coagulation equation)
- ...

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = n^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$ with the centered Gaussian initial condition with $\mathbb{E}\left[\langle\zeta_0^\infty, \varphi\rangle^2\right] = \langle\rho_0(1-\rho_0)\varphi, \varphi\rangle$

Then for large enough $l \in \mathbb{N}$

$$\sup_{t\in[0,T]} \left|\mathbb{E}f\left(\langle\vec{\varphi}, \tilde{\zeta}_t^n\rangle\right) - \mathbb{E}f\left(\langle\vec{\varphi}, \zeta_t^\infty\rangle\right)\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}_l^3} \|\vec{\varphi}\|_{\mathrm{C}^l}$$

for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$.

Main result

Theorem 3 [Gess, K. '24]

Let

- the initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ be smooth enough,
- η_t^n be SSEP with $\eta_0^n(x) \sim B(\rho_0(x))$ and independent,
- $\rho_t^n = \mathbb{E}\eta_t^n$, $\zeta_t^n = n^{d/2}(\eta_t^n - \rho_t^n)$
- ζ_t^∞ solves $d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$ with the centered Gaussian initial condition with $\mathbb{E}\left[\langle \zeta_0^\infty, \varphi\rangle^2\right] = \langle \rho_0(1-\rho_0)\varphi, \varphi\rangle$

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for all $n \geq 1$, $f \in \mathrm{C}_b^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^l(\mathbb{T}^d)\right)^m$.

The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ is optimal: $\frac{1}{n}$ – lattice discretization error, $\frac{1}{n^{\frac{d}{2}}}$ – particle approximation error

Main tool

Idea of proof: Compare two (time-homogeneous) Markov processes X_t, Y_t taking values in the same state space and $X_0 = Y_0 = x$ using

$$\begin{aligned}\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) &= \int_0^t P_s^X \left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(x) ds,\\ &= \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P_{t-s}^Y F(X_s)\right] ds,\end{aligned}$$

[see e.g. Ethier, Kurtz '86]

Splitting of the problem

Recall

$$\mathbb{E}F(X_t) - \mathbb{E}F(Y_t) = \int_0^t \mathbb{E}\left[\left(\mathcal{G}^X - \mathcal{G}^Y\right) P^Y_{t-s} F(X_s)\right] ds,$$

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We consider the Markov processes:

- particle means and fluctuation field: $(\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$
- solution to heat equation and generalized OU process $(\rho^\infty_t, \zeta^\infty_t)$.

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The processes starts from different initial conditions!

We will compare:

- $(\tilde{\rho}^n_t, \tilde{\zeta}^n_t)$ an $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ **[comparison of dynamics]**
  where the generalized OU process $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ started from $(\tilde{\rho}^n_0, \tilde{\zeta}^n_0)$;
- $(\rho^{\infty,n}_t, \zeta^{\infty,n}_t)$ and $(\rho^\infty_t, \zeta^\infty_t)$ **[comparison of initial conditions]**
  (both are defined by the same equation).

Generators

We start from the formal computation for cylindrical functions:

$$F(\tilde{\rho}, \tilde{\zeta}) := f\left(\langle \psi, \tilde{\rho} \rangle, \langle \varphi, \tilde{\zeta} \rangle\right)$$

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$$\begin{aligned} \mathcal{G}_n^{FF} F(\tilde{\rho}, \tilde{\zeta}) = &\frac{1}{2}\partial_1 f \langle \Delta_n \varphi, \tilde{\rho} \rangle + \frac{1}{2}\partial_2 f \langle \Delta_n \varphi, \tilde{\zeta} \rangle \\ &+ \frac{1}{4}\partial_2^2 f \left\langle \left|\partial_{n,j}\varphi\right|^2, \tau_j \tilde{\eta} + \tilde{\eta} - 2\widetilde{\eta \tau_j \eta} \right\rangle + O\left(1/n^{\frac{d}{2}+1}\right), \end{aligned}$$

where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$.

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\end{aligned}$$

where $\tilde{\zeta} = n^{d/2}(\tilde{\eta} - \tilde{\rho})$. Moreover,

$$\begin{aligned}
\mathcal{G}^{OU} F(\rho^\infty, \zeta^\infty) &= \frac{1}{2}\partial_1 f \langle \Delta\varphi, \rho^\infty \rangle + \frac{1}{2}\partial_2 f \langle \Delta\varphi, \zeta^\infty \rangle \\
&\quad + \frac{1}{2}\partial_2^2 f \left\langle |\partial_j \varphi|^2, \rho^\infty - (\rho^\infty)^2 \right\rangle.
\end{aligned}$$

The difficulties

- Generators have to be compared on $U := P^{OU}_{t-s} f\left(\langle\psi,\cdot\rangle,\langle\varphi,\cdot\rangle\right)$, that is not cylindrical function.
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- We compare the generators on particle configurations $\tilde{\rho}^n_t$, $\tilde{\zeta}^n_t$, that are empirical distributions. But U is not differentiable at $\tilde{\rho}^n_t$ because of the term $\sqrt{\rho(1-\rho)}$ in the SPDE
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  $\rightsquigarrow$ Interpolation has to be smooth enough
- Control of $\mathbb{E}\left[f(\langle\varphi,\tilde{\zeta}^n_t\rangle) \left\langle \widetilde{\zeta^n_t\tau\zeta^n_t}, |\partial_{n,j}\varphi|^2 \right\rangle \right]$ via $\mathbb{E}\prod_{i=1}^m \left(\eta^n_t(x_i) - \rho^n_t(x_i)\right)$ does not give the optimal rate.

Discrete and continuous Fourier transform

Replace $\tilde{\rho} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho(x)\delta_x$ and $\tilde{\zeta} = \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \zeta(x)\delta_x$ by a smooth interpolation.

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- Let $L_2(\mathbb{T}_n^d)$ be the Hilbert space of all functions on $\mathbb{T}_n^d$ with inner product

$$\langle \rho_1, \rho_2 \rangle_n := \frac{1}{n^d} \sum_{x \in \mathbb{T}_n^d} \rho_1(x)\rho_2(x)$$

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- $\varsigma_k(x) = e^{2\pi i k \cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{T}^d \supset \mathbb{T}_n^d$
  – basis vectors on $L_2(\mathbb{T}_n^d)$ and $L_2(\mathbb{T}^d)$, and
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$$L_2(\mathbb{T}_n^d) \ni \rho = \sum_{k \in \mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \text{ on } \mathbb{T}_n^d, \quad L_2(\mathbb{T}^d) \ni g = \sum_{k \in \mathbb{Z}^d} \langle g, \varsigma_k \rangle \varsigma_k \text{ on } \mathbb{T}^d$$

New (smooth) lifting of discrete space

For functions $\rho \in L_2(\mathbb{T}_n^d)$ and $\varphi \in L_2(\mathbb{T}^d)$ define

$$\mathrm{ex}_n\rho := \sum_{k\in\mathbb{Z}_n^d} \langle \rho, \varsigma_k \rangle_n \varsigma_k \quad \text{on } \mathbb{T}^d, \quad \mathrm{pr}_n\varphi := \sum_{k\in\mathbb{Z}_n^d} \langle \varphi, \varsigma_k \rangle \varsigma_k \quad \text{on } \mathbb{T}^d$$

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Basic properties of $\mathrm{ex}_n f$ and $\mathrm{pr}_n g$

- $\mathrm{ex}_n\rho = \rho$ on $\mathbb{T}_n^d$ and $\mathrm{ex}_n\rho \in \mathrm{C}^\infty(\mathbb{T}^d)$
- $\mathrm{pr}_n\varphi$ is well defined on $\mathbb{T}_n^d$ for each $\varphi \in H_J$ for $H_J := \left\{\varphi : \ \|\varphi\|_{H_J}^2 := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^J \left|\langle\varphi,\varsigma_k\rangle\right|^2\right\}$, $J \in \mathbb{R}$.
- $\langle \rho_1, \rho_2\rangle_n = \langle \mathrm{ex}_n\rho_1, \mathrm{ex}_n\rho_2\rangle$ and $\langle \rho, \mathrm{pr}_n g\rangle_n = \langle \mathrm{ex}_n\rho, g\rangle$
- $\|\mathrm{pr}_n g - g\|_{H_J} \le \frac{1}{n}\|g\|_{H_{J+1}}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \le \frac{C}{n}\|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$

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- $\|\mathrm{pr}_n g - g\|_{H_J} \le \frac{1}{n}\|g\|_{H_{J+1}}$, $\|\mathrm{ex}_n\varphi - \varphi\|_{H_J} \le \frac{C}{n}\|\varphi\|_{\mathrm{C}^{J+2+\frac{d}{2}}}, \dots$

$$\langle \varphi, \tilde{\rho}\rangle = \frac{1}{(2n+1)^d}\sum_{x\in\mathbb{T}_n^d}\varphi(x)\rho(x) = \langle \varphi, \rho\rangle_n$$

$$= \langle \mathrm{pr}_n\varphi, \rho\rangle_n + O(1/n) = \langle \varphi, \mathrm{ex}_n\rho\rangle + O(1/n)$$

Comparison of generators for smooth interpolation

Now for $F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) := f\left(\langle\psi, \mathrm{ex}_n\rho\rangle, \langle\varphi, \mathrm{ex}_n\zeta\rangle\right) = f\left(\langle \mathrm{pr}_n\psi, \rho\rangle_n, \langle \mathrm{pr}_n\varphi, \zeta\rangle_n\right)$,

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$$\mathcal{G}_n^{FF} F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) = \frac{1}{2}\partial_1 f \langle \Delta_n \mathrm{pr}_n\psi, \rho\rangle_n + \frac{1}{2}\partial_2 f \langle \Delta_n \mathrm{pr}_n\varphi, \zeta\rangle_n$$
$$+ \frac{1}{2}\partial_2^2 f \sum_{j=1}^{d} \left\langle \left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2, \tau_j\eta + \eta - 2\eta\tau_j\eta \right\rangle_n + O\left(\frac{1}{n^{d/2+1}}\right)$$

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$$\left|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F\right| \lesssim \frac{1}{n^{d/2}} \left|\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle\right| + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \mathrm{ex}_n\partial_2^2 f\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + \ldots$$

We now need only to control:

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Now for $F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) := f\left(\langle\psi, \mathrm{ex}_n\rho\rangle, \langle\varphi, \mathrm{ex}_n\zeta\rangle\right) = f\left(\langle \mathrm{pr}_n\psi, \rho\rangle_n, \langle \mathrm{pr}_n\varphi, \zeta\rangle_n\right)$, we get

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$$\left|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F\right| \lesssim \frac{1}{n^{d/2}} \left|\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle\right| + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \mathrm{ex}_n\partial_2^2 f\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + \ldots$$

We now need only to control:

- The expectations:

$$\mathbb{E}\left|\left\langle \mathrm{ex}_n\zeta_t^n, \mathrm{ex}_n\left(\tau_j\rho_t^n\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle\right| \lesssim \|\mathrm{ex}_n\rho_t^n\|_{\mathrm{C}^J}\,\mathbb{E}\|\mathrm{ex}_n\zeta_t^n\|_{H_{-l}}$$

Comparison of generators for smooth interpolation

Now for $F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) := f\left(\langle \psi, \mathrm{ex}_n\rho\rangle, \langle \varphi, \mathrm{ex}_n\zeta\rangle\right) = f\left(\langle \mathrm{pr}_n\psi, \rho\rangle_n, \langle \mathrm{pr}_n\varphi, \zeta\rangle_n\right)$, we get

$$\mathcal{G}_n^{FF} F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta) = \frac{1}{2}\partial_1 f \langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\psi, \mathrm{ex}_n\rho\rangle + \frac{1}{2}\partial_2 f \langle \mathrm{ex}_n\Delta_n \mathrm{pr}_n\varphi, \mathrm{ex}_n\zeta\rangle$$
$$+ \frac{1}{2}\partial_2^2 f \sum_{j=1}^{d} \left\langle \mathrm{ex}_n\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2, \mathrm{ex}_n\left(\tau_j\eta + \eta - 2\eta\tau_j\eta\right)\right\rangle + O\left(\frac{1}{n^{d/2+1}}\right)$$

$$\left|\mathcal{G}_n^{FF} F - \mathcal{G}_n^{OU} F\right| \lesssim \frac{1}{n^{d/2}} \left|\left\langle \mathrm{ex}_n\zeta, \mathrm{ex}_n\left(\tau_j\rho\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle\right| + \left\langle \mathrm{ex}_n\left(\zeta\tau_j\zeta\right), \mathrm{ex}_n\partial_2^2 f\left(\partial_{n,j}\mathrm{pr}_n\varphi\right)^2\right\rangle^2 + \dots$$

We now need only to control:

- The expectations:

$$\mathbb{E}\left|\left\langle \mathrm{ex}_n\zeta_t^n, \mathrm{ex}_n\left(\tau_j\rho_t^n\left|\partial_{n,j}\varphi\right|^2\right)\right\rangle\right| \lesssim \|\mathrm{ex}_n\rho_t^n\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta_t^n\|_{H_{-l}}$$

- Using the Fourier analysis, the term $\mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_t^n\tau_j\zeta_t^n\right), \mathrm{ex}_n\partial_2^2 f(\dots)^2\right\rangle$ can be controlled via

$$\mathbb{E}\prod_{i=1}^{4}\left(\eta_t^n(x_i) - \rho_t^n(x_i)\right) \lesssim \frac{1}{n}$$

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- We can compare generators on Frechet diff. functions on $H_J \times H_{-l}$

Differentiability of $P_t^{OU}F(\mathrm{ex}_n\rho, \mathrm{ex}_n\zeta)$

A solution to

$$d\rho_t^\infty = \frac{1}{2}\Delta\rho_t^\infty dt$$
$$d\zeta_t^\infty = \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right)$$

exists for all $\rho_0^\infty \in L_2(\mathbb{T}^d; [0,1])$ and $\zeta_0^\infty \in H_{-l}$ for $l > \frac{d}{2} + 1$.

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For $F \in \mathrm{C}(H_J \times H_{-l})$ (e.g. $F = f(\langle \psi, \cdot \rangle, \langle \varphi, \cdot \rangle)$) define $U_t(\rho_0^\infty, \zeta_0^\infty) := \mathbb{E}F\left(\rho_t^\infty, \zeta_t^\infty\right)$

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For $F \in \mathrm{C}(H_J \times H_{-I})$ (e.g. $F = f(\langle\psi,\cdot\rangle,\langle\varphi,\cdot\rangle)$) define $U_t(\rho_0^\infty,\zeta_0^\infty) := \mathbb{E}F\left(\rho_t^\infty,\zeta_t^\infty\right)$

Proposition 1 [Gess, K. '24]

Let $I > \frac{d}{2}+1$ and $F \in \mathrm{C}_b^{2,4}(H_{-I})$. Then $U_t(\rho_0^\infty,\zeta_0^\infty) = \mathbb{E}F\left(\zeta_t^\infty\right) \in \mathrm{C}_b^{1,3}(H_J \times H_{-I})$ for $J > \frac{d}{2}$. Moreover,

$$D_1 U_t(\rho_0^\infty,\zeta_0^\infty)[h] = \frac{1}{2}\mathbb{E}\left[D^2F(\zeta_t^\infty) : DV_t(\rho_0^\infty)\left[h\right]\right]$$

with

$$V_t(\rho_0^\infty)(\varphi,\psi) = \mathrm{Cov}\left(\langle\varphi,\zeta_t^\infty\rangle,\langle\psi,\zeta_t^\infty\rangle\right)$$
$$= \frac{1}{2}\int_0^t \left\langle \nabla P_{t-s}^{HE}\varphi \cdot \nabla P_{t-s}^{HE}\psi, \rho_s^\infty\left(1-\rho_s^\infty\right)\right\rangle ds$$

Comparison of dynamics

Recall that (ρ_t^n, ζ_t^n) is the mean process together with the fluctuation field of SSEP.

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$(\rho_t^{\infty,n}, \zeta_t^{\infty,n})$ is a solution to

$$\begin{aligned} d\rho_t^\infty &= \frac{1}{2}\Delta\rho_t^\infty dt \\ d\zeta_t^\infty &= \frac{1}{2}\Delta\zeta_t^\infty dt + \nabla \cdot \left(\sqrt{\rho_t^\infty(1-\rho_t^\infty)}dW_t\right) \end{aligned}$$

started from $(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n)$.

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started from $(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n)$.

Then for each I, J large enough and $F \in \mathrm{C}_b(H_J \times H_{-I})$

$$|\mathbb{E}F(\mathrm{ex}_n\rho_t^n, \mathrm{ex}_n\zeta_t^n) - \mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n})| \leq \int_0^t \left|\mathbb{E}\left[\left(\mathcal{G}^{FF} - \mathcal{G}^{OU}\right) P_{t-s}^{OU} F(\mathrm{ex}_n\rho_t^n, \mathrm{ex}_n\zeta_t^n)\right]\right| ds$$

$$\lesssim \|F\|_{\mathrm{C}^{1,3}} \int_0^t \left(\frac{1}{n} + \frac{1}{n^{d/2}}\|\mathrm{ex}_n\rho_s^n\|_{\mathrm{C}^J}\mathbb{E}\|\mathrm{ex}_n\zeta_s^n\|_{H_{-I}} + \mathbb{E}\left\langle \mathrm{ex}_n\left(\zeta_s^n\tau_j\zeta_s^n\right), \dots\right\rangle^2 + \dots\right) ds$$

Berry-Esseen bound for the initial fluctuations

- It remains only to compare

$$\mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^{\infty}, \zeta_t^{\infty}) = P_t^{OU}F(\mathrm{ex}_n\rho_0^n, \mathrm{ex}_n\zeta_0^n) - P_t^{OU}F(\rho_0, \zeta_0)$$

  where ρ_t^∞ started from the initial profile ρ_0 and ζ_t started from the centered Gaussian distribution with

$$\mathbb{E}\langle\zeta_0, \varphi\rangle^2 = \langle\rho_0(1-\rho_0)\varphi, \varphi\rangle.$$

- It is enough to compare only

$$\mathbb{E}F(\mathrm{ex}_n\zeta_0^n) - \mathbb{E}F(\mathrm{pr}_n\zeta_0),$$

  for $F \in \mathrm{C}^3(H_{-l})$, where

$$\mathrm{ex}_n\zeta_0^n := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0^n, \varsigma_k\rangle_n\varsigma_k, \quad \mathrm{pr}_n\zeta_0 := \sum_{k\in\mathbb{Z}_n^d}\langle\zeta_0, \varsigma_k\rangle\varsigma_k$$

- Is enough to compare for $f \in \mathrm{C}^3\left(\mathbb{R}^{\mathbb{Z}_n^d}\right)$

$$\mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0^n, \varsigma_k\rangle_n\right)_{k\in\mathbb{Z}_n^d}\right) - \mathbb{E}f\left(\left((1+|k|^2)^{-l/2}\langle\zeta_0, \varsigma_k\rangle\right)_{k\in\mathbb{Z}_n^d}\right).$$

- Apply multidimensional Berry-Essen theorem [Meckes '09]

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Thank you!