Topics in random matrix theory. Lecture 9. Fixed t multi-point intensity functions for the real Ginibre ensemble.

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23.05.2013

As an application of the formalism of spin variables we will show how to re-derive the result of Borodin and Sinclair [1] for all the one-dimensional densities of real eigenvalues of the real Ginibre ensemble in the limit $N \to \infty$. Let us fix time $t = 1$ and a set of multiple space points $x_1 < x_2 < \ldots < x_K$. The answer for an arbitrary time t can be obtained from the $t = 1$ answer by the diffusive re-scaling $x \to \frac{x}{\sqrt{t}}$. As we have done before, we will first compute the modified density $\tilde{\rho}_N = \tilde{\rho}_N(x_1, x_2, \ldots, x_K)$ defined by

$$\tilde{\rho}_N(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_K = N\mathbb{E}\left(\prod_{k=1}^{K} s_{x_k}(M_1)\Lambda^{M_1}(dx_k)\right)$$

As before, we choose right hand limits for the density, that is where the intervals dx_k denote infinitesimal intervals just to the right of the point x_k. The answers are zero for odd K, so we take even K throughout. Moreover it is convenient to consider only even N throughout (which avoids us tracking various $\pm$ signs). Recall that correlation functions of spins can be computed by integrating $\tilde{\rho}$ with respect to space variables:

$$\mathbb{E}\left(\prod_{k=1}^{K} s_{x_k}(M_1)\right) = (-2)^K \left(\prod_{k=1}^{K} \int_{-\infty}^{x_k} dy_k\right) \tilde{\rho}_N(y_1, y_2, \ldots, y_k). \qquad (1)$$

Finally, the probability density functions of real eigenvalues can be computed as explained in Lecture 7.

Equivalence with a correlation function of characteristic polynomials. The integral over the Gaussian density

$$\tilde{\rho}_N(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_K = \int_{R^{N^2}} dM \gamma_1(M) \prod_{k=1}^{K} s_{x_k}(M)\Lambda^M(dx_k) \qquad (2)$$

can be treated using the Edelman transform for the eigenvalue lying in dx_K, and after integrating over the half sphere S_{N-1}^+, we obtain

$$\begin{aligned}
&\tilde{\rho}_N(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_{K-1} \\
&= \frac{1}{2}|S_{N-1}|\pi^{-\frac{N-1}{2}} e^{-x_K^2} \mathbb{E}_{N-1}\left(\det(M_1 - x_K I) \prod_{k=1}^{K-1} s_{x_k}(M_1)\Lambda^{M_1}(dx_k)\right),
\end{aligned}$$

where the subscript $N-1$ on the $\mathbb{E}_{N-1}$ means that the averaging occurs over the $(N-1)\times(N-1)$ Ginibre distribution. Another Edelman transform about the eigenvalue lying in dx_{K-1} yields

$$\begin{aligned}
&\tilde{\rho}_N(x_1, x_2, \ldots, x_K)\, dx_1 \ldots dx_{K-2} \\
&= \frac{1}{4}|S_{N-1}||S_{N-2}|\pi^{-\frac{N-1}{2}-\frac{N-2}{2}} e^{-x_K^2 - x_{K-1}^2}(x_{K-1} - x_K) \\
&\quad \mathbb{E}_{N-2}\left(\det(M_1 - x_K I)\det(M_1 - x_{K-1} I) \prod_{k=1}^{K-2} s_{x_k}(M_1)\Lambda^{M_1}(dx_k)\right).
\end{aligned}$$

A further $(K-2)$ applications of Edelman transform will lead to the following expression for the modified density:

$$\tilde{\rho}_N(x_1, x_2, \ldots, x_K) = \frac{\Delta(\mathbf{x})}{2^K} \prod_{k=1}^{K}\left(|S_{N-k}|\pi^{-\frac{N-k}{2}} e^{-x_k^2}\right) \mathbb{E}_{N-K}\left(\prod_{m=1}^{K} \det(M_1 - x_m I)\right) \quad (3)$$

where $\Delta(\mathbf{x}) = \prod_{1\le i<j\le K}(x_j - x_i)$ is the Van-der-Monde determinant. Therefore, the problem of computing the modified density has been reduced to the computation of the expectation of the product of characteristic polynomials of the random matrix M_1.

Integral representation for product of characteristic polynomials. Such a computation has been carried out in [13] by exploiting Berezin integrals. Firstly, as we explained in Lecture 8, for any $N \times N$ matrix A,

$$det(A) = \int_{\mathbf{R}^{0|2N}} d\psi d\overline{\psi} e^{\overline{\psi} A \psi}$$

Representing all determinants in (3) as Gaussian Berezin integrals, we get a Gaussian integral over $(N-K)\times(N-K)$ matrices M. This can be computed using

$$\int_{\mathbf{R}^{n\times n}} dM e^{-\frac{1}{2}TrMM^T + TrMJ^T} = (2\pi)^{n^2/2} e^{TrJJ^T}$$

The result has a term

$$e^{Tr(A\overline{A}^T)},$$

where A, $\overline{A}$ are skew-symmetric $K \times K$ matrices constructed out of anti-commuting vectors:

$$A_{ij} = \psi_i \cdot \psi_j, \overline{A}_{ij} = \overline{\psi}_i \cdot \overline{\psi}_j.$$

The integration over $\psi, \overline{\psi}$ is carried out using Hubbard-Stratonovich transformation:

$$e^{Tr(A^T B)} = \int_{\mathbf{C}^{K(K-1)/2}} dH d\overline{H} e^{-Tr(HH^\dagger)+Tr(HA^T)+Tr(H^\dagger B)}$$

, where

$$dH d\overline{H} = \prod_{i<j} \frac{dImH_{ij} dReH_{ij}}{\pi}$$

and the integral is over the space of complex skew-symmetric $K \times K$ matrices. The resulting integral over anti-commuting variables is Gaussian which we compute using

$$Pf(M) = \int_{R^{0|2n}} d\chi e^{\chi^T M \chi}.$$

After all these steps we arrive at the expression for the modified density as a $K(K-1)$-dimensional integral over Hubbard-Stratonovich variables:

$$\mathbb{E}_N \left(\prod_{m=1}^{K} \det\left(M_1 - x_m I\right) \right) = \prod_{1 \le p < q \le K} \left[\int_{\mathbf{R^2}} \frac{dz_{pq} d\overline{z}_{pq}}{\pi} e^{-|z_{pq}|^2} \right] Pf \begin{pmatrix} \frac{1}{\sqrt{2}} Z & X \\ -X & \frac{1}{\sqrt{2}} Z^\dagger \end{pmatrix}^N . \quad (4)$$

Here X is a diagonal $K \times K$ matrix with entries $(x_1, x_2, \ldots x_K)$; and Z is a skew symmetric complex $K \times K$ matrix:

$$Z_{ij} = \begin{cases} z_{ij} & i > j \\ 0 & i = j \\ -z_{ij} & i < j \end{cases}$$

Expression (4) can be neatly written as a matrix integral:

$$\mathbb{E}_N \left(\prod_{m=1}^{K} \det\left(M_1 - x_m\right) \right) = \pi^{-\frac{K(K-1)}{2}} \int_{Q^{(K)}} \lambda(dZ, dZ^\dagger) e^{-\frac{1}{2} Tr Z Z^\dagger} Pf \begin{pmatrix} \frac{1}{\sqrt{2}} Z & X \\ -X & \frac{1}{\sqrt{2}} Z^\dagger \end{pmatrix}^N , \quad (5)$$

where $Q^{(K)} = \{Z \in \mathbf{C}^{K \times K} \mid Z^T = -Z^T\}$ is the space of skew-symmetric complex matrices, $\lambda(Z, Z^\dagger)$ is the Lebesgue measure on $Q^{(K)}$ as described above. Note that the dimension of the integral in the right hand side of (5) is N-independent. The size of the original matrix only enters the integral as the power of the Pfaffian in the integrand. This allows one to calculate the large N-limit of (5) using Laplace method. To facilitate the application of asymptotic methods, let us re-scale the integration variables using $(Z, Z^\dagger) \to \sqrt{N}(Z, Z^\dagger)$, which gives

$$\mathbb{E}_N \left(\prod_{m=1}^{K} \det\left(M_1 - x_m\right) \right) = \pi^{-\frac{K(K-1)}{2}} 2^{-\frac{NK}{2}} N^{\frac{NK}{2}} N^{\frac{K(K-1)}{2}} J_N \quad (6)$$

where

$$J_N = \int_{Q^{(K)}} \lambda(dZ, dZ^\dagger) e^{-\frac{N}{2} Tr Z Z^\dagger} Pf \begin{pmatrix} Z & \sqrt{\frac{2}{N}} X \\ -\sqrt{\frac{2}{N}} X & Z^\dagger \end{pmatrix}^N . \quad (7)$$

The integrand in J_N is now of the form $\exp\{NF_N(Z,Z^\dagger)\}$, where F_N is a slow function of N.

Applying the Laplace method. We will show that the integral J_N localizes onto the subset

$$C^{(K)} = \{Z \in Q^{(K)} \mid ZZ^\dagger = I\}. \tag{8}$$

To do this it is convenient to split J into two parts,

$$J_{N,0} = \int_{Q^{(K)}\cap S} \qquad J_{N,1} = \int_{Q^{(K)}\setminus S}$$

where

$$S = \{Z \mid \mu_k(Z) \in [1/2, 2] \text{ for } k = 1, \ldots, K\}$$

and $(\mu_k(Z) : 1 \le k \le K\}$ are the singular values of Z. Note that S is compact and contains only non-singular matrices. We first bound the integral $J_{N,1}$, aiming to show that it is of smaller order that $J_{N,0}$. For $c \ge 0$ write $(\lambda_k(c) : 1 \le k \le 2K)$ for the singular values of the matrix

$$\begin{pmatrix} Z & cX \\ -cX & Z^\dagger \end{pmatrix}$$

One may bound the difference of singular values via $|\mu_k(A) - \mu_k(B)| \le \|A - B\|$ so that

$$|\lambda_k(c) - \lambda_k(0)| \le \|cX\| = cx_*$$

where $x_* = \max_k |x_k|$. Then

$$\begin{aligned}
\left| Pf \begin{pmatrix} Z & cX \\ -cX & Z^\dagger \end{pmatrix} \right| &= \left| \det \begin{pmatrix} Z & cX \\ -cX & Z^\dagger \end{pmatrix} \right|^{1/2} \\
&= \left| \prod_{k=1}^{2K} \lambda_k(c) \right|^{1/2} \\
&\le \left| \prod_{k=1}^{2K} (\lambda_k(0) + cx_*) \right|^{1/2} \\
&= \prod_{k=1}^{K} (\mu_k(Z) + cx_*).
\end{aligned}$$

Using this in $J_{N,1}$ we find

$$\begin{aligned}
J_{N,1} &\le \int_{Q^{(K)}\setminus S} \lambda(dZ, dZ^\dagger) e^{-\frac{N}{2} TrZZ^\dagger} \left| \prod_{k=1}^{K} \left(\mu_k(Z) + (2/N)^{1/2} x_*\right) \right|^N \\
&= \int_{Q^{(K)}\setminus S} \lambda(dZ, dZ^\dagger) e^{-TrZZ^\dagger} e^{-N \sum_{k=1}^{K} H_N(\mu_k(Z))}
\end{aligned}$$

where

$$H_N(z) = \left(\frac{1}{2} - \frac{1}{N}\right) z^2 - \ln\left(z + \left|\frac{2}{N}\right|^{1/2} x_*\right).$$

Note that $H_N(z) \to H(z) = \frac{1}{2}z^2 - \ln(z)$ and that $H(z)$ has a minimal value of $H(1) = \frac{1}{2}$. On $Q^{(K)} \setminus S$ there must exist at least one singular value μ lying outside $[\frac{1}{2}, 2]$, and for this value $H(\mu) \geq \frac{1}{2} + 2\delta$ for an easily calculated $\delta > 0$. For large N, when $Z \in Q^{(K)} \setminus S$, we have

$$\sum_{k=1}^{K} H_N(\mu_k(Z)) \geq \frac{K}{2} + \delta$$

and the value of $J_{N,1}$ is bounded by $C_k e^{-NK/2} e^{-N\delta}$. This is is exponentially smaller than that of $J_{N,0}$, which we will see is, to leading exponential order, $O(e^{-NK/2})$.

Next, we will calculate the asymptotic expansion of $J_{N,0}$ for large N. The Nth power of the Pfaffian in the integrand of $J_{N,0}$ can be simplified using the Taylor expansion for the Pfaffian:

$$\begin{aligned} \frac{Pf\left(A + \frac{1}{\sqrt{N}}B\right)}{Pf\,(A)} &= 1 + \frac{1}{2\sqrt{N}} TrBA^{-1} \\ &+ \frac{1}{8N}\left(TrBA^{-1}TrBA^{-1} - 2TrBA^{-1}BA^{-1}\right) + O\left(N^{-\frac{3}{2}}\right) \end{aligned} \tag{9}$$

While we cannot pinpoint the exact reference for the original derivation of the above expansion, it can be easily derived using the Berezin integral representation of the Pfaffian. It is interesting to note that unlike the analogous determinant expansion formula, the series (9) contains finitely many terms. With the help of (9), and the fact that the terms with $Tr(BA^{-1})$ are zero in our case, we can re-write the Nth power of the Pfaffian in the integrand of $J_{N,0}$ as follows:

$$\begin{aligned} Pf^N \begin{pmatrix} Z & \sqrt{\frac{2}{N}}X \\ -\sqrt{\frac{2}{N}}X & Z^\dagger \end{pmatrix} &= det^{\frac{N}{2}}(ZZ^\dagger) \cdot \left(1 + \frac{1}{N}Tr\left(Z^\dagger XZX\right) + O(N^{-2})\right)^N \\ &= det^{\frac{N}{2}}(ZZ^\dagger) \cdot e^{Tr\left(Z^\dagger XZX\right)}\left(1 + O(N^{-1})\right) \end{aligned}$$

This allows us to express $J_{N,0}$ in a form well suited for the application of Laplace formula:

$$J_{N,0} = \int_{Q^{(K)} \cap S} \Lambda(dZ, dZ^\dagger) e^{-\frac{N}{2}\left(TrZZ^\dagger - \ln\det\left(ZZ^\dagger\right)\right)} e^{Tr\left(Z^\dagger XZX\right)}(1 + O(N^{-1})). \tag{10}$$

The fact that S does not contain degenerate matrices, and the compactness of S allows one to pass the correction term through the integral. In the limit $N \to \infty$, the main contribution to (10) comes from the neighborhood of the points of global minimum of the function

$$F(Z) = TrZZ^{\dagger} - \ln det\left(ZZ^{\dagger}\right) = \sum_{k=1}^{K}\left(\mu_k^2(Z) - 2\ln(\mu_k(Z))\right). \qquad (11)$$

The global minimum value of F is K and it is attained on the set $C^{(K)}$ of skew-symmetric unitary $K \times K$ matrices, which is a smooth sub-manifold of $Q^{(K)}$. We will show that $C^{(K)}$ is a non-degenerate critical set, which means that the Hessian of F has the maximal possible rank at every point of $C^{(K)}$. Therefore we can use Laplace theorem [4] to calculate the asymptotic expansion of $J_{N,0}$: let (w, y) be local co-ordinates on $Q^{(K)}$ such that the sub-manifold $C^{(K)}$ is locally determined by the set of equations $y = 0$; then

$$\begin{aligned}&\int_{Q^{(K)}\cap S}\Lambda(dZ, dZ^{\dagger})e^{-\frac{N}{2}\left(TrZZ^{\dagger}-\ln\det\left(ZZ^{\dagger}\right)\right)}e^{Tr\left(Z^{\dagger}XZX\right)}\\ &= e^{-NF|_{C^{(K)}}}\left(\frac{1}{\sqrt{2\pi N}}\right)^{\dim(Q^{(K)})-\dim(C^{(K)})}\int_{C^{(K)}}\mu(dw)e^{Tr\left(Z^{\dagger}XZX\right)}. \qquad (12)\end{aligned}$$

Here $\mu(dw)$ is the measure on $C^{(K)}$ generated by the embedding $C^{(K)} \subset Q^{(K)}$ and integration over transverse co-ordinates y. Explicitly,

$$d\mu(w) = \frac{\rho(w, y=0)}{\sqrt{\det Hess(F)\mid_C^{(K)}(w)}}\prod_{k=1}^{dim(C^{(K)})}dw_k, \qquad (13)$$

where $\rho(w, y)$ is the density of Lebesgue measure $\Lambda(dZ, dZ^{\dagger})$ with respect to local co-ordinates (w, y), the Hessian is defined as the matrix of second derivatives with respect to transverse co-ordinates y. In writing (12) we used the fact that the critical manifold lies a positive distance away from the boundary of $Q^{(K)} \cap S$. Noting that F takes the value K on $C^{(K)}$ and that

$$\begin{aligned}\dim(Q^{(K)}) - \dim(C^{(K)}) &= K(K-1) - (\dim(U(K)) - \dim(Sp(K)))\\ &= K(K-1) - K^2 + \frac{1}{2}K(K+1) \qquad (14)\\ &= \frac{1}{2}K(K-1),\end{aligned}$$

we reach

$$J_{N,0} = e^{-\frac{NK}{2}}(2\pi N)^{-\frac{K(K-1)}{4}}\int_{C^{(K)}}\mu(dw)e^{Tr\left(Z^{\dagger}XZX\right)}(1+O(N^{-1})). \qquad (15)$$

Collecting together (3), (6) and (15) we find

$$\tilde{\rho}_N(x_1, x_2, \ldots, x_K) = c_2(N, K)\Delta(\mathbf{x})\prod_{k=1}^{K}e^{-x_k^2}\int_{C^{(K)}}\mu(dw)e^{Tr\left(Z^{\dagger}XZX\right)}\left(1+o(1)\right),$$

where

$$
\begin{aligned}
c_2(N,K) &= C(K)\prod_{k=1}^{K}\left(|S_{N-k}|\pi^{-\frac{N-k}{2}}\right)\pi^{-\frac{K(K-1)}{2}}2^{-\frac{(N-K)K}{2}} \\
&\qquad (N-K)^{\frac{(N-K)K}{2}}(N-K)^{\frac{K(K-1)}{2}}e^{-\frac{(N-K)K}{2}}(2\pi(N-K))^{-\frac{K(K-1)}{4}}
\end{aligned}
$$

and $C(K)$ denotes a constant only depending on K. It is lengthy but straightforward to check that $c_2(N,K)\to c_3(K)>0$ as $N\to\infty$ and hence that limiting modified density $\tilde{\rho}(x_1,x_2,\ldots,x_K)=\lim_{N\to\infty}\tilde{\rho}_N(x_1,x_2,\ldots,x_K)$ exists and is given by

$$
\tilde{\rho}(x_1,x_2,\ldots,x_K)=c_3(K)\Delta(\mathbf{x})\prod_{k=1}^{K}e^{-x_k^2}\int_{C^{(K)}}\mu(dw)e^{Tr\left(Z^{\dagger}XZX\right)}. \tag{16}
$$

The explicit value of $c_3(K)$ will be determined later by using properties of the densities $\tilde{\rho}$. In the next subsection we will find a parameterisation of the integral in the right hand side of (16), which will allow us to calculate it very efficiently using the standard tools of random matrix theory.

Recasting as an integral over the unitary group. An important property of the function F is its invariance with respect to the following action of the unitary group $U(K)$ on $Q^{(K)}$:

$$
\begin{aligned}
U(K)\times Q^{(K)} &\to U(K) && (17)\\
(U,Z) &\mapsto UZU^T && (18)
\end{aligned}
$$

Namely, for any $A\in U(K)$,

$$
F(AZA^T)=F(Z).
$$

The decomposition theorem for skew symmetric unitary matrices [8] states that

$$
Z=UJU^T, \tag{19}
$$

where U is a unitary matrix, J is the canonical symplectic matrix. Notice that (19) does not determine the unitary matrix U uniquely: indeed $Z\to Z$ if $U\to US$, where S is a unitary matrix: $SJS^T=J$. The set of such matrices is a subgroup of $U(K)$ called symplectic group $Sp(K)$:

$$
Sp(K)=\{S\in U(K)\mid SJS^T=J\}. \tag{20}
$$

It can be checked that the critical manifold $C^{(K)}$ can be identified with the factor space of $U(K)$ with respect to the action of $Sp(K)$ on $U(K)$ via right multiplications:

$$
C^{(K)}\cong U(K)/Sp(K) \tag{21}
$$

The $U(K)$-action (17) on $Q^{(K)}$ preserves the critical manifold and induces the $U(K)$-action on C. Using parameterisation (19) of $C^{(K)}$ this induced action can be written explicitly:

$$\begin{array}{rcl} U(K) \times C^{(K)} & \to & C^{(K)}, \\ (A, [U]) & \mapsto & [AU], \end{array} \tag{22}$$

where $[U]$ is an equivalence class of $U \in U(K)$ with respect to right multiplications by elements of $Sp(K) \subset U(K)$. In the vicinity of a critical point $Z_c \in C^{(K)}$,

$$F(Z_c + \delta Z) = K + \frac{1}{2} Tr(\delta Z Z_c^\dagger + Z_c \delta Z^\dagger)^2 + \dots \tag{23}$$

We notice that the quadratic form describing the second order term in the above Taylor expansion of F is $U(K)$-invariant and has the maximal possible rank equal to $\frac{1}{2}K(K-1) = \dim(Q^{(K)}) - \dim(C^{(K)})$. We rewrite the integral from (16), using the mapping (21), as

$$\int_{C^{(K)}} \mu(dw) e^{Tr\left(Z^\dagger X Z X\right)} = \int_{U(K)/Sp(K)} \hat{\mu}(dU) e^{Tr\left(Z(U)^\dagger X Z(U) X\right)}) \tag{24}$$

where $Z(U)$ is given by (19) and $\hat{\mu}(dU)$ is the pull back of the measure μ on the critical manifold. We can work out an explicit expression for μ in local co-ordinates on $C^{(K)}$ using the general formula (13). We will not do that. Instead we will characterise $\hat{\mu}$ up to a multiplicative constant by establishing its symmetry with respect to the $U(N)$-action on $C^{(K)}$: recall that the measure μ is determined by the Lebesgue measure on $Q^{(K)}$ and the determinant of the quadratic form in the right hand side of (23). As it is easy to check,

(i.) The Lebesgue measure Λ and the quadratic form $Tr(\delta Z Z^\dagger + Z \delta Z^\dagger)^2$ on $Q^{(K)}$ are invariant with respect to the $U(K)$-action (17).

(ii.) The critical manifold $C^{(K)}$ is invariant with respect to the $U(K)$-action.

(iii.) The restriction of the quadratic form $Tr(\delta Z Z^\dagger + Z \delta Z^\dagger)^2$ on $Q^{(K)}$ to $C^{(K)}$ has maximal rank.

A calculation employing elementary tools of differential geometry [3] shows that the above three observations imply the invariance of the measure $\hat{\mu}$ with respect to the induced action of $U(K)$ on the critical manifold $U(K)/Sp(K)$ defined by (22). Therefore $\hat{\mu}$ is a **Haar measure** on the symmetric space $U(K)/Sp(K)$, which is unique up to normalization. It is generally easier to work with integrals over the whole unitary group rather than a factor space. As we have established already the measure $\hat{\mu}$ is invariant with respect to the action of $U(K)$ on $C^{(K)}$. Note also that the function $Tr\left(Z(U)^\dagger XZ(U)X\right)$ which determines the integrand of (24) is also $U(K)$-invariant. Therefore, by Weyl's theorem [6], Chapter X,

$$\int_{U(K)/Sp(K)} \hat{\mu}(dU) e^{Tr\left(Z(U)^\dagger XZ(U)X\right)} = \int_{U(K)} \mu_H(dU) e^{Tr\left(Z(U)^\dagger XZ(U)X\right)} \tag{25}$$

where μ_H is an appropriately normalized Haar measure on the unitary group. We will determine the normalization factor later using the properties of spin-spin correlation functions. Substituting (19) into (25) we find that

$$\int_{U(K)} \mu_H(dU) e^{Tr\left(Z(U)^\dagger XZ(U)X\right)} = \int_{U(K)} \mu_H(dU) e^{-Tr\left(JHJH^T\right)} \tag{26}$$

where H is a Hermitian matrix with eigenvalues $x_1, x_2, \ldots, x_K$ given by $H = UXU^\dagger$. Tracing back we find that the large N behaviour of

$$\mathbb{E} \prod_{m=1}^{K} \det\left(M_1 - x_m\right)$$

- the expected value of the product of K characteristic polynomials in the real Ginibre ensemble - turns out to be determined by the integration of the symplectic-invariant Gaussian weight $\exp\{-TrJHJH^T\}$ with respect to unitary degrees of freedom. See [12] for a discussion of the origin of the connection between real Ginibre and symplectic ensembles. We may rewrite the integral in (26) as

$$\int_{U(K)} \mu_H(dU) e^{Tr\left(HH^R\right)} \tag{27}$$

where $H = UXU^\dagger$ and

$$H^R = JH^T J^T \tag{28}$$

is a 'symplectic' involution on the space of complex $K \times K$ matrices, see [9] for details. Following Mehta, we will call matrix M self-dual if $M = M^R$ and anti-self-dual if $M = -M^R$. It is easy to check that any even-dimensional matrix can be uniquely represented as a sum of a self-dual and anti-self-dual matrices. Let $ASD^{(K)}$ be the linear space of all anti-self dual $K \times K$ matrices. In order to perform the integration over the unitary group in the right hand side of (27), we use use the following transformation found in [11], [10]:

$$e^{TrHH^R} = Z_K e^{TrH^2} \int_{ASD^{(K)}} \Lambda(dA) e^{TrA^2 + 2\sqrt{2}TrHA}, \tag{29}$$

where $\Lambda(dA)$ is a Lebesgue measure on $ASD^{(K)}$, Z_K is a normalization constant. The absolute convergence of the above integral can be checked using the decomposition theorem for anti-self-dual matrices, see [8]: for any $A \in ASD^{(K)}$ there exists $V \in U(K)$ and a diagonal matrix Θ with entries $\pm\theta_1, \pm\theta_2, \dots \pm \theta_{\frac{K}{2}}$, where $\theta_1 \geq 0, \theta_2 \geq 0, \dots \theta_{K/2} \geq 0$, so that

$$A = iV\Theta V^{\dagger}. \tag{30}$$

Substituting (29) into (27) and using the invariance property of the Haar measure μ_H we get:

$$\begin{aligned} &\int_{U(K)} \mu_H(dU) e^{Tr\left(HH^R\right)} \\ = \;& C_K e^{\sum x_k^2} \int_{U(K)} \mu_H(dU) \int_{\mathbf{R}_+^{K/2}} \nu(d\theta) e^{-Tr\left(\Theta^2\right)} e^{i2\sqrt{2} Tr U X U^{\dagger} \Theta} \end{aligned} \tag{31}$$

where $\nu(d\theta)$ is the measure on the eigenvalues of unitary matrices induced by the marginalization of the Lebesgue measure on $ASD^{(K)}$ over the unitary degrees of freedom, explicitly [9]

$$\nu(d\theta) = \Delta\left(\pm\theta_1, \pm\theta_2, \dots \pm \theta_{\frac{K}{2}}\right) \prod_{k=1}^{K/2} \theta_k \, d\theta_k. \tag{32}$$

We now allow the constants C_K, depending only on K, to change value from line to line. Now we can integrate over the unitary group $U(K)$ using Harish-Chandra-Itzykson-Zuber formula [5], [7]. The result is

$$\begin{aligned} &\int_{U(K)} \mu_H(dU) e^{Tr\left(HH^R\right)} \\ = \;& C_K e^{\sum x_k^2} \Delta(x)^{-1} \int_{\mathbf{R}_+^{K/2}} \prod_{k=1}^{K/2} \theta_k e^{-\theta_k^2} \, d\theta_k \; det \left[e^{i2\sqrt{2} x_i \Theta_{jj}} \right]_{1 \leq i,j \leq K}. \end{aligned} \tag{33}$$

The remaining integration over the singular values $\theta_1, \theta_2, \dots, \theta_{K/2}$ is carried out using de Bruijn formula [2]:

$$\int_{U(K)} \mu_H(dU) e^{Tr\left(HH^R\right)} = C_K e^{\sum x_k^2} \Delta(x)^{-1} Pf \left[(x_i - x_j) e^{-2(x_i - x_j)^2} \right]_{1 \leq i,j \leq K}. \tag{34}$$

Combined with (16) this yields

$$\tilde{\rho}(x_1, x_2, \dots, x_K) = C_k Pf \left[(x_i - x_j) e^{-2(x_i - x_j)^2} \right]_{1 \leq i,j \leq K}. \tag{35}$$

Integrating all variables x_k for $k = 1, \dots, K$ as in (1) we find the spin-spin correlation

$$\mathbb{E} \prod_{k=1}^{K} s_{k_k}(M_1) = C_k Pf \left[\int_{x_i - x_j} e^{-2z^2} dz \right]_{1 \leq i,j \leq K}. \tag{36}$$

Now the constants C_k can be found inductively in k by allowing $x_{2k} \downarrow x_{2k-1}$, and noting that $\rho(x_1, x_1) = 1$. Expression (36) coincides with the continuous limit of single time correlation function of spin variables for the system of one dimensional annihilating Brownian motions under the maximal entrance law, see [14] for details. Differentiating (36) with respect to spatial variables and using the points according to

$$\rho_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n) = \left(-\frac{1}{2}\right)^n \left(\prod_{k=1}^{n} \frac{\partial}{\partial y_k}\right) \mathbb{E}\left(\prod_{m=1}^{n} s_{x_m}(M_{t_m})\, s_{x_m+y_m}(M_{t_m})\right)\Bigg|_{y_l=+0,\, l=1,2\ldots,n}, \quad (37)$$

we get the first statement of the Corollary 9 of [1].

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