AN INTRODUCTION TO CONVEX INTEGRATION

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In this note we briefly discuss the method of \textit{convex integration}. We first prove a theorem regarding divergence-free vector fields, which demonstrates the main idea of the method while being elementary. We then move to the context of the three-dimensional incompressible Euler equations, where we follow De Lellis & Székelyhidi Jr. (2009) to construct infinitely many compactly supported weak solutions to the Euler equations.

1. Divergence-free vector fields

Let $\Omega \subset \mathbb{R}^3$ be open and bounded. In this note we will prove the following.

\textbf{Theorem 1.} There exist infinitely many $u \in L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that
\begin{align}
\text{div } u &= 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \\
|u(x)| &= 1_{\Omega}(x) \quad \text{for almost every } x \in \mathbb{R}^3. 
\end{align}

Here $1_{\Omega}$ denotes the indicator function of $\Omega$ and $\mathcal{D}'(\mathbb{R}^3)$ denotes the space of distributions on $\mathbb{R}^3$, so that the first line of (1) is equivalent to
\[ \int_{\mathbb{R}^3} u \cdot \nabla \phi = 0 \]
being satisfied for all $\phi \in C_0^\infty(\mathbb{R}^3)$.

We will need the following two lemmas.

\textbf{Lemma 1.} If $x_n \to x$ in a real Hilbert space, then $x_n \to x$ if and only if $\|x_n\| \to \|x\|$.

\textit{Proof.} Writing
\[ \|x_n - x\|^2 = \|x_n\|^2 - 2(x_n, x) + \|x\|^2 \]
we see that $(x_n, x) \to \|x\|^2$ as $n \to \infty$, and so $\|x_n - x\|^2 \to 0$ as $n \to \infty$ if and only if the norms $\|x_n\|$ converge to $\|x\|$.

\textbf{Lemma 2.} If $(X, d)$ is a complete metric space and $J: X \to \mathbb{R}$ is a pointwise limit of continuous functions on $X$ (that is there exist $J_k \in C(X)$ such that $J_k(u) \to J(u)$ for each $u \in X$) then the set $S$ of points of continuity of $J$ is dense in $X$.

The lemma is a standard result in functional analysis, and the main element of the proof (which can be found below Theorem 4.6 in Székelyhidi Jr. (2013)) uses Baire's theorem (that is: A countable union of nowhere dense sets in a complete metric space is nowhere dense as well.)

We can now prove the main theorem.

\textit{Proof of Theorem 1.}
Step 1. Specify the functional setup.

Let
\[ X_0 := \{ u \in C_0^\infty(\Omega): \text{div} u = 0 \text{ and } |u| < 1 \} \]
denote the set of subsolutions. Let
\[ X := \text{weak closure of } X_0 \text{ in } L^2(\Omega). \]
Note that \( X \) is bounded in \( L^2(\Omega) \) (by \( \sqrt{|\Omega|} \)) and so \( X \), together with the weak \( L^2 \) topology, is metrizable (since \( B(0, \sqrt{|\Omega|}) \subset L^2(\Omega) \) equipped with this topology is metrizable (see for example Theorem 5.1 in Section V.5 in Conway (1990)) and \( X \) is its closed subset). Let \( d \) denote the corresponding metric. Then \( (X, d) \) is in fact a compact metric space (by the Banach-Alaoglu theorem, see for example Theorem 3.1 in Section V.3 in Conway (1990)).

Let \( I: X \to \mathbb{R}, \)
\[ I(u) := \int_\Omega \left( 1 - |u|^2 \right) . \]
Clearly, \( I \) is continuous on \( X \) with respect to the strong \( L^2 \) topology. Note that any \( u \in X \) satisfying \( I(u) = 0 \) is a solution to (1). In the following steps we will show that
\[ \{ I(u) = 0 \} \text{ is dense in } X. \]
This shows that the set of solutions to (1) is dense in \( X \), and so in particular proves the theorem.

Step 2. Given \( u \in X_0 \) and \( \tilde{\Omega} \subset \Omega \), add oscillations to \( u \) on \( \tilde{\Omega} \).

Namely, we will construct a sequence \( \{ u_k \} \) such that
(a) \( u_k \in X_0 \) for sufficiently large \( k \),
(b) \( u_k \rightharpoonup u \) in \( L^\infty(\Omega) \),
(c) \( \liminf_{k \to \infty} \| u_k \|^2 \geq \| u \|^2 + I_{\tilde{\Omega}}(u)^2/8|\tilde{\Omega}| \), where \( \| \cdot \| \) denotes the \( L^2(\Omega) \) norm and
\[ I_{\tilde{\Omega}}(u) := \int_{\tilde{\Omega}} \left( 1 - |u|^2 \right) . \]
Note that (b) gives in particular \( u_k \rightharpoonup u \) in \( L^2(\Omega) \), and that (c) gives in particular that \( u_k \) does not converge strongly to \( u \) (see Lemma 1).

In order to construct \( u_k \) let \( \xi, \eta \in \mathbb{R}^3 \) be such that \( \xi \perp \eta, |\xi| = |\eta| = 1, \phi \in C_0^\infty(\Omega; [0,1]) \) such that \( \phi = 1 \) on \( \tilde{\Omega} \) and
\[ v_k := \frac{\eta}{2k}(1 - |u(x)|^2)\phi(x) \sin(kx \cdot \xi). \]
We will show that
\[ u_k := u + \text{curl } v_k \]
satisfies (a), (b), (c). As for (a) note that \( u_k \in C_0^\infty(\Omega), \text{div} u_k = 0 \) (recall the identity \( \text{div curl} = 0 \)) and that \( |u_k| = |u| < 1 \) outside supp \( \phi \). Moreover, let \( \delta > 0 \) be such that \( |u| \leq 1 - \delta \text{ on } \text{supp } \phi \), and note that
\[ \text{curl } v_k = \frac{(\xi \times \eta)}{2}(1 - |u(x)|^2)\phi(x) \cos(kx \cdot \xi) + O(1/k). \]
Thus on supp $\phi$

$$|u_k| \leq |u| + \frac{1}{2}(1 - |u|)(1 + |u|) + \frac{C}{k}$$

$$\leq |u| + (1 - |u|)(1 - \delta/2) + \frac{C}{k}$$

$$= 1 + \frac{\delta}{2}(|u| - 1) + \frac{C}{k}$$

$$\leq 1 - \frac{\delta^2}{2} + \frac{C}{k},$$

and so $|u_k| < 1$ for sufficiently large $k$, which gives (a).

Claim (b) follows immediately, since the first part of the right side of (3) is an oscillatory term with the frequency increasing with $k$ (and so tends to zero weakly-$\star$ in $L^\infty$ by the Riemann-Lebesgue lemma, see for example Lemma 1.4 in Duoandikoetxea (2001)) and the second part tends strongly to 0 in $L^\infty$ as $k \to \infty$.

Claim (c) follows by a direct calculation. Indeed, similarly as in (b) we obtain $(\text{curl } v_k, u) \to 0$ as $k \to \infty$ and

$$\|u_k\|^2 = \|u\|^2 + 2(\text{curl } v_k, u) + \|\text{curl } v_k\|^2$$

$$= \|u\|^2 + \frac{1}{4} \int_\Omega (1 - |u(x)|)^2 \phi(x)^2 \cos(kx \cdot \xi)^2 \, dx + o(1)$$

$$\geq \|u\|^2 + \frac{1}{8} \int_\Omega (1 - |u(x)|)^2 \, dx + o(1),$$

where we used the identity $\cos^2 \alpha = (1 + \cos 2\alpha)/2$, the Cauchy-Schwarz inequality and we denoted any term tending to 0 as $k \to \infty$ by $o(1)$. Taking $\liminf_{k \to \infty}$ gives (c).

**Step 3.** Observe that the set

$$S := \{ u \in X : \text{ any sequence weakly convergent to } u \text{ in } L^2(\Omega) \text{ converges strongly} \}$$

is dense in $(X, d)$.

Indeed, letting $J(u) := \|u\|$ we see (using Fubini’s theorem) that $J$ is a pointwise limit of

$$J_k(u) := \|\rho_{1/k} \ast u\|,$$

where we extended $u$ by zero outside $\Omega$, the symbol $\ast$ denotes the convolution, $\rho_{\epsilon}(x) := \epsilon^{-3}\rho(x/\epsilon)$ and $\rho$ is a standard mollifying kernel. Moreover, each $J_k$ is a continuous function on $(X, d)$ (that is if $w_n \to w$ then $J_k(w_n) \to J_k(w)$ as $n \to \infty$ for each $k$). Thus the claim follows from Lemma 2.

**Step 4.** Show that $S \subset \{ I(u) = 0 \}$.

(Note that this inclusion together with Step 3 prove (2), as required.)

Suppose otherwise that there exists $u \in S$ such that $I(u) > 0$. Then there exists $\hat{\Omega} \Subset \Omega$ such that $I_{\hat{\Omega}}(u) > 0$. Since $u$ is a weak limit of a sequence $\{u_k\} \subset X_0$ (by definition of $X$),
we can apply Step 2 for each \( u_k \) to obtain \( u_k' \in X_0 \) such that
\[
d(u_k', u_k) \leq 1/k \quad \text{and} \quad \|u_k'\|^2 \geq \|u_k\|^2 + I_{\tilde{\Omega}}(u_k)^2 / 8|\tilde{\Omega}| - 1/k.
\]
Noting that in fact \( u_k \) converges to \( u \) strongly in \( L^2 \) (by definition of \( S \)) we see that \( u_k' \) converges strongly to \( u \) as well, and so taking the limit \( k \to \infty \) in the last inequality gives
\[
\|u\|^2 \geq \|u\|^2 + I_{\tilde{\Omega}}(u)^2 / 8|\tilde{\Omega}| > \|u\|^2,
\]
a contradiction. \( \square \)

2. Euler equations

We will write "previously" or "as before" to relate back to the problem (1). In this section we paraphrase the main result from De Lellis & Székelyhidi Jr. (2009), using also some tools from De Lellis & Székelyhidi Jr. (2010).

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^n \times \mathbb{R} \) be open and bounded. Then there exists infinitely many pairs \( v, p \in L^\infty(\mathbb{R}^3 \times \mathbb{R}) \) such that
\[
\partial_t v + (v \cdot \nabla)v + \nabla p = 0 \quad \text{in} \quad D'(\mathbb{R}^3 \times \mathbb{R}),
\]
\[
div v = 0 \quad \text{in} \quad D'(\mathbb{R}^3 \times \mathbb{R}),
\]
\[
|v| = 1_\Omega \quad \text{a.e. in} \quad \mathbb{R}^3 \times \mathbb{R},
\]
\[
p = 0 \quad \text{outside} \quad \Omega.
\]

**Proof.**

**Step 1.** Reformulate the problem.

Namely, problem (4) is equivalent to finding \( v, u, q \in L^\infty(\mathbb{R}^3 \times \mathbb{R}) \) supported in \( \Omega \) such that
\[
\begin{cases}
\partial_t v + \text{div} u + \nabla q = 0, \\
\text{div} v = 0,
\end{cases}
\]
(5)

\((v, u) \in K := \{(\tau, \pi) \in S^{n-1} \times S^n_0 : u = v \otimes v - |v|^2 I_n / n \} \quad \text{a.e. in} \Omega, \)

(6)

where \( v \otimes v = vv^T \) denotes the matrix, whose \( i, j \)-th entry is \( u_{ij} \), \( S^{n-1} = \partial B(0, 1) \) denotes the \( n-1 \) dimensional sphere, \( S^n_0 \) denotes the space of \( n \times n \) symmetric matrices with zero trace and \( I_n \) denotes the \( n \)-dimensional identity matrix.

The equivalence (4) \( \Leftrightarrow \) (5, 6) is clear by substituting
\[q = p + |v|^2 / n,\]
since then
\[
\text{div} u + \nabla q = \text{div} (v \otimes v) - \text{div} (|v|^2 I_n / n) + \nabla q = (v \cdot \nabla)v - \nabla(|v|^2 / n) + \nabla q = (v \cdot \nabla)v + \nabla p.
\]

**Step 2.** Observe that
\[
K^\text{co} = \{(\tau, \pi) \in B(0, 1) \times S^n_0 : e(v, u) \leq 1/2 \},
\]
(7)

where \( K^\text{co} \) denotes the convex hull of \( K \) and \( e : \mathbb{R}^n \times S^n_0 \to \mathbb{R} \) is defined by
\[e(v, u) := \frac{n}{2} \lambda_{\max} (v \otimes v - u)\]
An easy calculation shows that $e$ is convex and for $v \in \mathbb{R}^n$, $u \in S^n_0$

$$|u|_\infty \leq \frac{2n-1}{n} e(v, u),$$

(8)

where $|.|_\infty$ denotes the operator norm of a matrix, and

$$\frac{|v|^2}{2} \leq e(v, u) \quad \text{with } "= " \text{ if and only if } u = v \otimes v - \frac{|v|^2}{n} I_n.$$ (9)

These facts are, essentially, a consequence of the fact that $u$ is trace-free, see Lemma 3 in De Lellis & Székelyhidi Jr. (2010) for an enlightening proof.

From (9) we see that

$$K = \{(\tilde{v}, \tilde{u}) \in S^{n-1} \times S^n_0 : e(v, u) = \frac{|v|^2}{2}\} \subset S_1,$$

where $S_1$ denotes the right-hand side of (7). Since $S_1$ is convex (as $e(v, u)$ is a convex function) we obtain

$$K^{co} \subset S_1.$$

It remains to show the opposite inclusion. To this end one can use the fact that $u$ has zero trace to show that the set $E$ of extremal points of $S_1$ is contained in $K$, see Lemma 3 (iv) in De Lellis & Székelyhidi Jr. (2010) for a proof. Thus, since (8), (9) give in particular that $S_1$ is compact, we see that $S_1$ is the closure of the convex hull of $E$ (by the Krein-Milman theorem (see Section XII.1 in Yosida (1965) for example)) and so

$$S_1 \subset E^{co} \subset K^{co} = K^{co},$$

where the last equality follows from the fact that $K$ is compact (recall that the convex hull of a compact set in $\mathbb{R}^n$ is compact).

**Step 3.** Show that $0 \in \mathcal{U} := \text{Int} K^{co}$.

Observe that Step 2 immediately gives $0 \in K^{co}$. In order to see that $0$ belongs to the interior of this set one needs to work a bit harder. Namely, one can prove it using Jensen’s inequality, the facts that $\int_{S^{n-1}} z dz = 0$, $\int_{S^{n-1}} (z \otimes z - I_n/n)dz = 0$, and the Open Mapping Theorem, see Lemma 4.2 in De Lellis & Székelyhidi Jr. (2009) for the details. Here we omit the proof.

**Step 4.** Specify the functional setup.

Let

$$X_0 := \{(v, u, q) \in C^\infty_0 (\mathbb{R}^n \times \mathbb{R}) : \text{supp } (v, u, q) \subset \Omega, (v, u, q) \text{ solves } (5) \text{ and } (v, u) \in \mathcal{U}, |q| < 1 \text{ a. e. } \}$$

be the set of subsolutions and let

$$X := \text{weak closure of } X_0 \text{ in } L^2.$$

Similarly as before, $X$, equipped with the $L^2$ weak topology, is metrizable (with some metric $d$) and the resulting space $(X, d)$ is a compact metric space.
Observe that any triple \((v, u, q) \in X\) is supported within \(\Omega\), solves (5) and \((v, u) \in K^\omega\), \(|q| \leq 1\) almost everywhere.

Let \(I: (X, d) \to \mathbb{R}_+\) be defined by

\[
I(v, u, q) := \int_\Omega (1 - |v|^2).
\]

Note that \(I\) is continuous with respect to the strong \(L^2\) topology. Moreover \(I(v, u, q) = 0\) if and only if \((v, u, q)\) is the required solution, that is solve (5), (6). Indeed, the \(\Leftarrow\) implication is trivial. As for the \(\Rightarrow\) implication note that \(I(v, u, q) = 0\) gives in particular that \(|v| = 1\) a.e. in \(\Omega\), and so

\[
\frac{1}{2} = \frac{|v|^2}{2} \leq e(v, u) \leq \frac{1}{2},
\]

where the two inequalities follow from (9) and Step 2, respectively. Thus (9) gives that \(u = v \otimes v - |v|^2I_n/n\). Thus, except for the equation (5) the triple \((v, u, q)\) also satisfies the constraint (6), as required.

In the following steps we will verify that

\[
\{(v, u, q): I(v, u, q) = 0\}
\]

is dense in \((X, d)\),

(10)

which (similarly as before) proves the theorem.

**Step 5.** Define the wave cone \(\Lambda\).

Let

\[
\Lambda := \left\{(\nu, \pi, \eta) \in \mathbb{R}^n \times S^0_\nu \times \mathbb{R}: \det \begin{pmatrix} \pi + \eta I_n & \pi \\ \overline{\nu}^T & 0 \end{pmatrix} \right\}.
\]

What is the wave cone? That \(\Lambda\) is a cone (that is \(\alpha a \in \Lambda\) for all \(\alpha > 0, a \in \Lambda\)) is clear. As for the “wave” part note that \((\nu, \pi, \eta) \in \Lambda\) if and only if there exists \(\xi \in \mathbb{R}^{n+1}\) such that

\[
\begin{pmatrix} \pi + \eta I_n & \pi \\ \overline{\nu}^T & 0 \end{pmatrix} \xi = 0.
\]

An elementary calculation shows that

\[
w(x, t) := (\nu, \pi, \eta) h(\xi \cdot (x, t))
\]

satisfies (5) for any choice of \(h: \mathbb{R} \to \mathbb{R}\). Thus it is helpful to think of the wave cone \(\Lambda\) as of the set of all directions \((\nu, \pi, \eta)\) in which there exist a planewave.

The wave cone is an object of crucial importance, since it is by adding infinitely many oscillations, each in a direction from the wave cone, that solutions to (5), (6) are obtained (that is the density of \(\{I = 0\}\) in \((X, d)\) will be shown).

In fact, the notion of the wave cone is used in a wide family of equations (the family of differential inclusions, of which the Euler equations and (1) are particular cases), see Section 5.3 in Székelyhidi Jr. (2013).

Note however that the plane waves of the form (11) are not compactly supported. Rather, the support of \(w\) is a “slice” in \(\mathbb{R}^3 \times \mathbb{R}\) that is orthogonal to \(\xi\) and whose width equals the length of \(\text{supp} h\).

This is an unfortunate property (since we are interested only in the oscillations supported within \(\Omega\)), and this problem is resolved in the next step.
Step 6. Construct localised oscillations in a direction from the wave cone.

Namely, given \((\tau, \zeta, \eta) \in \Lambda, k > 0\) and \(\Omega_1 \subseteq \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}\), there exists \((v_k, u_k, q_k) \in C_0^\infty(\Omega_2)\) that solves (5) and
\[
\text{dist}((v_k, u_k, q_k)(x,t), J_{(\tau, \zeta, \eta)}) < 1/k \quad \text{for } (x,t) \in \Omega_2,
\]
\[
\int_{\Omega_2} |v_k| \geq \alpha |\tau||\Omega_1|,
\]
\[(v_k, u_k, q_k) \rightharpoonup 0 \quad \text{in } L^\infty,
\]
where \(\alpha > 0\) is a dimensional constant and \(J_z = [−z, z]\) denotes the closed line segment joining \(z\) and \(−z\).

The proof of this step consists of a number of elementary calculations (and, as the reader might expect, a choice of a cutoff function \(\phi \in C_0^\infty(\Omega_2; [0,1])\) such that \(\phi = 1\) on \(\Omega_1\)), and thus is omitted in this note. We refer the reader to Proposition 2.2 in De Lellis & Székelyhidi Jr. (2009) for the proof in the case \(\Omega_1 = B(0,1/2), \Omega_2 = B(0,1)\).

Step 7. Show that there is sufficiently many directions in \(\Lambda\).

Namely show that for every \((v_0, u_0) \in \mathcal{U}\) there exists \((\tau, \zeta, 0) \in \Lambda\),
\[
|\tau| \geq C(1 − |v_0|^2) \quad \text{and}
\]
\[
(v_0, u_0) + J_{(\tau, \zeta, \eta)} \subset \mathcal{U} \quad \text{with} \quad \text{dist}((v_0, u_0) + J_{(\tau, \zeta, \eta)}, \partial \mathcal{U}) \geq \frac{1}{2} \text{dist}((v_0, u_0), \partial \mathcal{U}).
\]  

In this note we only sketch the main ideas of the proof of this step (see Lemma 4.3 in De Lellis & Székelyhidi Jr. (2009) for the details).

First use Carathéodory’s Theorem to choose \((\tau, \zeta)\) such that (12) holds and moreover
\[
(v_0, u_0) + J_{2(\tau, \zeta)} \subset K^c.
\]  

Then observe that for every \(d > 0\)
\[
(v_0, u_0) + J_{(\tau, \zeta)} + B(0, d/2) \subset (B((v_0, u_0), d) \cup J_{2(\tau, \zeta)})^c
\]
which is a simple geometric fact described in Figure 1. Taking \(d := \text{dist}((v_0, u_0), \partial \mathcal{U})\) we see from (14) that
\[
(v_0, u_0) + J_{2(\tau, \zeta)} + B(0, d/2) \subset K^c,
\]
which is equivalent to (13).

Finally, one can perform a simple matrix calculation to see that \((\tau, \zeta, 0) \in \Lambda\).

Step 8. Given \((v, u, q) \in X_0\) and \(\tilde{\Omega} \Subset \Omega\) add oscillations to \((v, u, q)\) on \(\tilde{\Omega}\).

Namely, we will construct a sequence \((v_k, u_k, q_k)\) such that
\begin{enumerate}
  \item \((v_k, u_k, q_k) \in X_0\) for sufficiently large \(k\),
  \item \((v_k, u_k, q_k) \rightharpoonup (v, u, q)\) in \(L^\infty\),
  \item \(\liminf_{k \to \infty} ||v_k||^2 \geq ||v||^2 + \beta I_\Omega(v)/|\tilde{\Omega}|\),
\end{enumerate}
Figure 1. Consider a ball $B(O, d)$ and a line segment whose middle point coincides with $O$. Then

$$J + B(0, d/2) \subset (B(O, d) \cup 2J)\|$,$$

which follows from the fact that the ball centred at the right endpoint of $J$ and radius $d/2$ is tangent to the line tangent to $B(O, d)$ and passing through $2J$ (since the former is a homothetic transform of $B(O, d)$ with respect to the endpoint of $2J$ and ratio $1/2$).

Thus (15) follows by taking $O := (v_0, u_0)$, $J := (v_0, u_0) + J_{(\overline{v}, \overline{u})}$.

where $\| \cdot \|$ denotes the $L^2(\Omega)$ norm, $\beta := 4^{-n-1} \alpha^2 C^2$ and

$$I_{\Omega}(v) := \int_\Omega (1 - |v|^2).$$

To this end observe that since $(v, u, q) \in X_0$ we have $(v, u)(x_0, t_0) \in U$, and, by continuity, there exists $\delta > 0$ such that

$$\text{dist} ((v, u)(x_0, t_0), \partial U) \geq \delta \quad \text{for} \quad (x_0, t_0) \in \tilde{\Omega}.$$

Moreover, Step 7 gives that for every $(x_0, t_0) \in \tilde{\Omega}$ there exists a pair $(\overline{v}, \overline{u}, 0)(x_0, t_0)$ such that $(\overline{v}, \overline{u}, 0)(x_0, t_0) \in \Lambda,$

$$|\overline{v}(x_0, t_0)| \geq C(1 - |v(x_0, t_0)|^2)$$

and

$$\text{dist} ((v, u)(x_0, t_0) + J_{(\overline{v}, \overline{u})(x_0, t_0)}, \partial U) \geq \delta/2.$$

Using the last inequality and the uniform continuity of $(v, u)$ we see that there exists $\varepsilon > 0$ such that

$$\text{dist} ((v, u)(x, t) + J_{(\overline{v}, \overline{u})(x_0, t_0)}, \partial U) \geq \delta/4$$

whenever $(x, t), (x_0, t_0) \in \tilde{\Omega}$ are such that $|(x, t) - (x_0, t_0)| \leq \varepsilon$.

Now let $\{B_i\}, B_i = B_i(x_i, t_i)$, be a (finite) family of pairwise disjoint $\varepsilon$-balls that are contained in $\tilde{\Omega}$ and

$$\int_{\Omega} (1 - |v|^2) \leq 2 \sum_i (1 - |v(x_i, t_i)|^2)|B_i|;$$

for this take $\varepsilon$ smaller, if required. For each $k \geq 1$ and $i$ apply step 6 (with $\Omega_1 := B_i/2$, $\Omega_2 := B_i$) to obtain $(v_{i,k}, u_{i,k}, q_{i,k}) \in C^\infty_0(B_i)$ such that

(i) $\text{dist} ((v_{i,k}, u_{i,k}, q_{i,k})(x, t), J_{(\overline{v}, \overline{u}, 0)(x_i, t_i)}) \leq 1/k$ for $(x, t) \in B_i$. 


\( \int_{B_i} |v_{i,k}| \geq \alpha |v(x_i, t_i)| |B_i|/2, \) and

\( (v_{i,k}, u_{i,k}, q_{i,k}) \to 0 \) in \( L^\infty(B_i) \) as \( k \to \infty \), for each \( i \).

We will show that

\[
(v_k, u_k, q_k) := (v, u, q) + \sum_i (v_{i,k}, u_{i,k}, q_{i,k})
\]

satisfies the claims (a), (b), (c).

Indeed, (i) gives \( |q_k| < 1 \) and (17) and (i) give \( (v_k, u_k) \in U \) for \( k \geq 4/\delta \). Thus (a) follows. Claim (b) follows from (iii) (recall that there are only finitely many \( i \)'s). As for (c) we write

\[
\|v_k\|^2 = \|v\|^2 + 2 \sum_i (v_{i,k}, v) + \left( \sum_i \|v_{i,k}\| \right)^2
\]

\[
\geq \|v\|^2 + \frac{1}{|\Omega|} \left( \sum_i \|v_{i,k}\|_{L^1(\Omega)} \right)^2 + o(1)
\]

\[
\geq \|v\|^2 + \frac{\alpha^2}{|\Omega|} \left( \sum_i |B_i/2| |v(x, t)| \right)^2 + o(1)
\]

\[
\geq \|v\|^2 + \frac{\alpha^2 C^2}{4n|\Omega|} \left( \sum_i |B_i|(1 - |v(x, t_i)|^2) \right)^2 + o(1)
\]

\[
\geq \|v\|^2 + \frac{\alpha^2 C^2}{4n+1|\Omega|} \left( \int_{\tilde{\Omega}} (1 - |v|^2) \right)^2 + o(1),
\]

where we denoted by \( o(1) \) any term tending to 0 as \( k \to \infty \) and we used (iii), the Cauchy-Schwarz inequality, (ii), (16) and (18). Taking \( \lim \inf_{k \to \infty} \) gives (c).

**Step 9.** Observe that

\[
S := \{(v, u, q) \in X: \text{any sequence weakly convergent to } (v, u, q) \text{ in } L^2(\Omega) \text{ converges strongly}\}
\]

is dense in \((X,d)\).

This step follows in the same way as step 3 in the previous problem.

**Step 10.** Show that \( S \subset \{(v, u, q): I(v, u, q) = 0\} \).

(Show that this and step 9 give (10), as required.)

Suppose otherwise that there exists \((v, u, q) \in S\) such that \( I(v, u, q) > 0 \). Then there exists a subdomain \( \tilde{\Omega} \subset \Omega \) such that

\[
I_{\tilde{\Omega}}(v) > 0.
\]

By definition of \( X \), \((v, u, q)\) is a \( L^2 \) weak limit of a sequence \( \{(v_k, u_k, q_k)\} \subset X_0 \). For each \( k \) we apply step 8 to obtain \((v'_k, u'_k, q'_k)\) \( \in X_0 \) such that

\[
d((v_k, u_k, q_k), (v'_k, u'_k, q'_k)) \leq 1/k
\]

(19)
and
\[ \|v'_k\|^2 \geq \|v_k\|^2 + \beta \frac{I_{\tilde{\Omega}}(v_k)^2}{|\tilde{\Omega}|} - 1/k \] (20)
Since \( v_k \to v \) in \( L^2(\Omega) \) the same is true for the sequence \( v'_k \) (by (19)), and so \( v'_k \to v \) in \( L^2(\Omega) \) (by the definition of \( S \)). Thus, taking the limit \( k \to \infty \) in (20) gives
\[ \|v\|^2 \geq \|v\|^2 + \beta \frac{I_{\tilde{\Omega}}(v)^2}{|\tilde{\Omega}|}, \]
a contradiction. \( \Box \)

References


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