Partial regularity theory for the incompressible Navier-Stokes equations and counterexamples

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The Navier–Stokes equations

\[ u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \]
\[ \text{div } u = 0, \]
\[ u(0) = u_0, \]

where:
\[ u : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \text{ - velocity field,} \]
\[ p : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R} \text{ - pressure function,} \]
\[ \nu > 0 \text{ - kinematic viscosity.} \]
Existence and uniqueness of solutions

$$\|\nabla u_0\|_{L^2}$$
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\[ \| \nabla u_0 \|_{L^2} \quad \| \nabla u(t) \|_{L^2} \]
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Existence and uniqueness of solutions

Leray weak solution

\[ \|\nabla u_0\|_{L^2} \|\nabla u(t)\|_{L^2} \]

\[ \|u_0\|_{L^2} \|u(t)\|_{L^2} \]

Leray (1934)  [O. & Pooley (2017)]
Existence and uniqueness of solutions
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\[ \| \nabla u_0 \|_{L^2} \]

\[ \| u(t) \|_{L^2} \]

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Leray weak solution

\[ \| \nabla u \|_{L^2} \]

\[ \| u(t) \|_{L^2} \]

\[ \| u_0 \|_{L^2} \]

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Existence and uniqueness of solutions

Leray weak solution

\[ \|\nabla u_0\|_{L^2} \]
\[ \|\nabla u(t)\|_{L^2} \]
\[ \|u_0\|_{L^2} \]
\[ \|u(t)\|_{L^2} \]

\[ t_0 \quad t_1 \quad t_2 \]
Existence and uniqueness of solutions

\[ \| \nabla u_0 \|_{L^2} \quad \| \nabla u(t) \|_{L^2} \]

\[ \| u_0 \|_{L^2} \quad \| u(t) \|_{L^2} \]

Leray weak solution

\[ t_0 \quad t_1 \quad t_2 \quad T \]
Existence and uniqueness of solutions

\[ \| \nabla u_0 \|_{L^2} \quad \| \nabla u(t) \|_{L^2} \]

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Leray weak solution

\[ \| \nabla u_0 \|_{L^2} \quad \| u(t) \|_{L^2} \]

\[ t_0 \quad t_1 \quad t_2 \quad T \]
Existence and uniqueness of solutions

\[ \dim(\mathcal{T}) \leq 1/2, \quad \text{where} \quad \mathcal{T} := \{ t > 0 : \| \nabla u(t) \|_{L^2} = \infty \} \]
Partial regularity theory

Let

\[ S := \{ (x, t) : u \text{ is unbounded in any neighbourhood of } (x, t) \}. \]
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Scheffer (1985 & 1987): constructions of weak solutions to the Navier–Stokes inequality,

\[
u \cdot (u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p) \leq 0,
\]

which show that the bound \( \dim(S) \leq 1 \) is sharp for such solutions.
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O. (2017): constructions satisfying the “approximate equality”

\[ -\vartheta \leq u \cdot (u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p) \leq 0 \]

for any preassigned \( \vartheta > 0. \)
The counterexample
by Scheffer (1987), see also Theorem 2 in O. (2017)

Theorem (Weak solutions of NSI with singularities)
There exist $\nu_0 > 0$ and a function $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ such that

(i) $u(t) \in C^\infty$, $\text{div } u(t) = 0$ and $\text{supp } u(t) \subset G$ for all $t$, where $G \subset \mathbb{R}^3$ is compact,

(ii) $u \in L^\infty(L^2) \cap L^2(H^1) \cap L^3(L^3)$, and $u p \in L^1(L^1)$, where

$$p(x, t) := \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \frac{\partial_i u_j(y, t) \partial_j u_i(y, t)}{4\pi|x-y|} \, dy.$$ 

(iii) $u$ satisfies the Navier–Stokes inequality for all $\nu \in [0, \nu_0]$,

(iv) $u$ is unbounded in every neighbourhood of $S \times \{T_0\}$, where $S \subset \mathbb{R}^3$ is a uniform Cantor set with $\dim(S) \geq \xi$ for any preassigned $\xi \in (0, 1)$. 

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\[ \text{supp } u(0)(\cdot, t) \]
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