The stability of Brownian maxima and extending the noise of splitting.

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Tsirelson's theory of noises

A continuous product of probability spaces is a separable probability space equipped with a two-parameter family of sub-sigma algebras $\left(\mathcal{F}_{s,t}\right)_{s\leq t}$ indexed by pairs of times and having the following properties. For all $s \leq t \leq u$

$$\mathcal{F}_{s,t} \text{ is independent of } \mathcal{F}_{t,u}, \tag{1}$$

and

$$\mathcal{F}_{s,u} = \mathcal{F}_{s,t} \vee \mathcal{F}_{t,u}. \tag{2}$$

A continuity property is also imposed. It is a *noise* if it is homogeneous in time, so that for each h one may shift time by h and this carries $\mathcal{F}_{s,t}$ to $\mathcal{F}_{s+h,t+h}$.

Classical and nonclassical noise

Two natural examples: when $\mathcal{F}_{s,t}$ is generated by the increments of a Brownian motion between times s and t, and when it is generated by the points of a Poisson point process that occur between the two times; more generally we can combine the two and consider the increments of a Lévy process. These are, in Tsirelson's terminolgy, *classical noises*.

The noise is *non-classical* if there is no Lévy process whose increments generate the noise.

Classical and nonclassical noise continued

A fairly general recipe for constructing a non-classical noise is to let $\mathcal{F}_{s,t}$ be generated by the evolution of a stochastic flow between times s and t, using a flow which is associated with an SDE having a weak solution which is not measurable with respect to the driving Brownian motion. The extra randomness in the flow, not in the driving Brownian motion, takes the form of *sensitive* random variables and from the existence of such it follows that the noise is non-classical.

Extending a noise: a characterization of classical noise

An important question, posed originally by Feldman, is whether we can extend a continuous product of probability spaces, in some natural way, to a larger family of σ-algebras $\mathcal{F}_E$ where E runs over all Borel measurable subsets of **R**.

This is rather easily done for classical noises .

A major achievement of Tsirelson's was to show that such an extension is never possible for a non-classical noise. Nevertheless one can try to extend the continuous product by defining $\mathcal{F}_E$ for some E, and this leads to the notion of *the completion* of the noise introduced by Tsirelson.

This is the largest noise-type Boolean lattice of σ-algebras which one can obtain by taking limits starting from the $\mathcal{F}_{s,t}$. Noise-type means distributive and all elements have independent complements.

Extending a noise: loss of sensitive information

If E is a union of disjoint intervals,

$$\mathcal{F}_E = \bigvee \mathcal{F}_{s_i,t_i}$$

If E is closed then it is a decreasing limit of E_n which are, in turn, unions of disjoint intervals, and,

$$\mathcal{F}_E = \bigwedge \mathcal{F}_{E_n}.$$

Question

For a closed E is it true that $\mathcal{F}_{-\infty,\infty} = \mathcal{F}_E \vee \mathcal{F}_{E^c}$?

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If E is too "small" then some sensitive random variables are lost. Then E does not belong to the completion. This is the key to Tsirelson's proof that only classical noises extend fully.

Tanaka's SDE 1

Challenge

Find an example of a noise for which we can identify the completion.

Tanaka's stochastic differential equation

$$X_t = x + \int_0^t \mathrm{sgn}(X_s) dW_s,$$

admits a weak solution but has no strong solution.

Applying Tanaka's equation to X we obtain

$$|X_t| = x + W_t + L_t^X$$

where L_t^X is the local time accrued by X at 0, and this can be moreover identified as being equal to $-\inf_{s \leq t}(W_s + x) \vee 0$.

Tanaka's SDE 2

So, $|X_t|$ is measurable with respect to the driving Brownian motion, but the signs of the excursions of X from 0 are, since X is itself distributed as a Brownian motion, independent random signs which are independent of W.

Notice, that as time passes these random signs manifest themselves at the instants excursions of X begin, and that these times are the times that W makes excursions above its running infimum process. They are in particular the times of (certain) local minima of W.

The noise of splitting 1

It is possible to construct a stochastic flow of maps $(X_{s,t}, s \leq t)$ so that each one point motion $t \mapsto X_{s,t}(x)$ for $t \geq s$ solves Tanaka's SDE, starting from x at time s, driven by a common Brownian motion W. Then the flow contains even more randomness than a single solution to the SDE does; the Brownian motion W is augmented with a family of independent random signs with a random sign associated to every time at which the Brownian motion has a local minima. Then the noise of splitting is defined by taking

$\mathcal{F}_{st}$ is generated by the evolution of the flow between times s and t.

Equivalently $\mathcal{F}_{st}$ to be generated by the increments of the Brownian motion w between s and t, together with the random signs associated with local minima occurring between these times.

The noise of splitting 2

In fact one may dispense with the flow. Let $(\tau_1, \tau_2, \ldots, \tau_i, \ldots)$ be an enumeration of the times of the local minima of W. Then for any random time τ which selects a local minima define $i(\tau)$ by $\tau_{i(\tau)} = \tau$. Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_i, \ldots)$ be a sequence of independent random signs, independent of W, and define

$$\mathcal{F}_{st} = \mathcal{F}^W_{st} \bigvee \sigma\left(\epsilon_{i(\tau)}, \, \tau \in \mathcal{T}_{st}\right)$$

where $\mathcal{F}^W_{st}$ is the σ-algebra generated by the increments of W between s and t and $\mathcal{T}_{st}$ is the collection of all $\mathcal{F}^W_{s,t}$-measurable random times which select local minima of W between times s and t.

First thoughts on an extension

Guess

For general E, $\mathcal{F}_E$ is generated by $\mathcal{F}_E^W$ together with the random signs attached to local minima of W which occur at times in E.

Issue

We should "address" the local minima by using $\mathcal{F}_E^W$-measurable random times. Can we?

Max unstable sets

Definition

Say that the set E is max (or min) unstable if coupled BMs W and W^E, satisfying

$$[W, W^E](t) = \int_0^t 1_E(s)ds$$

have local min times $\tau_{s,t} = \text{argmin}\{W_h, h \in [s,t]\}$ and $\tau^E_{s,t}$ such that

$$\mathbf{P}\left(\tau_{s,t} = \tau^E_{s,t}\right) = 0$$

for all $s < t$.

Note that this is equivalent to saying the conditional distribution of τ_{st} given $\mathcal{F}^W_E$ contains no atoms.

Loss of the random signs

Proposition

E is max unstable if and only if

$$\mathcal{F}_E = \mathcal{F}_E^W.$$

Proof.

Suppose E closed. Let E_n be a finite union of closed intervals, with E_n decreasing to E. Then we consider coupled Brownian motions (W, W^{E_n}). These satisfy

$$\mathbf{P}(\tau_{s,t} = \tau_{s,t}^{E_n}) = \mathbf{E}\left[\mathbf{E}[\epsilon(\tau_{s,t})|\mathcal{F}_{E_n}]^2\right].$$

□

Max stable sets

Definition

Say that the set E is max (or min) stable if the local minima of W which occur at times belonging to E are all the times of local minima of $W_E = \int_0^{\cdot} 1_E(s) dW_s$.

Note that this is implies that the conditional distribution of τ_{st} given $\mathcal{F}_E^W$ is purely atomic.

The completion of the noise of splitting

Theorem

The completion of the noise of splitting consists of σ-algebras $\mathcal{F}_E$ associated with precisely those $E \subset \mathbf{R}$ for which both E and E^c are max stable.

Local density of E

Suppose that E is a measurable subset of the real line, and let A_{tu} be the Lebesgue measure of $E \cap [t, u]$ (or $E \cap [u, t]$ when $u < t$). According to the Lebesgue density theorem, for almost every $t \in E$,

$$\frac{1}{|u-t|} A_{tu} \to 1, \text{ as } u \to t.$$

We are interested in the how the speed at which this convergence occurs controls the stability of the set E.

Criteria for max stability and instability

Proposition

If, for almost every $t \in E$,

$$|h| - A_{t,t+h} \geq C(t)\frac{|h|}{(\log(1/|h|))^2}$$

then E is unstable.
If there exists an $\epsilon > 0$, so that for almost every $t \in E$,

$$|h| - A_{t,t+h} \leq C(t)\frac{|h|}{(\log(1/|h|))^{2+\epsilon}}$$

then E is stable.

The range of subordinators

To exhibit explicit examples of stable and unstable sets we consider E to be the range of a subordinator having Lévy measure Π and drift $d > 0$.

Proposition

If the tail of the Lévy measure satisfies

$$\overline{\Pi}(x) \geq \frac{C}{x(\log(1/x))^3} \text{ for all sufficiently small } x > 0$$

then E is unstable.
If the tail satisfies, for some $\epsilon > 0$,

$$\overline{\Pi}(x) \leq \frac{C}{x(\log(1/x))^{3+\epsilon}} \text{ for all sufficiently small } x > 0$$

then E is stable.