Introduction

We consider a surface growth model,

\[ u_t = -u_{xxx} - \varepsilon^2 |u_x|^2 \quad (1) \]

on the one-dimensional torus \( T = (-\pi, \pi) \). We prove two partial regularity results for this model. As with so many results for this equation, these parallel those available for the three-dimensional Navier-Stokes equations.

We show that the set of space-time singularities \( S \) has 1-dimensional (pseudo-)Hausdorff measure zero. We use the rescaling approach of Ladyzhenskaya & Seregin (1999). See Oziński & Robinson (2017) for a detailed presentation of the following results.

Notation

A weak solution to the surface growth model (1) on the time interval \((0, T)\) is any \( u \in L^\infty(0, T), L^2(0, T, B^2) \) satisfying (1) in the sense of distributions.

A suitable weak solution is a weak solution satisfying the local energy inequality

\[ \int_0^t \int_Q u^2 \, dt \leq \int_0^t \int_Q \left( \frac{1}{2} u_{xx}^2 + 2u_x u_{xx} \right) + 2u_x u_{xxx} \, dt \]  

\( \phi \leq \frac{1}{2} u^2 - \frac{1}{2} u_x^2 \) for all nonnegative \( \phi \in C_0^\infty(T \times (0, T)) \). Fix a suitable weak solution \( u \) and let \( S \) denote its singular set.

We write \( z = (x, t) \) and denote the \( r \)-cylinder by \( Q_r(z) = \{ (x, t) \mid |x - x_0| < r, |t - t_0| < r \} \); see Fig. 1.

\[ Q_r(z) \]

\[ 2r \]

\[ 2r \]

\[ z = (x, t) \]

\[ 2r^4 \]

\[ 2r \]

\[ 2r \]

We write the mean of \( u \) over a cylinder \( Q_r(z) \) as

\[ \bar{u}_r = \frac{1}{|Q_r(z)|} \int_{Q_r(z)} u \, dx \]

and we also write

\[ Y(x, r) = \frac{1}{|Q_r(z)|} \int_{Q_r(z)} |u|^2 \]

Preliminary results

Lemma 1 (Parabolic Poincaré inequality (PPI))

Suppose that \( u \) satisfies the surface growth equation

\[ u_t = -u_{xxx} - \varepsilon^2 |u_x|^2 \quad (u \leq 1) \]  

in the sense of distributions then

\[ \frac{1}{\varepsilon} \int_{Q_r(z)} \left| u - u_0 \right|^2 \leq C \left( \varepsilon(Y(x, r) + \varepsilon Y(x, r)^2) \right) \]  

Note there is no time derivative on the right-hand side.

Lemma 2 (Inner boundedness of the biharmonic flow)

Suppose that \( v \) is a solution to the biharmonic equation

\[ \nabla^2 v = \nabla^2 \phi \quad (v \leq 1) \]  

in the sense of distributions, and such that \( v \in L^2(Q_r(z)) \). Then

\[ \int_{Q_r(z)} \left( |v|^2 \leq C \left( \int_{Q_r(z)} v^2 + 1 \right) \right) \]

\[ \text{Corollary 5. There exist } \varepsilon > 0 \text{ and } R \text{ such that if } r < R \text{ and } \]

\[ \varepsilon \left( \int_{Q_r(z)} |u|^2 \right) \leq \frac{1}{\varepsilon} \left( \int_{Q_r(z)} |u_x|^2 \right) \]

then

\[ \frac{1}{\varepsilon} \int_{Q_r(z)} |u_x|^2 \leq C \varepsilon \frac{1}{\varepsilon} \left( \int_{Q_r(z)} |u_x|^2 \right) \]

for all \( \varepsilon \leq 1 \) and \( x \in Q_r(z) \).

The 1st regularity result

Lemma 3 (Campanato lemma)

If \( p \geq 1, r > 0, \alpha \in [0, 1] \) and \( u \) satisfies

\[ \left( \frac{1}{2} \right) \int_{Q_r(z)} \left( |u - u_0|^2 \right)^{1/2} \leq C \varepsilon p \]

for all \( \varepsilon \leq 1 \) and \( x \in Q_r(z) \), then \( u \) is Hölder continuous in \( Q_r(z) \) with

\[ |u(x, t) - u(y, s)| \leq C \left( |x - y| + |t - s|^{1/4} \right) \]

for all \( (x, t), (y, s) \in Q_r(z) \).

The main iteration

Proposition 4. Given \( \theta \in (0, 1/4) \) there exist \( \varepsilon > 0 \) such that if \( r < R \) and

\[ Y(x, r) \leq \theta Y(x, r) \]

then

\[ Y(x, r) \leq \theta Y(x, r) \]

where \( \varepsilon \) is a universal constant.

Proof. Choose a particular value of \( \varepsilon \) (see Step 1 below) and suppose the claim is false, that is, for some \( \varepsilon > 0 \) there exist \( x \in Q_r(z) \) and \( \varepsilon_0 > 0 \) such that

\[ Y(x, r) > \varepsilon_0 \]

and \( Y(x, r) \) is not singular.

The 2nd local regularity result

Theorem 2. (2nd local regularity of the surface growth model)

There exists \( \varepsilon > 0 \) such that if

\[ \lim_{r \to 0} \int_{Q_r(z)} u^2 \leq \varepsilon \]

then \( u \) is regular at \( z \).

Proof. Set

\[ A(r) = \sup_{r < s < T} \frac{1}{T-s} \int_s^T |u^2| ds \]

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References


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