On approximations of finite-dimensional point densities for Arratia flows with drift

M. Vovchanskyi

Institute of Mathematics, NAS of Ukraine

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Definition

A Brownian web is a collection of random processes

$$\{X_{t,\cdot}(u) \in C([t;+\infty)) \mid u \in \mathbb{R}, t \in \mathbb{R}_+\}$$

such that for any $(u_1, t_1), \ldots, (u_N, t_N) \in \mathbb{R} \times \mathbb{R}_+$ the processes

$$X_{t_1,\cdot}(u_1), \ldots, X_{t_N,\cdot}(u_N)$$

are coalescing Brownian motions w.r.t. the joint filtration.

Definition

The Brownian web X with drift a is a family of $D(\mathbb{R},\mathbb{R})-$valued random elements $\{X_{s,t}(\cdot) \mid s \le t\}$ such that

1. for any $s \le t \le r$ $\mathsf{P}\{X_{s,r} = X_{t,r} \circ X_{s,t}\} = 1$; $X_{s,s} = \mathrm{Id}$ a.s.;
2. for any $t_1 \le t_2 \le \ldots \le t_n$ $X_{t_1,t_2}, \ldots, X_{t_{n-1},t_n}$ are independent;
3. for any $s, t \in \mathbb{R}, h > 0$ $Law\,(X_{s,t}) = Law\,(X_{s+h,t+h})$;
4. as $h \to 0+$, $X_{0,h} \to Id$ in probability in $D(\mathbb{R},\mathbb{R})$;
5. for any $x \in \mathbb{R}, s \ge 0$

   $$X_{s,t}(x) = x + \int_s^t a(X_{s,r}(x))dr + w_{x,s}(t), \quad t \ge s,$$

   where $w_{x,s}$ is a Brownian motion started at 0;
6. for any $x, y \in \mathbb{R}, s \ge 0$

   $$\langle w_{x,s}, w_{y,s}\rangle(t) = \int_s^t 1\!\!\mathrm{I}\,[X_{s,r}(x) = X_{s,r}(y)]\,dr, \quad t \ge s.$$

Brownian web (Arratia flows) (with drift)

Arratia flow: $\{X_{0,s}(x) \mid s \geq 0, x \in \mathbb{R}\}$
Particles move independently before they meet and merge afterwards.

- a limit of random walks: Arratia;
- reflecting Wiener processes: Soucaliuc, Tóth, Werner;
- a limit of homeomorphic stochastic flows: Dorogovtsev;
- **adding drift: Dorogovtsev;**
- a flow of kernels: Le Jan, Raimond;
- a random element in a specific space of compact sets of the space of trajectories: Fontes, Isopi, Newman, Ravishankar...;
- physical coalescing and annihilating systems of particles: Masser, ben-Avraham, Tribe, Zaboronski ... ;
- a universal limit object: Roy, Saha, Sarkar, Birkner, Gantert, Steiber, Norris, Turner...;
- related models: Konarovskyi, von Renesse

Contents:

1. What point densities are
2. Cornerstones
3. Representations
4. Some multidimensional results
5. Approximations

References:

- A. A. Dorogovtsev and N. B. Vovchanskii, "Representations of the finite-dimensional point densities in Arratia flows with drift", *Theory Stoch. Process.*, vol. 25, no. 1, pp. 25–36, 2020
- A. A. Dorogovtsev and M. B. Vovchanskii, "On the approximations of point measures associated with the Brownian web by means of the fractional step method and discretization of the initial interval", *Ukrain. Math. J.*, vol. 72, pp. 1358–1376, 2021
- A. A. Dorogovtsev та M. B. Vovchanskyi, "On 1-point densities for Arratia flows with drift," *Stochastics*, т. 95, № 8, с. 1429—1445, 2023
- preprint

Correlation functions ($n-$point densities)

- The Arratia flow $X^a = \{X_t^a(u) | u \in \mathbb{R}, t \in [0; T]\}$ with drift $a \in L_\infty(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$
- A random locally finite measure

$$\mu_t^a(\Delta) = \left| X_t(\mathbb{R}) \cap \Delta \right|, \ \Delta \in \mathcal{B}(\mathbb{R}),$$

- Different approaches
  - (physical) diffusion particles with instant interaction
  - theory of random matrices
  - as a stochastic dynamic system (cocycle)

$$X_t^a = \{X_t^a(v) \mid v \in [0;1]\};$$

Definition (correlation functions: F.J.Dyson 1962)

The k-point density p_t^k is a function on $\mathbb{R}^k$ such that for any bounded $f : \mathbb{R}^k \to \mathbb{R}$

$$\mathrm{E}\,\mathbb{1}\left(\mid X_t^a \mid \geq k\right) \sum_{\substack{v_1,\dots,v_k \in X_t^a, \\ v_1,\dots,v_k \text{ all distinct}}} f(v_1,\dots,v_k) = \int_{\mathbb{R}^k} p_t^{a,k}(y) f(y) dy.$$

An alternative definition

$$p_t^n = \lim_{\delta \to 0+} \delta^{-k} \prod_{k=\overline{1,n}} N_t([x_k; x_k + \delta))$$
$$= \lim_{\delta \to 0+} \delta^{-k} \mathsf{P}\left(N_t([x_k; x_k + \delta)) > 0, k = \overline{1,n}\right),$$

where $N_t(A)$ is the number of particles in the set A

The original proof utilizes the idea from (Munasinghe, Rajesh, Tribe та O. Zaboronski 2006) and the Girsanov theorem for the Arratia flow. A different constructive proof can be given.

In: R. Tribe and O. V. Zaboronski, “Pfaffian formulae for one dimensional coalescing and annihilating systems”, *Electron. J. Probab.*, vol. 16, no. 76, pp. 2080–2103, 2011

Theorem

The point process for X^0 *at time* t *is the Pfaffian point process* M *with kernel* $t^{-1/2}K(t^{-1/2}u, t^{-1/2}v)$*, that is, for all* $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}), A_i \cap A_j = \emptyset, i \neq j$*, and numbers* $k_1, \ldots, k_m \in \mathbb{N} \cup \{0\}\colon \sum k_i = k$ *we have*

$$\mathrm{E} \prod_{j=1}^{m} M_t(A_j) \cdot \ldots \cdot (M_t(A_j) + 1 - k_j) = \int_{A_1^{k_1} \times \ldots \times A_m^{k_m}} p_t^k(x_1, \ldots, x_k) dx_1 \ldots dx_k,$$

where the k-point density $p_t^k(x_1, \ldots, x_k)$ *is the Pfaffian of the* $2k \times 2k$ *matrix built of* k^2 *blocks, ecch block being given in the terms of Gaussian density and its first 2 derivatives.*

The Karlin-McGregor determinant:

$$g_t^{KM}(y,x) = \det \|g_t(y_j - x_k)\|_{j,k},$$

In: R. Munasinghe, R. Rajesh, R. Tribe, and O. Zaboronski, "Multi-scaling of the n-point density function for coalescing Brownian motions", *Comm. Math. Phys.*, vol. 268, no. 3, pp. 717–725, 2006

Theorem

Since

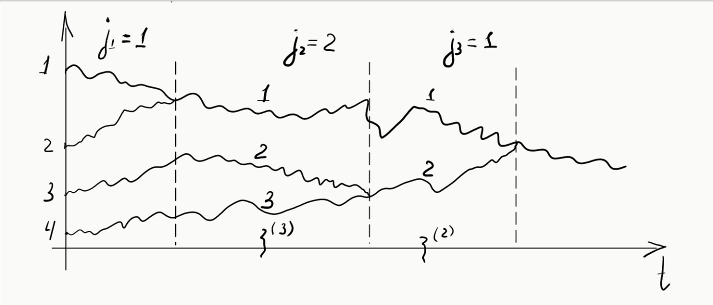
$$c_n \prod_k g_t(x_k, y_{n+1-k}) \leq \frac{g_t^{KM}(y,x)}{h_n(t^{-1/2}x)h_n(t^{-1/2}y)} \leq \prod_k g_t(x_k, y_k),$$

where

$$h_n(u) = \prod_{k<j}(u_j - u_k),$$

we have

$$p_t^n(x) \leq \frac{1}{(\pi t)^{n/2}}.$$



- $\xi = (\xi_1, \ldots, \xi_n)$ is a continuous process process with coalescence and no triple collisions
- $\varkappa = n - |\{\xi_j(T) \mid j = \overline{1, n}\}|$
- Hitting times: $\tau_1 < \tau_2 < \ldots < \tau_\varkappa$
- A coalescence scheme $S(\xi) = (j_1, \ldots, j_\varkappa)$ ($S(\xi) = \emptyset$ if $\varkappa = 0$)
- $\xi^{(n-1)}$ is obtained by removing the j_1-th coordinate and so on

$$j_1 = \min\{i \mid \exists j \neq i \ \xi_j(\tau_1) = \xi_i(\tau_1)\},$$

$$j_2 = \min\{i \mid \exists \ j \neq i \ \xi_j^{(n-1)}(\tau_2) = \xi_i^{(n-1)}(\tau_2)\}, \ldots$$

$$\ldots$$

$$X_t^a(u) = \{X_t^a(u_k) \mid k = \overline{1,n}\},$$
$$\Delta_n = \{u \in \mathbb{R}^n \mid u_1 < \ldots < u_n\}$$

Definition

The $(n,k)-$point density corresponding to $u \in \Delta_n$ and $k \in \{1,\ldots,n\}$, is a function $p_t^{a,n,k}(u;\cdot)$ on $\mathbb{R}^k$ such that for any bounded $f\colon \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{E}\,1\!\!1\,(|X_t^a(u)| \geq k) \sum_{\substack{v_1,\ldots,v_k \in X_t^a(u), \\ v_1,\ldots,v_k \text{ all distinct}}} f(v_1,\ldots,v_k) = \int_{\mathbb{R}^k} p_t^{a,n,k}(u;y) f(y)dy.$$

Definition

The $(n,k)-$point density corresponding to $u \in \Delta_n$, a coalescence scheme s with $\varkappa = j$ and $k \leq n-j$, is a function $p_t^{a,n,s,k}(u;\cdot)$ on $\mathbb{R}^k$ such that for any non-negative $f\colon \mathbb{R}^k \to \mathbb{R}$

$$\mathrm{E}\,1\!\!1(S(X^a(u)) = s) \sum_{\substack{v_1,\ldots,v_k \in X_t^a(u), \\ v_1,\ldots,v_k \text{ all distinct}}} f(v_1,\ldots,v_k) = \int_{\mathbb{R}^k} p_t^{a,n,s,k}(u;y) f(y)dy.$$

Lemma

1. *For any* s *with* $\varkappa = k$, $u \in \Delta_n$ *and* $j \leq n - k$ *the density* $p_t^{a,n,s,j}(u;\cdot)$ *exists.*
2. *For any* $n \in \mathbb{N}, u \in \Delta_n$ *i* $k \in \{1, \ldots, n\}$ *the density* $p_t^{a,n,k}(u;\cdot)$ *exists, and a.e.*

$$p_t^{a,n,k}(u;\cdot) = \sum_{l=0}^{n-k} \sum_{s \in \mathcal{S}_{n,l}} p_t^{a,n,s,k}(u;\cdot),$$

where $\mathcal{S}_{n,l}$ *is a set of all coalescence schemes for* n *particles with* l *collisions.*

The Girsanov theorem for the Arratia flow

- A set $U = \{u_k \mid k \in \mathbb{N}\}$ is dense in $[0;1]$; $u^{(n)} = (u_1, \ldots, u_n), n \in \mathbb{N}$
- Define collision times

$$\tau_1 = T,$$
$$\tau_k = \inf \left\{ T; s \mid \prod_{j=1}^{k-1} (X_s(u_k) - X_s(u_j)) = 0 \right\}, \quad k \geq 2$$

- Define

$$I_n\left(u^{(n)}\right) = \sum_{k=1}^{n} \int_0^{\tau_k} a(X_t(u_k)) dX_t(u_k),$$
$$J_n\left(u^{(n)}\right) = \sum_{k=1}^{n} \int_0^{\tau_k} a^2(X_t(u_k)) dt, \quad n \in \mathbb{N}$$

Theorem (Dorogovtsev 2007)

There exist

$$I = L_2\text{-}\lim_{n\to\infty} I_n\left(u^{(u)}\right),$$
$$J = L_2\text{-}\lim_{n\to\infty} J_n\left(u^{(u)}\right)$$

Theorem (Dorogovtsev 2007)

1. *Let* $n \in \mathbb{N}$. *For all* $u \in \mathbb{R}^n$ *the distribution of* $X^a(u,\cdot)$ *is absolutely continuous w.r.t the distribution of* $X(u,\cdot)$ *in* $C([0;T],\mathbb{R}^n)$ *with density*

$$\widetilde{\mathcal{E}}^a_{T,n}(u) = \exp\left\{I_n(u) - \frac{1}{2}J_n(u)\right\}.$$

2. *The distribution of* X^a *as a random element in* $D([0;1], C([0;T]))$ *is absolutely continuous w.r.t the distribution of* X *with density*

$$\widetilde{\mathcal{E}}^a_T = \exp\left\{I - \frac{1}{2}J\right\}.$$

- w is a standard BM in $\mathbb{R}^n$, $\theta = \inf\{r \mid w(r) \notin \Delta_n\}$
- The Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial r} F(z,r) &= -\frac{1}{2}\Delta_z F(z,r) \text{ in } \Delta_n \times (s;t), \\ F(z,t) &= 0,\ z \in \overline{\Delta}_n, \\ F(z,r) &= \varphi(z),\ z \in \partial\Delta_n,\ r \in (s;t), \end{aligned}$$

  has a solution

$$F(z,r) = \mathrm{E}_{r,z}\, \varphi(w(\theta)) 1\!\mathrm{I}(\theta > t)$$

- $\partial\Delta_{n,j} = \{(u_1, \ldots, u_n) \mid u_1 < \ldots < u_j = u_{j+1} < \ldots < u_n\},\ j = \overline{1, n-1}$
- m is a surface measure on $\bigcup_{j=1}^{n-1} \partial\Delta_{n,j}$
- $\frac{\partial}{\partial \nu_y}$ is an outward normal derivartive
- $\rho_t^{a,m}(u;\cdot)$ is the density of the BM with drift $(a, \ldots, a)$ killed on $\partial\Delta_m$

Theorem (Dorogovtsev, V., 2020)

For all $n \in \mathbb{N}, x \in \Delta_n, t \in [0;T], k \in \{1, \ldots, n\}$, *any coalescence scheme* s *with* $\varkappa = n - k$, *and any* $j \in \{1, \ldots, k\}$

$$
\begin{aligned}
p_t^{a,n,s,j}(x;y) = (-1)^k 2^{-k} \int_{0 \le t_1 \le \ldots \le t_{n-k} \le t} & dt_1 \ldots dt_{n-k} \\
& \int_{\partial \Delta_{n,j_1}} m(dz_1) \int_{\partial \Delta_{n,j_2}} m(dz_2) \ldots \ldots \int_{\partial \Delta_{k+1, j_{n-k}}} m(dz_{n-k}) \\
& \times \frac{\partial}{\partial \nu_{z_1}} \rho_{t_1}^{a,n}(x, z_1) \times \frac{\partial}{\partial \nu_{z_2}} \rho_{t_2 - t_1}^{a,n-1}\left(R_{j_1}^n z_1, z_2\right) \times \ldots \times \\
& \times \frac{\partial}{\partial \nu_{z_{n-k}}} \rho_{t_{n-k} - t_{n-k-1}}^{a,k+1}\left(R_{j_{n-k-1}}^{k+2} z_{n-k-1}, z_{n-k}\right) \times \\
& \times \sum_{\substack{L = \{l_1, \ldots, l_j\} \subset \\ \{1, \ldots, k\}}} \int_{\mathbb{R}^{k-j}} dv^{-L} \rho_{t - t_{n-k}}^{a,k}\left(R_{j_{n-k}}^{k+1} z_{n-k}, v\right) \Big|_{\substack{v \in \mathbb{R}^k, \\ v^L = y}},
\end{aligned}
$$

where $R_j^m \colon \partial \Delta_{m,j} \to \Delta_{m-1}$ *removes the* $j+1$*–th coordinate, and* $x^L = (x_i), i \in L$, $x^{-L} = (x_i), i \notin L$

Construction of finite systems in Arratia flows

- $W = (w_1, \ldots, w_n)$ is a standard BM in $\mathbb{R}^n$, $W(0) = u$.
- $\widetilde{W}$ is obtained from W by merging coordinates after a collision
- $\{\widetilde{\theta}_k\}$ are the corresponding meeting times
- Define

$$\mathcal{E}^a_{T,n}(W,u) = \exp\Big\{ \sum_{k=1}^{n} \int_0^{\widetilde{\theta}_k} a(w_k(t))dw_k(t) - \\ - \frac{1}{2}\sum_{k=1}^{n} \int_0^{\widetilde{\theta}_k} a^2(w_k(t))dt \Big\}.$$

Lemma

In $C([0;T], \mathbb{R}^n)$

$$(X_{0,\cdot}(u_1), \ldots, X_{0,\cdot}(u_n)) \stackrel{d}{=} \widetilde{W}$$

and

$$\widetilde{\mathcal{E}}^a_{T,n}(u) \stackrel{d}{=} \mathcal{E}^a_{T,n}(W,u).$$

- Brownian bridges $\eta = (\eta_1, \dots, \eta_n)$:

$$w_k(t) = \frac{t}{T} w_k(T) + \eta_k(t),\ t \in [0;T], k = \overline{1,n}$$

- For any k define

$$d\eta_k(t) = d\beta_k(t) - \frac{\eta_k(t)}{T-t} dt,\ t \in [0;T),$$
$$\eta_k(0) = \eta_k(T) = 0,$$

- For any $y \in \mathbb{R}^n$ define

$$\eta^{u,y}(t) = \eta(t) + \left(1 - \frac{t}{T}\right) u + \frac{t}{T} y,\ t \in [0;T].$$

- $\{\theta_{ij}(u)\}$ are meeting time for the process W
- $\{\tau_{ij}(u,y)\}$ are meeting times for the process $\eta^{u,y}$
- $\theta_{ij}(u) = \tau_{ij}(u, w(T)),\ j = \overline{1, i-1}, i = \overline{2, n}$
- Non-random numbers $\{\lambda_{ij}(s) \mid i = 1, 2, j = \overline{1, n}\}$:

$$\widetilde{\theta}_k(u) = \tau_{\lambda_{1k}(s)\lambda_{2k}(s)}(u, W(T))$$

  on $\{S(W) = s\}$ for a coalescence scheme s

Define on $\{S(W) = s\}$

$$a_k(t, u, y, s) = 1\!\mathrm{I}(t \le \tau_{\lambda_{1k}(s)\lambda_{2k}(s)}(u, y)) \cdot a\left(\eta_k^{u,y}(t)\right),$$

$$\mathfrak{e}^a_{T,n}(u, y, s) = \exp\Big\{ \sum_{k=1}^{n} \int_0^T a_k(t, y, s) d\eta_k(t) +$$

$$+ \sum_{k=1}^{n} \int_0^T a_k(t, u, y, s)\Big(\frac{y_k - u_k}{T} - \frac{1}{2} a_k(t, u, y, s)\Big) ds\Big\}.$$

Lemma

We have:

- *for any* s, y *and starting point* u

$$\mathrm{E}_u\left(1\!\!\mathrm{I}(S(W)=s)\mathcal{E}^a_{T,n}(W)/W(T)=y\right)= \\ =\mathrm{E}\,1\!\!\mathrm{I}(S(\eta_{u,y})=s)\mathfrak{e}^a_{T,n}(u,y,s);$$

- *for any* y, s, u *and* $p>0$

$$\mathrm{E}\left(\mathfrak{e}^a_{T,n}(u,y,s)\right)^p\le C_1 e^{C_2\|y\|};$$

- *for any* s *the mapping* $y\mapsto \mathrm{E}\,1\!\!\mathrm{I}(S(\eta_{u,y})=s)\mathfrak{e}^a_{T,n}(u,y,s)$ *is continuous.*

- Every coalescence scheme generates a partition of $\{1,\dots,n\}$ of blocks of merged particles. Set $I(s)$ to be the set of smallest elements in all blocks.
- For a set K of indexes in $\{1,\dots,n\}$
  - $z^K = (z_i), i \in K$
  - $z^{-K} = (z_i), i \notin K$
- $g_t^{(n)}$ is $n-$dimensional Gaussian density for $\mathcal{N}(x, t\mathrm{Id})$

Theorem (Dorogovtsev, V. 2020)

Consider $n \in \mathbb{N}$, $u \in \Delta_n$ and a coalescence scheme s with $\varkappa = k$. Then for all $j \in \{1,\dots,k\}$ and $y \in \Delta_k$

$$p_t^{a,n,s,j}(u;y) = \sum_{L=\{l_1,\dots,l_j\}\subset\{1,\dots,k\}} g_t^{(j)}\left(u^{I(s),L} - z^{I(s),L}\right)\times$$

$$\times \int_{\mathbb{R}^{k-j}} dz^{I(s),-L} g_t^{(k-j)}\left(u^{I(s),-L} - z^{I(s),-L}\right) \int_{\mathbb{R}^{n-k}} dz^{-I(s)} g_t^{(n-k)}\left(u^{-I(s)} - z^{-I(s)}\right)$$

$$\left(\mathrm{E}\,1\!\!1(S(\eta^{u,z}) = s)\mathfrak{e}^a_{T,n}(u,z,s)\right)\Big|_{\substack{z\in\mathbb{R}^n,\\ z^{I(s),L}=y}}.$$

Alternative proof of existence

$$B_\varepsilon^+(y) = [y; y+\varepsilon), \quad y \in \mathbb{R}, \varepsilon \in \mathbb{R}_+,$$
$$N_t(A) = \left|\{X_t^a(x) \mid X_t^a(x) \in A, x \in \mathbb{R}\}\right|, \quad A \in \mathcal{B}(\mathbb{R}),$$
$$N_t(B; A) = \left|\{X_t^a(x) \mid X_t^a(x) \in A, x \in B\}\right|, \quad A, B \in \mathcal{B}(\mathbb{R})$$

Lemma

For any u, y, s and some $k \leq \varkappa(s)$

$$p^{a,n,s,k}(u; y) = \lim_{\varepsilon \to 0+} \mathrm{E} \prod_{j=\overline{1,n}} 1\!\!\mathrm{I}\left[N_t(u; B_\varepsilon^+(y_j)) > 0\right] 1\!\!\mathrm{I}\left[s(X^a(u)) = s\right].$$

Corollary

In particular,

$$p_t^{a,n}(u; y) = \lim_{\varepsilon \to 0+} \varepsilon^{-k} \, \mathsf{P}\left(N_t(u; B_\varepsilon(x_k)) > 0, k = \overline{1,n}\right).$$

Finite point approximations

$$u_{m,j} = \frac{j}{m}, \quad j = \overline{0,m}, m \in \mathbb{N},$$
$$u_m = (u_{m,0}, \dots, u_{m,m}),$$
$$U_m = \{u_{m,j} \mid j = \overline{0,m}\}$$

Theorem (V. 2024)

For any n and $x \in \Delta_n$

$$\lim_{m\to\infty} m^2 \left(p_t^{a,n}(x) - p_t^{a,m,n}(u_m; x)\right) = C_n > 0,$$

where

$$C_n = \lim_{m\to\infty} \lim_{\varepsilon\to 0+} \varepsilon^{-n} \sum_{k=\overline{1,n}} \mathsf{P}\Big(S(X^{-a}(x_\varepsilon)) = \emptyset;$$
$$(X_t^{-a}(x_k), X_t^{-a}(x_k+\varepsilon)) \cap U_m = \emptyset;$$
$$\forall i \neq k \ (X_t^{-a}(x_i), X_t^{-a}(x_i+\varepsilon)) \cap U_m \neq \emptyset\Big),$$
$$x_\varepsilon = (x_1, x_1+\varepsilon, x_2, x_2+\varepsilon, \dots, x_n, x_n+\varepsilon)$$

The proof relies on the following observations:

1. using dual flows to estimate the probability of two particles getting close yet not merging
2. relations between point densities and PDEs
3. estimates for point densities with drift

Preliminary transformations

$$B^+_\varepsilon(y) = [y; y+\varepsilon), \quad y \in \mathbb{R}, \varepsilon \in \mathbb{R}_+,$$
$$N_t(A) = \left|\{X^a_t(x) \mid X^a_t(x) \in A, x \in \mathbb{R}\}\right|, \quad A \in \mathcal{B}(\mathbb{R}),$$
$$N_t(B; A) = \left|\{X^a_t(x) \mid X^a_t(x) \in A, x \in B\}\right|, \quad A, B \in \mathcal{B}(\mathbb{R})$$

We need to study

$$\limsup_{\varepsilon \to 0+} \varepsilon^{-n} \int_{B^+_\varepsilon(x_1) \times \ldots B^+_\varepsilon(x_n)} (p^{a,n}_t(y) - p^{a,n}_t(u_m; y)) dy$$
$$= \limsup_{\varepsilon \to 0+} \varepsilon^{-n} \Big[\prod_{k=\overline{1,n}} N_t(B^+_\varepsilon(x_j)) - \prod_{k=\overline{1,n}} N_t(U_m; B^+_\varepsilon(x_j)) \Big]$$
$$= \limsup_{\varepsilon \to 0+} \varepsilon^{-n} \Big[\mathsf{P}\left(N_t(B^+_\varepsilon(x_j)) > 0, j = \overline{1,n}\right) - \mathsf{P}\left(N_t(U_m; B^+_\varepsilon(x_j)) > 0, j = \overline{1,n}\right) \Big]$$

Indeed, it is well known that the error between two last lines can be estimated in the terms

$$\sum_{k=\overline{1,n}} \int_{B^+_\varepsilon(x_1) \times \ldots \times B^+_\varepsilon(x_k) \times B^+_\varepsilon(x_k) \times \ldots} p^{a,n+1}_t(y) dy,$$

so basic estimates suffice.

The method of the dual flow (the method of intervals)

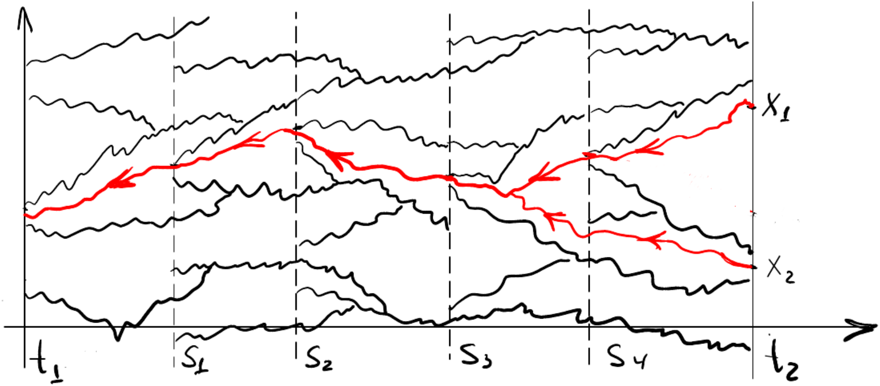
We need to estimate

$$\mathsf{P}\left(N_t(B_\varepsilon^+(x_j)) > 0, j = \overline{1,n}\right) - \mathsf{P}\left(N_t(U_m; B_\varepsilon^+(x_j)) > 0, j = \overline{1,n}\right)$$

There exists a dual Brownian web $\{\widetilde{X}_{s,t}(u) \mid u \in \mathbb{R},\ s \le t\}$:

- lives in the reversed time
- coalescing Brownian motions, independent before the meeting
- the trajectories of X^a and $\widetilde{X}$ do not intersect a.s.
- $\widetilde{X} = X^{-a}$ actually:
  - Riabov 2020

$$\widehat{X}_{t_1,t_2}(x) = \inf\left\{X_{r,T-t_2}(y) \mid X_{r,T-t_1}(y) > x, y \in \mathbb{R}, r \in [0;T-t_2]\right\}$$



Example: 1d case, $a \equiv 0$

We have: $B_\varepsilon^+(x)$ is non-empty but $X_{0,t}(u_m)$ misses $B_\varepsilon^+(x)$

$$
\begin{aligned}
&\mathsf{P}\big(X_{0,t} \cap B_\varepsilon^+(x) \neq \varnothing\big) - \mathsf{P}\big(X_{0,t}(u_m \cap B_\varepsilon^+(x) \neq \varnothing\big) = \\
&\quad \leq \mathsf{P}\Big(\widetilde{X}_{0,t}(x+\varepsilon) \neq \widetilde{X}_{0,t}(x),\ \exists\, j \in \{1,\dots,n-1\}: \\
&\qquad \Big(\widetilde{X}_{0,t}(x); \widetilde{X}_{0,t}(x+\varepsilon)\Big) \subset \big(u_{m,j}; u_{m,j+1}\big)\Big) \leq \\
&\quad \leq \int\limits_{0 \leq y_2 - y_1 < \max_j (u_{n,j+1} - u_{n,j})} dy_1\, dy_2\ p_t^{0,2,\varnothing,2}\big((x, x+\varepsilon); (y_1, y_2)\big),
\end{aligned}
$$

where (trivially)

$$
p_t^{0,2,\varnothing,2}(a;b) = \frac{1}{2\pi t} e^{-\frac{\|a-b\|^2}{2t}} \big(1 - e^{-(b_2-b_1)(a_2-a_1)}\big).
$$

Passing to the dual flow gives

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0+}\varepsilon^{-n}\sum_{k=\overline{1,n}}\mathsf{P}\Big(S(X^{-a}(x_\varepsilon))=\emptyset;$$

$$(X_t^{-a}(x_k),X_t^{-a}(x_k+\varepsilon))\cap U_m=\emptyset;$$

$$\forall i\neq k\ (X_t^{-a}(x_i),X_t^{-a}(x_i+\varepsilon))\cap U_m\neq\emptyset\Big),$$

$$x_\varepsilon=(x_1,x_1+\varepsilon,x_2,x_2+\varepsilon,\ldots,x_n,x_n+\varepsilon)$$

where we want to get rid of all collisions and do not allow multiple hits between points of the initial discretization, that is, to estimate properly expressions of the form

$$\mathsf{P}\Big(A;X_t^{-a}(x_j),X_t^{-a}(x_j+\varepsilon),X_t^{-a}(x_{j+1}),X_t^{-a}(x_{j+1}+\varepsilon)\in(u_{m,k};u_{m,k+1})\Big),$$

$$\mathsf{P}\Big(A;X_t^{-a}(x_j),X_t^{-a}(x_j+\varepsilon)\in(u_{m,k_1};u_{m,k_1+1}),$$

$$X_t^{-a}(x_{j+1}),X_t^{-a}(x_{j+1}+\varepsilon)\in(u_{m,k_2};u_{m,k_2+1})\Big)$$

where $k_1\neq k_2$ and

$$A=\{\omega\mid X_t(x_k+\varepsilon)>X_t(x_k),k=\overline{1,n}\}$$

Let Ξ_n be the set of all non-trivial coalescence scheme such that only collisions of the form $(2j;2j+1)$ are possible.
Then

$$\mathsf{P}\Big(S(X(x_\varepsilon)) \neq \emptyset;$$
$$X_t^{-a}(x_1), X_t^{-a}(x_1+\varepsilon) \in (u_{m,k_1}; u_{m,k_1+1}),$$
$$X_t^{-a}(x_2), X_t^{-a}(x_2+\varepsilon) \in (u_{m,k_2}; u_{m,k_2+1});$$
$$\leq \sum_{s\in\Xi_n} \int_{u_{m,k_1}}^{u_{m,k_1+1}} \int_{u_{m,k_2}}^{u_{m,k_2+1}} p^{a,2n,s,2}(x_\varepsilon, y)\, dy_1 dy_2.$$

Estimates for the Karlin-McGregor determinant and its derivatives

$$g_t^{KM}(y,x) = \det \|g_t(y_j - x_k)\|_{j,k}$$

In (Munasinghe, Rajesh, Tribe та O. Zaboronski 2006):

$$c_n \prod_k g_t(x_k, y_{n+1-k}) \leq \frac{p_t(x,t)}{h_n(t^{-1/2}x)h_n(t^{-1/2}y)} \leq \prod_k g_t(x_k, y_k),$$

In (Katori та Tanemura 2007):

$$g_t^{KM}(y,x) = \frac{1}{(2\pi t)^{n/2}} \mathrm{e}^{-\frac{\|x\|^2+\|y\|^2}{2t}} h_n(x)h_n(y) \sum_{\lambda: l(\lambda)\leq n} \frac{s_\lambda(x)s_\lambda(y)}{\prod_{k=1,N}(\lambda_k+n-k)!}$$

where h is a Vandermonde determinant and the sum is over Schur polynomials (over variables $x_1, \ldots, x_n$).

A representation of $n-$point densities as a series

Define

$$\nabla_y^a = \sum_{k=\overline{1,2n}} a(y_k)\partial_{y_k}$$

$$D_{2n} = \left\{x \in \mathbb{R}^{2n} \mid x_1 < \ldots < x_{2n}\right\}$$

$$\partial D_{2n,j} = \{y \in \partial D \mid y_j = y_{j+1}\}, \quad j = \overline{1, 2n-1}$$

$$\partial_{\nu_y} = \frac{1}{\sqrt{2}}(\partial_{y_{j+1}} - \partial_{y_j})$$

Every $\sigma \in \Xi_n$ defines some boundary condition f_σ as a sum of indicators of some hyperplanes.
Consider for some $f = f_\sigma$

$$\partial_t W_f = \frac{1}{2}\Delta W_f - \nabla^a W_f,$$

$$W_f(x, 0) = 1, \quad x \in D_{2n},$$

$$W_f(x, t) = f, \quad x \in \partial D_{2n}, \; t > 0.$$

Theorem (V. 2024, 1d: Dorogovtsev, V. 2023)

Assume $a \in L_\infty(\mathbb{R})$. We have

$$p_t^n(x) = \lim_{\varepsilon \to 0+} \varepsilon^{-n} \sum_{\sigma \in \Xi_n} W_{n,f_\sigma}\left((x_1, x_1+\varepsilon, \ldots, x_n, x_n+\varepsilon), t\right)$$
$$= \Big[\frac{\partial^n}{\partial u_2 \cdots \partial u_{2n}} \sum_{f \in \Xi_n} W_{n,f_\sigma}(u,t)\Big]\Big|_{u=(x_1,x_1,\ldots,x_n.x_n)}$$

where each function

$$W_{n,f_\sigma} \in C(\overline{D}_{2n} \times (0;\infty)) \cap C(D_2 \times [0;\infty)) \cap C^{1,0}(D_2 \times [0;\infty))$$

is a distributional solution in $\mathcal{D}'(D_2 \times (0;t))$ to the original IBVP, satisfies IC and BC and admits the representation as a series.

The case $n = 1$

$$p_t^{a,1}(u) = \lim_{\delta \to 0+} \delta^{-1} \mathsf{P}\left(X_t^a(\mathbb{R}) \cap [u; u+\delta) \neq \emptyset\right)$$

Let $\xi_x^a = (\xi_{x_1}^a, \xi_{x_2}^a)$ be the unique weak solution of

$$d(\xi_x^a)_k(t) = a\left((\xi_x^a)_k(t)\right) dt + dw_k(t),$$
$$(\xi_x^a)_k(0) = x_k, \quad k = 1, 2$$

where w_1, w_2 are independent standard Wiener processes. Define

$$\theta_x^a = \inf\{s \mid \xi_x^a \in \partial D_2\},$$

Then

$$p_t^{a,1}(u) = \lim_{\delta \to 0+} \delta^{-1} \mathsf{P}\left(\theta_{(u,u+\delta)}^{-a} > t\right).$$

Since $\partial_t W_f = \frac{1}{2}\Delta W_f - \nabla^a W_f$ we have formally

$$W(x,t) = \int_{D_{2n}} dy\, g_t^{KM}(x,y) - \int_0^t ds \int_{D_{2n}} dy\, g_{t-s}^{KM}(x,y)\nabla_y^a W(y,s)$$
$$- \int_0^t ds \int_{\partial D_{2n}} dS(y)\, \partial_{\nu_y} g_{t-s}^{KM}(x,y) f(y,s).$$

Iterating:

$$W = \sum_{n\in\mathbb{N}} W_n$$

$$W_0^a(x,s) = \int_{D_2} dy_0 \;\, g_s^{KM}(x,y_0),$$
$$W_n^a(x,s) = (-1)^n \int_{\Delta_n(s)} dr_1 \ldots dr_n \int_{D_2^{n+1}} dy_0 \ldots dy_n$$
$$g_{s-r_n}^{KM}(x,y_n) \prod_{j=1}^n \nabla_{y_j}^a g_{r_j - r_{j-1}}^{KM}(y_j, y_{j-1}),$$

$$n \geq 1,$$

Scheme of proof: step 1

Proposition

For all $n \geq 1$ in the sense of Schwartz distributions

$$\partial_s W_n^a = \frac{1}{2}\Delta W_n^a - \nabla_x^a W_{n-1}^a$$

in $\Delta_2 \times (0;\infty)$. Consequently,

$$\partial_s W^a = \frac{1}{2}\Delta W^a - \nabla_x^a W^a$$

in $\Delta_2 \times (0;\infty)$.

Proof.

$$W_n^a(x,s) = \int_0^s \int_{\Delta_2} drdy\ g_{s-r}^{KM}(x,y) f_n(r,y),$$
$$f_n(r,y) = -D_y^a W_{n-1}^a(y,r), \quad n \geq 1,$$

where $\sup_{n\geq 1} \sup_{r\in(0;t),y\in\Delta_2} |f_n(r,y)| < Cr^{-1/2}$.

For $h > 0$

$$W_n^a(x, s+h) - W_n^a(x, s) = \int_0^s \int_{D_2} drdy \ \left(g_{s+h-r}^{KM}(x,y) - g_{s-r}^{KM}(x,y)\right) f_n(r,y) + \int_s^{s+h} \int_{D_2} drdy \ \rho_{s+h-r}(x,y) f_n(r,y) = H_1(h,x,s) + H_2(h,x,s).$$

For every test function v

1. $$h^{-1} \int_{D_2} dx \ v(x) H_1(h,x,s) \to -\frac{1}{2} \int_{D_2} \int_0^s \int_{D_2} dxdrdy \ \nabla_x v(x) \cdot f_n(r,y) \nabla_x g_{s-r}^{KM}(x,y) = -\frac{1}{2} \int_{D_2} dx \ \nabla_x v(x) \cdot \nabla_x W_n^a(x,s), \quad h \to 0+$$

2. $$h^{-1} H_2(h,x,s) \to f_n(x,s) = -\nabla_x^a W_{n-1}^a(x,s), \quad h \to 0+.$$

H_2

- Recalling: ξ_x^a is the solution of

$$d(\xi_x^a)_k(t) = a((\xi_x^a)_k(t))dt + dw_k(t),$$
$$\theta_x^a = \inf\{s \mid \xi_x^a \in \partial\Delta_2\}$$

- Then

$$h^{-1}H_2(h, x, s) = h^{-1}\int_s^{s+h} dr\ k_{x,s}(h, r).$$

where

$$k_{x,s}(h, r) = \mathrm{E}\, f_n\left(r, \xi_x^a(s + h - r)\right) 1\!\mathrm{I}\left[\theta_x^a > s + h - r\right],$$
$$\forall s \quad k_{x,s}(h_0, s) \to f_n(s, x), \quad h_0 \to 0+,$$

- By the Girsanov theorem

$$k_{x,s}(h, r) = \mathrm{E}\, f_n\left(r, \xi_x^0(s + h - r)\right) 1\!\mathrm{I}\left[\theta_x^0 > s + h - r\right] \mathcal{E}_t^a,$$

Scheme of proof: step 2

Proposition

For all $s > 0$ and $x \in \Delta_2$

$$W^a(x,s) = \mathsf{P}\left(\theta_x^a > s\right).$$

Let $a \in C^\infty(\mathbb{R})$. Since the operator $\frac{1}{2}\Delta - \nabla_x^a$ is hypoelliptic,

$$W^a \in C^\infty((0;\infty) \times \Delta_2).$$

The exhaustion method and the property

$$W^a \in C(\overline{\Delta}_2 \times (0;\infty)) \cap C(\Delta_2 \times [0;\infty))$$

yield

$$W^a(x,s) = \mathsf{P}\left(\theta_x^a > s\right)$$

for a mollified a.

Back to the main result

Theorem

Let $a \in L_\infty(\mathbb{R})$*. Then for all* $x \in D_2$ *and* $t > 0$

$$p_t^{a,1}(x) = \partial_{x_2} W^a(x,t) = \sum_{n \geq 0} \partial_{x_2} W_n^a(x,t).$$

Theorem

Assume $a_n \in L_\infty(\mathbb{R}), n \geq 0; \sup_{n\geq 0} \|a_n\|_{L_\infty(\mathbb{R})} < \infty$*. Let one of the following conditions hold:*

1. $a_n \to a_0, n \to \infty,$ *in* $L_\infty(\mathbb{R})$;
2. $a_n \in L_1(\mathbb{R}), n \geq 0,$ *and* $a_n \to a_0, n \to \infty,$ *in* $L_1(\mathbb{R})$.

Then for all $x \in D_2$ *and* $t > 0$

$$p_t^{a_n,1}(x) \to p_t^{a_0,1}(x), \quad n \to \infty.$$

The multidimensional representation of the perturbed semigroup is also available.
In particular, it requires iterating double layer heat potentials.