Space-time localisation for the dynamic Φ^4_3 model

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PDE and their Applications Seminar

Joint with Hendrik Weber

The Φ^4 equation

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$$(\partial_{\widehat{t}} - \Delta)\widehat{\phi} = -\lambda^{4-d}\widehat{\phi}^3 + \widehat{\xi}.$$

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Local theory

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- HAIRER: Regularity Structures. For x close to y, $\phi(y) \approx \mathbf{I}(x) + v(x) - (\mathbf{\Psi}(y) - \mathbf{\Psi}(x)) - 3v(x)(\mathbf{Y}(y) - \mathbf{Y}(x)) + \nu(x).(y - x)$

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- GUBINELLI-HOFMANOVÀ: Global Solution on $\mathbb{R}^3$.

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Product is well defined : $C^\alpha \times C^{-\beta} \quad \Leftrightarrow \quad \alpha - \beta > 0.$

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$$C_1 = \mathbb{E}[\<1>^2] \sim \frac{1}{\delta} \Rightarrow \<2> := \<1>^2 - C_1$$

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Description

$$(\partial_t - \Delta)v = -v^3 - 3v^2 \Psi - 3v \Psi^{2} - \Psi^{3}$$

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$$(\partial_t - \Delta) v = -v^3 - 3v^2 \stackrel{\bullet}{|} - 3v \stackrel{\bullet\bullet}{\vee} - \stackrel{\bullet\bullet\bullet}{\Psi}$$

We keep expanding with the worst term:

$$(\partial_t - \Delta) \stackrel{\bullet\bullet\bullet}{\Upsilon} = \stackrel{\bullet\bullet\bullet}{\Psi}, \quad \stackrel{\bullet\bullet\bullet}{\Upsilon} \in C^{\frac{1}{2} - 3\epsilon},$$

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$$(\partial_t - \Delta)\<30> = \<3>, \quad \<30> \in C^{\frac{1}{2}-3\epsilon},$$

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- $v + \<30> \in C^{1-2\epsilon} \Rightarrow v(v + \<30>)\<1>$ OK , $\<30>\<1>$ needed

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Freezing of coefficients at base-point $x \in \mathbb{R} \times R^3$:

$$(\partial_t - \Delta)(v + \<30> + 3v(x)\<20>) = -v^3 - 3v^2 \<1> - 3(v - v(x))\<2>.$$

- $v + \<30> \in C^{1-2\epsilon} \Rightarrow v(v + \<30>)\<1>$ OK , $\<30>\<1>$ needed

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$$(\partial_t - \Delta)v = -v^3 - 3v^2\Psi - 3v\Psi^2 - \Psi^3$$

We keep expanding with the worst term:

$$(\partial_t - \Delta)\mathcal{I}(\Psi^3) = \Psi^3, \quad \mathcal{I}(\Psi^3) \in C^{\frac{1}{2}-3\epsilon}, \qquad (\partial_t - \Delta)\mathcal{I}(\Psi^2) = \Psi^2, \quad \mathcal{I}(\Psi^2) \in C^{1-2\epsilon}$$

Freezing of coefficients at base-point $x \in \mathbb{R} \times R^3$:

$$(\partial_t - \Delta)(v + \mathcal{I}(\Psi^3) + 3v(x)\mathcal{I}(\Psi^2)) = -v^3 - 3v^2\Psi - 3(v - v(x))\Psi^2.$$

- $v + \mathcal{I}(\Psi^3) \in C^{1-2\epsilon} \Rightarrow v(v + \mathcal{I}(\Psi^3))\Psi$ OK , $\mathcal{I}(\Psi^3)\Psi$ needed
- $v + \mathcal{I}(\Psi^3) + v(x)\mathcal{I}(\Psi^2) \in C^{\frac{3}{2}-5\epsilon}$

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Freezing of coefficients at base-point $x \in \mathbb{R} \times R^3$:

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More renormalisation

$$C_2 = \mathbb{E}[\text{VY}] \sim -\log(\delta)$$

More renormalisation

$$C_2 = \mathbb{E}[\mathrm{V}\,\mathrm{Y}] \sim -\log(\delta)$$

We introduce more diagrams, which requires negative renormalisation.

$$\mathrm{V}_x(y) = \quad (y \quad)\mathrm{V}(y)$$

$$\mathrm{V}\!\!\mathrm{V}(y) = \quad \mathrm{V}(y)\ \mathrm{Y}(y) \qquad - C_2$$

$$\mathrm{Y}\!\!\mathrm{V}(y) = \quad \mathrm{I}(y)\ \mathrm{Y}(y)$$

$$\mathrm{V}\!\!\mathrm{Y}(y) = \quad \mathrm{V}(y)\ \mathrm{Y}(y) \qquad - 3C_2\,\mathrm{I}(y)$$

More renormalisation

$$C_2 = \mathbb{E}[\vee\curlyvee] \sim -\log(\delta)$$

We introduce more diagrams, which requires negative and positive renormalisation.

$$\vee_x(y) = \quad (y - x)\vee(y)$$

$$\overset{\vee}{\vee}(y) = \quad \vee(y)(\curlyvee(y) - \curlyvee(x)) - C_2$$

$$\overset{\Psi}{\vee}(y) = \quad \dagger(y)(\Psi(y) - \Psi(x))$$

$$\overset{\Psi}{\overset{}{\vee}}(y) = \quad \vee(y)(\Psi(y) - \Psi(x)) - 3C_2\dagger(y)$$

More renormalisation

$$C_2 = \mathbb{E}[\mathsf{V}\mathsf{Y}] \sim -\log(\delta)$$

We introduce more diagrams, which requires negative and positive renormalisation. We make the following assumptions:

$$\mathsf{V}_x \qquad \left| \int (y - x)\mathsf{V}(y)\Psi_T(y-x)dy \right| \lesssim T^{-2\epsilon},$$

$$\mathsf{V}\mathsf{Y} \qquad \left| \int (\mathsf{V}(y)(\mathsf{Y}(y) - \mathsf{Y}(x)) - C_2)\Psi_T(y-x)dy \right| \lesssim T^{-4\epsilon},$$

$$\mathsf{I}\mathsf{\Psi} \qquad \left| \int (\mathsf{I}(y)(\mathsf{\Psi}(y) - \mathsf{\Psi}(x)))\Psi_T(y-x)dy \right| \lesssim T^{-4\epsilon},$$

$$\mathsf{V}\mathsf{\Psi} \qquad \left| \int (\mathsf{V}(y)(\mathsf{\Psi}(y) - \mathsf{\Psi}(x)) - 3C_2\mathsf{I}(y))\Psi_T(y-x)dy \right| \lesssim T^{-\frac{1}{2}-5\epsilon},$$

More renormalisation

$$C_2 = \mathbb{E}[\RSV\RSY] \sim -\log(\delta)$$

We introduce more diagrams, which requires negative and positive renormalisation. We make the following assumptions:

$$\RSVx \qquad \left|\int (y-x)\RSV(y)\Psi_T(y-x)dy\right| \lesssim T^{-2\epsilon},$$

$$\RSVVY \qquad \left|\int (\RSV(y)(\RSY(y)-\RSY(x))-C_2)\Psi_T(y-x)dy\right| \lesssim T^{-4\epsilon},$$

$$\RSIW \qquad \left|\int (\RSI(y)(\RSW(y)-\RSW(x)))\Psi_T(y-x)dy\right| \lesssim T^{-4\epsilon},$$

$$\RSVW \qquad \left|\int (\RSV(y)(\RSW(y)-\RSW(x))-3C_2\RSI(y))\Psi_T(y-x)dy\right| \lesssim T^{-\frac{1}{2}-5\epsilon},$$

$$\begin{aligned}(\partial_t-\Delta)(v+\RSW+3v(x)\RSY) = -v^3 &- 3(v^2-v^2(x)-v(x)(\RSW-\RSW(x)))\RSI \\ -3(v-v(x)-(\RSW-\RSW(x)) &- 3v(x)(\RSY-\RSY(x))-\nu(x)(y-x))\RSV \\ -3v^2(x)\RSI - 3v(x)\RSIW &- 3\RSVW - 9v(x)\RSVVY - 3\nu(x)\RSVx .\end{aligned}$$

Main result

$$(\partial_t - \Delta)v = -v^3 - 3v^2\mathsf{I} - 3v\mathsf{V} - 9C_2(v + \mathsf{I}) - \Psi. \qquad (1)$$

Main result

$$(\partial_t - \Delta)v = -v^3 - 3v^2 \<1> - 3v \<2> - 9C_2(v + \<1>) - \<3>. \qquad (1)$$

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Main result

$$(\partial_t - \Delta)v = -v^3 - 3v^2 \treeOne - 3v \treeTwo - {\color{red} 9C_2(v + \treeOne)} - \treeThree. \qquad (1)$$

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Theorem (M-Weber, 2018)

If v solves (1) pointwise on $P = (0,1) \times \{|x| < 1\}$, then we have:

$$\|v\|_{P_R} \leqslant C \max\left\{ \frac{1}{R}, [\tau]_{|\tau|}^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}, \tau \in \{\treeOne, \treeTwo, \treeThree, \treeTwoTwo, \treeThreeOne, \treeThreeTwo\} \right\}.$$

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Discussion

Main result

$$(\partial_t - \Delta)v = -v^3 - 3v^2\Psi - 3v\Psi^2 - 9C_2(v + \Psi) - \Psi^3. \qquad (1)$$

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Discussion

- Uniform over any space-time boundary condition.

Main result

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Discussion

- Uniform over any space-time boundary condition.
- Depends only on the noise on P.

Main result

$$(\partial_t - \Delta)v = -v^3 - 3v^2 \RSI - 3v \RSV - 9C_2(v + \RSI) - \RSW. \qquad (1)$$

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Discussion

- Uniform over any space-time boundary condition.
- Depends only on the noise on P.
- Gives stretched exponential bounds:

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Discussion

- Uniform over any space-time boundary condition.
- Depends only on the noise on P.
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One dimensional case

More regular situation

$$(\partial_t - \Delta)u = -|u|^{m-1}u + \xi \qquad m > 1. \tag{RD}$$

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- **Example:** $\xi =$ space-time white noise over $\mathbb{R}_x \times \mathbb{R}_t$. Then $\alpha = \frac{1}{2} - \varepsilon$.

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Theorem M., Weber

$$\|u\|_{P_R} \lesssim \max\left\{R^{-\frac{2}{m-1}}, [\zeta]_{\alpha-2,P_0}^{\frac{2}{2+(m-1)\alpha}}\right\}.$$

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- Interplay between regularity of ξ and exponent. Ultimately corresponds to stochastic integrability of u.

Maximum Principle

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$$\begin{aligned}(\partial_t - \Delta) v_{T_0} = &- (v_{T_0})^3 - 3(v^2 \mathsf{I})_{T_0} - 3(v \mathsf{V})_{T_0} - 9C_2(v_{T_0} + \mathsf{I}_{T_0}) - (\mathsf{W})_{T_0} \\ &+ ((v_{T_0})^3 - (v^3)_{T_0}).\end{aligned}$$

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$$\begin{aligned}\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{ &\frac{1}{R}, \|(v_{T_0})^3 - (v^3)_{T_0}\|_P^{\frac{1}{3}}, \|(v^2 \<1>)_{T_0}\|_P^{\frac{1}{3}}, \\ &\|(v \<2>)_{T_0} + 3C_2(v_{T_0} + \<1>_{T_0})\|_P^{\frac{1}{3}}, \|(\<3>)_{T_0}\|_P^{\frac{1}{3}} \Big\}.\end{aligned}$$

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$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{\frac{1}{R}, \|v\|^{\frac{2}{3}} [v]_{\frac{1}{2}-3\epsilon}^{\frac{1}{3}} T_0^{\frac{1}{6}-\epsilon}, \|(v^2 \dot{|})_{T_0}\|_P^{\frac{1}{3}}, \|(v \ddot{\vee})_{T_0} + 3C_2(v_{T_0} + \dot{|}_{T_0})\|_P^{\frac{1}{3}}, [\dddot{\Psi}]_{-\frac{3}{2}-3\epsilon}^{\frac{1}{3}} T_0^{-\frac{1}{2}-\epsilon}\Big\}.$$

Choice of T_0

$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{ \frac{1}{R}, \|v\|^{\frac{2}{3}} [v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, \|(v^2 \stackrel{\bullet}{|})_{T_0}\|_P^{\frac{1}{3}},$$

$$\|(v \stackrel{\bullet\bullet}{\vee})_{T_0} + 3C_2(v_{T_0} + \stackrel{\bullet}{|}_{T_0})\|_P^{\frac{1}{3}}, [\stackrel{\bullet\bullet\bullet}{\Psi}]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon} \Big\}.$$

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$$\|(v\<2>)_{T_0} + 3C_2(v_{T_0} + \<1>_{T_0})\|^{\frac{1}{3}}_{P}, [\<3>]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon} \Big\}.$$

$$\|v\|_{P_{R-T_0}} \lesssim \max\Big\{ \frac{1}{R}, \|v\|^{\frac{2}{3}} [v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, [v]_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{2}-3\epsilon}, \|(v^2 \<1>)_{T_0}\|^{\frac{1}{3}}_{P},$$

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Choice of T_0

$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{\frac{1}{R}, \|v\|^{\frac{2}{3}}[v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, \|(v^2\mathbf{I})_{T_0}\|_P^{\frac{1}{3}},$$

$$\|(v\mathbf{V})_{T_0} + 3C_2(v_{T_0} + \mathbf{I}_{T_0})\|_P^{\frac{1}{3}}, [\mathbf{W}]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon}\Big\}.$$

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$$\|v\|_{P_R} \leqslant C \max\left\{\frac{1}{R}, [\tau]_{|\tau|}^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}, \tau \in \{\mathbf{I}, \mathbf{V}, \mathbf{W}, \tau_4, \tau_5, \tau_6\}\right\}.$$

Choice of T_0

$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{\frac{1}{R}, \|v\|^{\frac{2}{3}}[v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, \|(v^2 \RSI)_{T_0}\|_P^{\frac{1}{3}},$$
$$\|(v\RSV)_{T_0} + 3C_2(v_{T_0} + \RSI_{T_0})\|_P^{\frac{1}{3}}, [\RSW]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon}\Big\}.$$

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$$\|(v\RSV)_{T_0} + 3C_2(v_{T_0} + \RSI_{T_0})\|_P^{\frac{1}{3}}, [\RSW]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon}\Big\}.$$

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Assume $[\tau]_{|\tau|} \leqslant c\|v\|_{P_R}^{n_\tau(\frac{1}{2}-\epsilon)}, \tau \in \{\RSI, \RSV, \RSW, \RSVV, \RSWV, \RSWW\}$,

Choice of T_0

$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{\frac{1}{R}, \|v\|^{\frac{2}{3}}[v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, \|(v^2 \<1>)_{T_0}\|^{\frac{1}{2}}_{P},$$

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Assume $[\tau]_{|\tau|} \leqslant c\|v\|^{n_\tau(\frac{1}{2}-\epsilon)}_{P_R}, \tau \in \{\<1>, \<2>, \<3>, \<4a>, \<4b>, \<5>\}$,then

$$[\<3>]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon} \leqslant c\|v\|^{\frac{1}{2}-\epsilon}_{P_R} T_0^{-\frac{1}{2}-\epsilon}$$

Choice of T_0

$$\|(v)_{T_0}\|_{P_R} \lesssim \max\Big\{\frac{1}{R}, \|v\|^{\frac{2}{3}}[v]^{\frac{1}{3}}_{\frac{1}{2}-3\epsilon} T_0^{\frac{1}{6}-\epsilon}, \|(v^2\dot{\mathrm{I}})_{T_0}\|^{\frac{1}{3}}_{P},$$

$$\|(v\mathrm{V})_{T_0} + 3C_2(v_{T_0} + \dot{\mathrm{I}}_{T_0})\|^{\frac{1}{3}}_{P}, [\Psi]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon}\Big\}.$$

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$$\|v\|_{P_R} \leqslant C \max\left\{\frac{1}{R}, [\tau]^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}_{|\tau|}, \tau \in \{\dot{\mathrm{I}}, \mathrm{V}, \Psi, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6\}\right\}.$$

Assume $[\tau]_{|\tau|} \leqslant c\|v\|^{n_\tau(\frac{1}{2}-\epsilon)}_{P_R}, \tau \in \{\dot{\mathrm{I}}, \mathrm{V}, \Psi, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6\}$,then

$$[\Psi]^{\frac{1}{3}}_{-\frac{3}{2}-3\epsilon} T_0^{-\frac{1}{2}-\epsilon} \leqslant c\|v\|^{\frac{1}{2}-\epsilon}_{P_R} T_0^{-\frac{1}{2}-\epsilon}$$

$T_0 = \frac{\mu}{\|v\|_{P_R}} \Rightarrow$ absorb in the left-hand side.

Reconstruction

Theorem

- $\gamma > 0$;
- $A \subset (-\infty, \gamma]$;

For $t \leqslant T$

$$\Big| \int \Psi_t(x_2 - y)(F(x_1, y) - F(x_2, y))dy \Big| \lesssim \sum_{\beta \in A} d(x_1, x_2)^{\gamma - \beta} t^{\beta},$$

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f is the reconstruction of F

Proof of the Reconstruction

Special choice of kernel Ψ defined by

$$\Psi_{2T} = \Phi_T * \Psi_T, \Psi = \Phi_{\frac{T}{2}} * \Phi_{\frac{T}{4}} * \Phi_{\frac{T}{8}} \dots$$

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$$\begin{aligned}&\left|[F, (\cdot)_{T2^{-n}}](x_1) - \Big([F, (\cdot)_{T2^{-n-1}}]\Big)_{T2^{-n},1}(x_1)\right| \\ &= \left| \int \int \Psi_{T2^{-n-1}}(x_2 - y)\Phi_{T2^{-n-1}}(x_1 - x_2)(F(x_1,y) - F(x_2,y)) dy dx_2 \right|\end{aligned}$$

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&\leqslant \sum_{\beta \in A} C_\beta \int \Phi_{T2^{-n-1}}(x_1 - x_2) d(x_1, x_2)^{\gamma - \beta} (T2^{-n-1})^\beta dx_2 \\
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&\leqslant \sum_{\beta \in A} C_\beta \int \Phi_{T2^{-n-1}}(x_1 - x_2) d(x_1, x_2)^{\gamma - \beta} (T2^{-n-1})^\beta dx_2 \\
&\leqslant \sum_{\beta \in A} C_\beta (T2^{-n-1})^\gamma.
\end{aligned}$$

$$\Rightarrow [F, (\cdot)_{T2^{-n}}] \overset{n \to \infty}{\longrightarrow} f : y \mapsto F(y, y)$$

Proof of the Reconstruction

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$$\begin{aligned} &\Big| [F, (\cdot)_T] - [F, (\cdot)_{T2^{-N}}]_{T,N-1} \Big| \\ &= \Big| \sum_{n=0}^{N} \Big([F, (\cdot)_{T2^{-n}}] - [F, (\cdot)_{T2^{-n-1}}]_{T2^{-n},1} \Big)_{T,n} \Big| \\ &\leqslant \sum_{n=0}^{N} \sum_{\beta \in A} C_\beta (T2^{-n-1})^\gamma \lesssim T^\gamma, \end{aligned}$$

$$((v - v(x))\overset{\bullet\bullet}{\vee})_{T_0}(x) + 3C_2(v_{T_0}(x) + \overset{\bullet}{\mid}_{T_0}(x)) =$$

$$((v - v(x))\vee)_{T_0}(x) + 3C_2(v_{T_0}(x) + \mathsf{I}_{T_0}(x)) =$$

$$(\underbrace{(v - v(x) + \Psi - \Psi(x) + 3v(x)(\Upsilon - \Upsilon(x)) - \nu(x).X(\cdot - x)}\vee)_{T_0}(x)$$

$$- 3C_2(v - v_{T_0})(x) - \underbrace{((\Psi - \Psi(x))\vee - 3C_2\mathsf{I})_{T_0}(x)}$$

$$- 3v(x)\underbrace{((\Upsilon - \Upsilon(x))\vee - 3C_2)_{T_0}(x)} + \nu(x).\underbrace{(X(\cdot - x)\vee)_{T_0}(x)},$$

$$
\begin{aligned}
&((v - v(x))\<2>)_{T_0}(x) + 3C_2(v_{T_0}(x) + \<1>_{T_0}(x)) = \\
&(\underbrace{(v - v(x) + \<30> - \<30>(x) + 3v(x)(\<20> - \<20>(x)) - \nu(x).X(\cdot - x))}\<2>)_{T_0}(x) \\
&- 3C_2(v - v_{T_0})(x) - \underbrace{((\<30> - \<30>(x))\<2> - 3C_2\<1>)_{T_0}(x)}_{\lesssim\, T_0^{-\frac{1}{2}-5\epsilon}[\<32>]_{-\frac{1}{2}-5\epsilon}} \\
&- 3v(x)\underbrace{((\<20> - \<20>(x))\<2> - 3C_2)_{T_0}(x)}_{\lesssim\, T_0^{-4\epsilon}[\<22>]_{-4\epsilon}} + \nu(x).\underbrace{(X(\cdot - x)\<2>)_{T_0}(x)}_{\lesssim\, T_0^{-2\epsilon}[\<2X>]_{-2\epsilon}},
\end{aligned}
$$

$$((v - v(x))\curlyvee)_{T_0}(x) + 3C_2(v_{T_0}(x) + \mathrm{I}_{T_0}(x)) =$$

$$(\underbrace{(v - v(x) + \Psi - \Psi(x) + 3v(x)(\Upsilon - \Upsilon(x)) - \nu(x).X(\cdot - x)}_{=:\, U(x,\cdot)}\curlyvee)_{T_0}(x)$$

$$- 3C_2(v - v_{T_0})(x) - \underbrace{((\Psi - \Psi(x))\curlyvee - 3C_2\mathrm{I})_{T_0}(x)}_{\lesssim\, {T_0}^{-\frac{1}{2}-5\epsilon}[\Psi\curlyvee]_{-\frac{1}{2}-5\epsilon}}$$

$$- 3v(x)\underbrace{((\Upsilon - \Upsilon(x))\curlyvee - 3C_2)_{T_0}(x)}_{\lesssim\, {T_0}^{-4\epsilon}[\Upsilon\curlyvee]_{-4\epsilon}} + \nu(x).\underbrace{(X(\cdot - x)\curlyvee)_{T_0}(x)}_{\lesssim\, {T_0}^{-2\epsilon}[\curlyvee_x]_{-2\epsilon}},$$

$$((v - v(x))\mathsf{V})_{T_0}(x) + 3C_2(v_{T_0}(x) + \mathsf{I}_{T_0}(x)) =$$

$$(\underbrace{(v - v(x) + \Psi - \Psi(x) + 3v(x)(\mathsf{Y} - \mathsf{Y}(x)) - \nu(x).X(\cdot - x)}_{=: U(x,\cdot)})\mathsf{V})_{T_0}(x)$$

$$- 3C_2(v - v_{T_0})(x) - \underbrace{((\Psi - \Psi(x))\mathsf{V} - 3C_2\mathsf{I})_{T_0}(x)}_{\lesssim T_0^{-\frac{1}{2}-5\epsilon}[\mathsf{W}]_{-\frac{1}{2}-5\epsilon}}$$

$$- 3v(x)\underbrace{((\mathsf{Y} - \mathsf{Y}(x))\mathsf{V} - 3C_2)_{T_0}(x)}_{\lesssim T_0^{-4\epsilon}[\mathsf{W}']_{-4\epsilon}} + \nu(x).\underbrace{(X(\cdot - x)\mathsf{V})_{T_0}(x)}_{\lesssim T_0^{-2\epsilon}[\mathsf{V}_x]_{-2\epsilon}},$$

To apply the reconstruction lemma we set

$$F(x_1, y) = (v(x_1) + \Psi(x_1) - \Psi(y) + 3v(x_1)(\mathsf{Y}(x_1) - \mathsf{Y}(y)) - \nu(x_1).X(x_1 - y))\mathsf{V}(y) - 3C_2 v(x_1).$$

We have

$$
\begin{aligned}
F(x_1, y) - F(x_2, y) = 3(v(x_1) - v(x_2))((\overset{\bullet\bullet}{\mathrm{Y}}(y) - \overset{\bullet\bullet}{\mathrm{Y}}(x_2))\overset{\bullet\bullet}{\mathrm{V}}(y) - C_2) \\
+ U(x_1, x_2)\overset{\bullet\bullet}{\mathrm{V}}(y) - (\nu(x_1) - \nu(x_2)).X(y - x_2)\overset{\bullet\bullet}{\mathrm{V}}(y).
\end{aligned}
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\end{aligned}
$$

This gives

$$
\int \Psi_t(x_2 - y)(F(x_1, y) - F(x_2, y))dy \leqslant 3[v]_{\frac{1}{2}-3\epsilon} d(x_1, x_2)^{\frac{1}{2}-3\epsilon}[\Upsilon\vee]_{-4\epsilon} t^{-4\epsilon}
$$

$$
+[U]_{\frac{3}{2}-5\epsilon} d(x_1, x_2)^{\frac{3}{2}-5\epsilon}[\vee]_{-1-2\epsilon} t^{-1-2\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon} d(x_1, x_2)^{\frac{1}{2}-5\epsilon}[\vee_x]_{-2\epsilon} t^{-2\epsilon}.
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In conclusion,

$$|((v - v(x))\vee)_{T_0}(x) + 3C_2(v_{T_0} + \mathbf{1}_{T_0})(x)| \lesssim \\ T_0{}^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon}[\Upsilon\vee]_{-4\epsilon} + [U]_{\frac{3}{2}-5\epsilon}[\vee]_{-1-2\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon}[\vee_x]_{-2\epsilon}\Big) \\ +[\Upsilon\Upsilon\vee]_{-\frac{1}{2}-5\epsilon} T_0{}^{-\frac{1}{2}-5\epsilon} + T_0{}^{-4\epsilon}\|v\|[\Upsilon\vee]_{-4\epsilon} + \|\nu\|[\vee_x]_{-2\epsilon} T_0{}^{-2\epsilon},$$

$$|((v - v(x))\mathsf{V})_{T_0}(x) + 3C_2(v_{T_0} + \mathsf{I}_{T_0})(x)| \lesssim$$

$$T_0^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon}[\mathsf{VV}]_{-4\epsilon} + [U]_{\frac{3}{2}-5\epsilon}[\mathsf{V}]_{-1-2\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon}[\mathsf{V}_x]_{-2\epsilon}\Big)$$

$$+[\mathsf{WW}]_{-\frac{1}{2}-5\epsilon}T_0^{-\frac{1}{2}-5\epsilon} + T_0^{-4\epsilon}\|v\|[\mathsf{VV}]_{-4\epsilon} + \|\nu\|[\mathsf{V}_x]_{-2\epsilon}T_0^{-2\epsilon},$$

Assume $[\tau]_{|\tau|} \leqslant c\|v\|_{P_R}^{n_\tau(\frac{1}{2}-\epsilon)}, \tau \in \{\mathsf{I}, \mathsf{V}, \mathsf{W}, \mathsf{VV}, \mathsf{WV}, \mathsf{WW}\}$ and $T_0 = \frac{\mu}{\|v\|_{P_R}}$, then

$$|((v - v(x))\mathsf{V})_{T_0}(x) + 3C_2(v_{T_0} + \mathsf{I}_{T_0})(x)| \lesssim$$

$$c\mu^{\frac{1}{2}-7\epsilon}\Big([v]_{\frac{1}{2}-3\epsilon}c\|v\|^{\frac{3}{2}+3\epsilon} + \|v\|^{\frac{1}{2}+5\epsilon}([U]_{\frac{3}{2}-5\epsilon} + [\nu]_{\frac{1}{2}-5\epsilon})\Big)$$

$$+c\|v\|^3\mu^{-\frac{1}{2}-5\epsilon} + c\mu^{-4\epsilon}\|v\|^3 + c\mu^{-2\epsilon}\|\nu\|\|v\|,$$

Schauder Estimate

Lemma (Otto-Sauer-Scott-Weber)

$1 < \kappa < 2, A \subset (-\infty, \kappa]$, $U : D \times D \to \mathbb{R}$

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Localised version

$$\sup_{d \leqslant d_0} d^\kappa [U]_{\kappa, D_d} \lesssim \widetilde{M}^{(1)} + \widetilde{M}^{(2)} + \sup_{d \leqslant d_0} \|U\|_{D_d, d}.$$

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$$|U(x,y) - U(x,z) - U(z,y)| = 3|v(x) - v(z)||\Upsilon(y) - \Upsilon(z)|$$

$$\leqslant 3[v]_{\frac{1}{2} - 3\epsilon, D_{\frac{d}{2}}, \frac{d}{2}} [\Upsilon]_{1-2\epsilon} d(x,z)^{\frac{1}{2} - 3\epsilon} d(y,z)^{1 - 2\epsilon}$$

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- Renormalisation depends on this description (as opposed to boundary conditions);
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Theorem (M-Weber, 2018)

If v solves (1) pointwise on $P = (0,1) \times \{|x| < 1\}$, then we have:

$$\|v\|_{P_R} \leqslant C \max \left\{ \frac{1}{R}, [\tau]_{|\tau|}^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}, \tau \in \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\} \right\}.$$