Non-uniqueness for the transport equation with Sobolev vector fields

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where

$$\begin{aligned} &u : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d \quad \text{given vector field,} \quad \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \\ &\rho : [0, T] \times \mathbb{T}^d \to \mathbb{R} \quad \text{unknown density.} \end{aligned}$$

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$$\operatorname{div} u = 0.$$

Then (T) is equivalent to the **continuity equation**:

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2. is the relation (1) still valid (in some sense)?

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Uniqueness:

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Thm. (M., Sattig, Székelyhidi, 2017-2018)

Let $p \in [1,\infty)$, $\tilde{p} \in [1,\infty)$. If

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d},$$

then there exist infinitely many incompressible vector fields

$$u \in C_t L_x^{p'} \cap C_t W_x^{1,\tilde{p}}$$

for which uniqueness of solutions to the transport equation

$$\partial_t \rho + \nabla \rho \cdot u = 0, \quad \operatorname{div} u = 0,$$

fails in the class of densities $\rho \in C_t L_x^p$.

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- if $p = 1, p' = \infty$, then $u \in C\big([0, T] \times \mathbb{T}^d\big) \cap C_t W_x^{1,\tilde{p}}$;
- the same result holds for the transport-diffusion equation

$$\partial_t \rho + \operatorname{div}(\rho u) = \Delta \rho, \quad \operatorname{div} u = 0,$$

  if, in addition, $p' < d$.