# IRRATIONALITY OF SOME $p$-ADIC INTEGERS

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*To Dipesh, my baby brother - with whom there is never a dull moment.*

Abstract. In [4], Beukers sketches a proof of the irrationality of $\zeta_2(2)$, $\zeta_3(2)$ and $L_2(2, \chi_8)$. Here we expand on Beukers' sketch, supplying full details.

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## 1. Introduction

Let $\zeta(s)$ be the usual zeta-function. Euler proved that for even positive integers $n$, the value $\zeta(n)$ is a non-zero rational multiple of $\pi^n$. Since $\pi$ is transcendental (as proved by Lindemann in 1882), it follows that the even values of $\zeta$ are all transcendental and therefore irrational. The odd values of $\zeta$ (that is the values $\zeta(n)$ where $n \geq 3$) are more mysterious. In 1978, Apéry announced a proof of the irrationality of $\zeta(3)$. It was later shown that infinitely many of the odd values must be irrational [7].

One can ask similar questions about the values of $p$-adic zeta and $L$-functions. Frank Calegari [5] proved the irrationality of $\zeta_2(2)$ and $\zeta_2(3)$ using $p$-adic modular forms. More recently, Beukers [4], sketched another proof of the irrationality of $\zeta_2(2)$ and $\zeta_3(2)$ using a more classical approach. Here we expand on Beukers' sketch, supplying full details. The main steps of the proof are as follows:

(1) A criterion for the irrationality of a $p$-adic number given as the limit of a sequence of rationals (Lemma 2.1).
(2) Some theory of linear differential equations, the power series expansion of their solutions and the arithmetic properties of their coefficients (Section 3 and Appendix B).
(3) Identities linking the infinite Laurent series $\Theta(x)$, $R(x)$ and $T(x)$ (Section 4 and Appendix C).
(4) We relate $\Theta(x)$, $R(x)$ and $T(x)$ with the $p$-adic Hurwitz zeta-function, and for specific values of $x$ we find relationships of $\Theta(x)$ with certain $p$-adic zeta and $L$-functions (Sections 5 and 6). This allows us to express $\zeta_2(2) = -\frac{1}{8}\Theta_2(1/2)$ and $\zeta_3(2) = -\frac{2}{27}\Theta_3(1/3)$. Thus it remains to prove the irrationality of $\Theta_2(1/2)$ and $\Theta_3(1/3)$. The indicies denote convergence in $\mathbb{Q}_2$ and $\mathbb{Q}_3$ respectively.
(5) The convergents of the continued fraction approximation of the function $\Theta(x)$ allows us to construct sequences of polynomials, namely $p_n(x)$ and $q_n(x)$ (Section 7).
(6) In the final section, we put everything together to deduce the irrationality of $\zeta_2(2)$ and $\zeta_3(2)$.

## 2. Key Fundamentals

To prove that a $p$-adic number $\alpha$ is irrational, we must construct a sequence of rational approximations $c_n/d_n$ which converge $p$-adically to $\alpha$ very quickly. The Lemma below precisely states this, however we have made a slight adjustment to that stated in Beukers' paper [4].

**Lemma 2.1.** *Let $\alpha$ be a $p$-adic number and let $\{c_n\}$ and $\{d_n\}$, with $n \in \mathbb{N}$, be two sequences of integers such that*

$$\lim_{n\to\infty} \max(|c_n|, |d_n|)\, |c_n + \alpha d_n|_p = 0$$

*and $c_n + \alpha d_n \neq 0$ infinitely often. Then $\alpha$ is irrational.*

*Proof.* Suppose $\alpha \in \mathbb{Q}$. Let $\alpha = A/B$ with $A, B \in \mathbb{Z}$ and $B > 0$. Then

$$|c_n + \alpha d_n|_p = \left| c_n + \frac{A}{B} d_n \right|_p = \frac{|Bc_n + Ad_n|_p}{|B|_p}.$$

Suppose $c_n + \alpha d_n \neq 0$. Let $Bc_n + Ad_n = \pm p^{a_n} t_n$, with $t_n \in \mathbb{Z}^+$ not divisible by $p$. By the triangle inequality we have

$$p^{a_n} t_n = |Bc_n + Ad_n| \leq 2 \max(|A|, B) \max(|c_n|, |d_n|).$$

Thus

$$\begin{aligned} |c_n + \alpha d_n|_p \max(|c_n|, |d_n|) &= \frac{p^{-a_n}}{|B|_p} \max(|c_n|, |d_n|) \\ &\geq \frac{p^{-a_n}}{|B|_p} \cdot \frac{p^{a_n} t_n}{2 \max(|A|, B)} \\ &\geq \frac{1}{2|B|_p \max(|A|, B)} \end{aligned}$$

and so the limit of $|c_n + \alpha d_n|_p \cdot \max(|c_n|, |d_n|)$ as $n$ tends to infinity cannot be zero, thereby arriving at a contradiction. Hence $\alpha$ must be irrational. $\square$

**Definition 2.1.** *For positive integer $n$, and rational number $\beta$, we define a rising factorial as stated in Wolfram Mathworld* [8].

$$(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1), \quad (\beta)_0 = 1.$$

**Lemma 2.2.** *Let $\beta \in \mathbb{Q}$ with $F \in \mathbb{Z}_{>1}$ as denominator. Then $(\beta)_n/n! \in \mathbb{Q}$, with denominator dividing $\mu_n(F)$ defined by*

$$\mu_n(F) = F^n \prod_{q|F} q^{\left[\frac{n}{q-1}\right]}$$

*where the product is taken over all the primes $q$ which divide $F$. Moreover, the number of times a prime $q_1$ appears in the denominator of $(\beta)_n/n!$ is at least*

$$n\left(r + \frac{1}{q_1 - 1}\right) - \frac{\log n}{\log q_1} - 1$$

*where $r$ is defined via the relation $|F|_{q_1} = q_1^{-r}$.*

*Proof.* Let $\beta = b/F$ where $b \in \mathbb{Z}$. Then

$$\begin{aligned} \frac{(\beta)_n}{n!} = \frac{(b/F)_n}{n!} &= \frac{1}{n!} \cdot (b/F)(b/F+1)\cdots(b/F+n-1) \\ &= \frac{1}{n! F^n} \cdot b(b+F)\cdots(b+(n-1)F) = \frac{1}{n! F^n} \prod_{k=0}^{n-1} (b + Fk) \end{aligned}$$

which shows that $(\beta)_n/n!$ is a rational number.

We proceed to show that the denominator of $(\beta)_n/n!$ divides $\mu_n(F)$. Suppose $q$ is a prime which divides $F$. Since $\gcd(b, F) = 1$, this implies that $q \nmid b$ and therefore $q \nmid (b + Fk)$ for any $k \in \mathbb{Z}$. Therefore, $q$ does not divide

$$\prod_{k=0}^{n-1} (b + Fk).$$

We consider the number of times $q$ appears in $n!$. Of the numbers $1, 2, \ldots, n$, precisely $[n/q]$ are divisible by $q$, $[n/q^2]$ are divisible by $q^2$ and so on. Therefore the number of appearances of $q$ in $n!$ is

$$\left[\frac{n}{q}\right] + \left[\frac{n}{q^2}\right] + \left[\frac{n}{q^3}\right] + \cdots$$

which is bounded above by
$$\frac{n}{q} + \frac{n}{q^2} + \cdots = \frac{n}{q-1}.$$
Shortly we shall see that for any prime $p$ which does not divide $F$, $p$ does not divide the denominator of $(\beta)_n/n!$, thus concluding that the denominator of $(\beta)_n/n!$ divides
$$F^n \prod_{q|F} q^{\left[\frac{n}{q-1}\right]} = \mu_n(F).$$
In order to arrive at the above conclusion, we suppose $p$ is a prime not dividing $F$. It suffices to show that the number of times $p$ appears in the numerator is greater than or equal to the number of times $p$ appears in the denominator. In other words we show for any $s \in \mathbb{Z}_{>0}$,
$$\begin{aligned} &\#\{1 \leq k \leq n;\ \text{such that } p^s \mid k\} \\ &\qquad \leq \#\{0 \leq k \leq n-1;\ \text{such that } p^s \mid (b+Fk)\}. \end{aligned}$$
We have already seen that the $LHS = [n/p^s]$. Let us suppose that the $RHS = r$. Let $k_1$ be the smallest element of the set
$$\Omega = \{0 \leq k \leq n-1;\ \text{such that } p^s \mid (b+Fk)\}.$$
Then
$$b + Fk_1 \equiv 0 \pmod{p^s}$$
which is equivalent to
$$k_1 \equiv \frac{-b}{F} \pmod{p^s}$$
where we are able to divide by F because $p \nmid F$. Clearly $0 \leq k_1 \leq p^s - 1$, therefore we obtain the remaining elements of $\Omega$ using $k_1$ and our assumption that $|\Omega| = r$. Consequently we find
$$k_1, k_1 + p^s, k_1 + 2p^s, \ldots, k_1 + (r-1)p^s \in \Omega.$$
Since $0 \leq k \leq n-1$ must be satisfied for all elements $k \in \Omega$, we have
$$0 \leq k_1 + (r-1)p^s \leq n-1.$$
Rearranging, we see
$$r \leq \frac{n-1-k_1}{p^s} + 1.$$
Using $0 \leq k_1 \leq p^s - 1$, and taking the maximum value of $r$, we make further calculations.
$$RHS = r = \frac{n-1-k_1+p^s}{p^s} \geq \frac{n-1-(p^s-1)+p^s}{p^s} = \frac{n}{p^s} \geq \left[\frac{n}{p^s}\right] = LHS$$
which shows that if $p \nmid F$ then $p$ does not divide the denominator of $(\beta)_n/n!$.

To complete the proof, we find a lower bound for
$$\left[\frac{n}{q_1}\right] + \left[\frac{n}{q_1^2}\right] + \cdots$$
to find the minimum number of times a prime $q_1$ appears in the denominator of $(\beta)_n/n!$. Finding the lower bound was not as straight forward as stated in Beukers' paper [4], but Professor Samir Siksek managed to shed some light on the problem with his simple, yet elegant proof stated below.

First we need a lemma!

**Lemma 2.3.** *If* $0 \le x \le 1$ *then* $q^x \le (q-1)x + 1$.

*Proof.* Let $f(x) = (q-1)x + 1 - q^x$. We must show that $f(x) \ge 0$ for $0 \le x \le 1$. Differentiating (and using the fact that $q^x = e^{x \log q}$) we find that

$$f''(x) = -(\log q)^2 q^x.$$

In particular $f''(x) < 0$ for all $0 \le x \le 1$. So the graph of $f$ is concave between 0 and 1. But $f(0) = f(1) = 0$. Hence $f(x) \ge 0$ for $0 \le x \le 1$. □

Let

$$z = \left[\frac{\log n}{\log q_1}\right] \qquad \text{and} \qquad \epsilon = \frac{\log n}{\log q_1} - z. \tag{2.1}$$

Clearly $0 \le \epsilon \le 1$. Moreover, $\log n = (z+\epsilon)\log q_1$, which can also be written as

$$n = q_1^{z+\epsilon}. \tag{2.2}$$

Now we can start to calculate some bounds.

$$\begin{aligned}
\sum_{k=1}^{z}\left[\frac{n}{q_1^k}\right] &\ge \sum_{k=1}^{z}\left(\frac{n}{q_1^k} - 1\right) \\
&\ge -z + n\left(\frac{1}{q_1} + \frac{1}{q_1^2} + \cdots + \frac{1}{q_1^z}\right) \\
&= -z + \frac{n}{q_1 - 1}\left(1 - \frac{1}{q_1^z}\right) && \text{geometric series} \\
&= -z + \frac{n}{q_1 - 1} - \frac{n}{(q_1-1)q_1^z} \\
&= -z + \frac{n}{q_1 - 1} - \frac{q_1^\epsilon}{q_1 - 1} && \text{using (2.2)} \\
&\ge -z + \frac{n}{q_1 - 1} - \frac{(q_1-1)\epsilon + 1}{q_1 - 1} && \text{by Lemma 2.3} \\
&= \frac{n-1}{q_1 - 1} - z - \epsilon \\
&= \frac{n-1}{q_1 - 1} - \frac{\log n}{\log q_1} && \text{by (2.1).}
\end{aligned}$$

The number of appearances of a prime $q_1$ in the denominator of $(\beta)_n/n!$ is the number of appearances in $F^n$ plus the number of appearances in $n!$, which is bounded below by

$$nr + \frac{n-1}{q_1 - 1} - \frac{\log n}{\log q_1} \ge n\left(r + \frac{1}{q_1 - 1}\right) - \frac{\log n}{\log q_1} - 1,$$

where $r$ is given by the relation $|F|_{q_1} = q_1^{-r}$. In Beukers' paper [4], this proof is given as a sketch and we have filled in some essential details.

□

## 3. Introducing Differential Equations

In this section we consider linear differential equations of order 2 and their corresponding solutions. We arrive at some interesting results with regards to the $n$-th coefficient of their power series solution.

**Definition 3.1.** *Let $R$ be a commutative ring with unity. A zero divisor is a non-zero element $x$ of $R$ such that there is some non-zero element $y$ of $R$ satisfying $xy = 0$. An integral domain is a commutative ring with unity that has no zero divisors.*

*Let $R$ be an integral domain. We define the characteristic of $R$ to be the least positive integer $n$ such that $n \cdot 1_R = 0$ if there is such an $n$. If no such $n$ exists, we say that $R$ has characteristic zero. We denote the quotient ring of $R$ by $Q(R)$.*

From now on, $R$ will be a domain of characteristic zero.

**Definition 3.2.** *$R[z]$ denotes the polynomial ring which consists of polynomials with coefficients in $R$. $R[[z]]$ denotes the ring of formal power series which consists of formal power series that have coefficients in $R$.*

**Definition 3.3.** *The logarithmic derivative of a function $f(z)$ is $f'(z)/f(z)$.*

**Lemma 3.1.** *Let $f(z) \in Q(R)[[z]]$. Then*

$$\int_0^z f(w)\log(w)dw = \log(z)\int_0^z f(w)dw - \int_0^z \frac{1}{x}\int_0^x f(w)dwdx$$

*and*

$$\begin{aligned}\int_0^z f(w)(\log w)^2dw = (\log z)^2\int_0^z f(w)dw - 2\log(z)\int_0^z \frac{1}{x}\int_0^x f(w)dwdx \\ + 2\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x f(w)dwdxdy.\end{aligned}$$

*Proof.* Let $F(z) = \int_0^z f(w)dw$. Then $F'(t) = f(t)$. Integrating by parts gives

$$\begin{aligned}\int_0^z f(w)\log(w)dw &= \left[\log(w)\int_0^z f(w)dw\right]_0^z - \int_0^z \frac{1}{x}\int_0^x f(w)dwdx \\ &= \log(z)F(z) - \underbrace{[\log(n)F(n)]}_{\to 0 \text{ as } n\to 0} - \int_0^z \frac{1}{x}\int_0^x f(w)dwdx \\ &= \log(z)\int_0^z f(w)dw - \int_0^z \frac{1}{x}\int_0^x f(w)dwdx.\end{aligned}$$

Notice that if $f(w) = a_0 + a_1w + a_2w^2 \cdots$ with coefficients $a_i \in Q(R)$ then $F(z) = a_0z + (a_1/2)z^2 + (a_2/3)z^3 \cdots$ and $(\log n)F(n) = (n\log n)(a_0 + (a_1/2)n + (a_2/3)n^2 \cdots)$ which tends to zero as $n$ tends to zero.

We can easily see that $(\log n)^2F(n) \to 0$ as $n \to 0$. In proving the second part of the lemma, we shall continue to use methods of partial integration.

$$\int_0^z f(w)(\log w)^2 dw = \left[(\log w)^2 \int_0^z f(w)dw\right]_0^z - \int_0^z \frac{2\log x}{x}\int_0^x f(w)dwdx$$
$$= (\log z)^2 F(z) - \underbrace{(\log n)^2 F(n)}_{\to 0 \text{ as } n\to 0} - \left[2\log(x)\int_0^z \frac{1}{x}\int_0^x f(w)dwdx\right]_0^z$$
$$+ 2\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x f(w)dwdxdy$$
$$= (\log z)^2 F(z) - 2\log(z)\int_0^z \frac{1}{x}\int_0^x f(w)dwdx + \underbrace{2\log(n)G(n)}_{\to 0 \text{ as } n\to 0}$$
$$+ 2\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x f(w)dwdxdy$$
$$= (\log z)^2 \int_0^z f(w)dw - 2\log(z)\int_0^z \frac{1}{x}\int_0^x f(w)dwdx$$
$$+ 2\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x f(w)dwdxdy$$

where we define

$$G(n) = \int_0^n \frac{F(x)}{x}dx.$$

□

**Lemma 3.2.** *Let $R$ be a domain of characteristic zero. Let $p, q, r \in R[z]$, with $p(0) = 1$. Let $L_2$ be the differential operator defined by*

$$L_2(y) := zp(z)y'' + q(z)y' + r(z)y.$$

*Suppose there exists $W_0 \in R[[z]]$ such that $W_0(0) = 1$ and the logarithmic derivative of $W_0/z$ is $-q(z)/zp(z)$. We remark that $W_0/z$ is called the* **Wronskian determinant** *of $L_2$.*

*Suppose that the equation $L_2(y) = 0$ has a formal power series solution, $y_0 \in R[[z]]$, with $y_0(0) = 1$. Since the space of solutions in $Q(R)[[z]]$ has dimension one, we observe that $y_0$ is unique.*

*Moreover, let*

$$y_1(z) = y_0(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt.$$

*Then $y_1(z)$ is a solution to $L_2(y) = 0$ with $y_0(z)$ and $y_1(z)$ linearly independent.*

*Proof.* First of all, we shall prove that if $y_0(z) \in Q(R)[[z]]$ then it is the unique solution to $L_2(y) = 0$. Suppose we have two distinct solutions in $Q(R)[[z]]$. Let $y(z)$ be the difference of these two solutions. Therefore,

$$y(z) = \not{C_0} + C_1 z + C_2 z^2 + \cdots,$$
$$y'(z) = C_1 + 2C_2 z + 3C_3 z^2 + \cdots,$$
$$y''(z) = 2C_2 + 6C_3 z + \cdots,$$

where $C_0 = 0$ because both solutions must satisfy the condition $y_0(0) = 1$. Let $f(z) = W_0(z)/z$ with $f'(z)/f(z) = -q(z)/zp(z)$. Then

$$\begin{aligned} W_0(z) &= 1 + d_1 z + d_2 z^2 + \cdots , \\ f(z) &= \frac{1}{z} + d_1 + d_2 z + \cdots , \\ f'(z) &= \frac{-1}{z^2} + 0 + d_2 + 2d_3 z + \cdots , \\ \frac{f'(z)}{f(z)} &= \frac{-1}{z} + \cdots = \frac{-q(z)}{zp(z)}, \\ \frac{q(z)}{p(z)} &= 1 + \cdots \Rightarrow \frac{q(0)}{p(0)} = 1. \end{aligned}$$

Since $p(0) = 1$ we deduce that $q(0) = 1$. We substitute our findings into $L_2(y) = 0$.

$$\begin{aligned} z(1+\cdots)&(2C_2 + 6C_3 z + \cdots) \\ &+ (1+\cdots)(C_1 + 2C_2 z + 3C_3 z^2 + \cdots) \\ &\qquad + r(z)(C_1 z + C_2 z^2 + \cdots) = 0. \end{aligned}$$

Evaluating at $z = 0$ we see that $C_1 = 0$. By induction we have $C_i = 0$ for all $i \in \mathbb{N}$, thereby confirming the uniqueness of $y_0(z)$.

It remains to show that $y_1(z)$ is another independent solution of $L_2(y) = 0$. We already have

$$y_1(z) = y_0(z)\int_0^z \frac{W_0(t)}{t y_0^2(t)} dt.$$

Differentiation with respect to $z$ gives

$$y_1'(z) = y_0'(z)\int_0^z \frac{W_0(t)}{t y_0^2(t)} dt + \left(\frac{W_0(z)}{z} \cdot \frac{1}{y_0(z)}\right)$$

and

$$y_1''(z) = y_0''(z)\int_0^z \frac{W_0(t)}{t y_0^2(t)} dt + \cancel{y_0'(z)\frac{W_0(z)}{z y_0^2(z)} + \frac{W_0(z)}{z}\left(\frac{-y_0'(z)}{y_0^2(z)}\right)} + \frac{1}{y_0(z)} \cdot \left(\frac{-q(z)W_0(z)}{z^2 p(z)}\right)$$

where we have calculated the derivative of $W_0(z)/z$ under the assumption of the existence of its logarithmic derivative. Let $f(z) = W_0(z)/z$. Then $f'(z)/f(z) = -q(z)/zp(z)$ and therefore

$$f'(z) = \frac{-q(z)}{zp(z)} f(z) = \frac{-q(z)W_0(z)}{z^2 p(z)}.$$

Substituting $y_1(z)$ and its derivatives into $L_2(y)$ gives

$$\begin{aligned}
L_2(y_1) &= zp(z)\left((y_0''(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt + \frac{1}{y_0(z)}\left(\frac{-q(z)W_0(z)}{z^2p(z)}\right)\right) \\
&\quad + q(z)\left(y_0'(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt + \frac{W_0(z)}{zy_0(z)}\right) + r(z)\left(y_0(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt\right) \\
&= \underbrace{(zp(z)y_0''(z) + q(z)y_0'(z) + r(z)y_0)}_{=0 \text{ since } y_0(z) \text{ solves } L_2(y)=0}\left(\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt\right) \\
&\quad + \frac{z\cancel{p(z)}}{y_0(z)}\cdot\left(\frac{-q(z)W_0(z)}{z^2\cancel{p(z)}}\right) + q(z)\frac{W_0(z)}{zy_0(z)} = 0.
\end{aligned}$$

Hence $y_1(z)$ is another independent solution of $L_2(y) = 0$. □

**Lemma 3.3.** *We can write $y_1(z) = y_0(z)\log(z) + \tilde{y}_0(z)$, where $\tilde{y}_0(z) \in Q(R)[[z]]$. Furthermore, the denominator of the $n$-th coefficient of $\tilde{y}_0(z)$ divides lcm$(1, 2, \ldots, n)$.*

*Proof.* Suppose

$$\begin{aligned}
W_0(z) &= 1 + d_1z + d_2z^2 + \cdots, \\
y_0(z) &= 1 + b_1z + b_2z^2 + \cdots, \\
y_0^2(z) &= 1 + c_1z + c_2z^2 + \cdots, \qquad \text{with } d_i, b_i, c_i \in R.
\end{aligned}$$

Then,

$$\begin{aligned}
y_1(z) &= y_0(z)\int_0^z \frac{1}{t}\left(\frac{1 + d_1t + d_2t^2 + \cdots}{1 + c_1t + c_2t^2 + \cdots}\right)dt \\
&= y_0(z)\int_0^z \frac{1}{t}\left(1 + e_1t + e_2t^2 + \cdots\right)dt \qquad \text{where } e_i \in Q(R) \\
&= y_0(z)\log(z) + y_0(z)\left(e_1z + \frac{e_2z^2}{2} + \frac{e_3z^3}{3} + \cdots\right).
\end{aligned}$$

Therefore,

$$\tilde{y}_0(z) = y_0(z)\left(e_1z + \frac{e_2z^2}{2} + \frac{e_3z^3}{3} + \cdots\right) \in Q(R)[[z]]$$

where we can choose a constant of integration such that $\tilde{y}_0(0) = 0$. If we suppose that $\tilde{y}_0(z)$ has the following power series expansion in $Q(R)[[z]]$

$$\tilde{y}_0(z) = a_1z + a_2z^2 + a_3z^3 + \cdots \text{ with } a_i \in Q(R),$$

then upon substitution of the power series expansion for $y_0(z)$ into $\tilde{y}_0(z)$, one notices

$$a_n = b_{n-1}e_1 + \frac{b_{n-2}e_2}{2} + \frac{b_{n-3}e_3}{3} + \cdots + \frac{b_0e_n}{n}.$$

It readily follows that the $n$-th coefficient of $\tilde{y}_0(z)$, $a_n$, has denominator dividing lcm$(1, 2, \ldots, n)$. □

**Corollary 3.1.** *We remark that*

$$\frac{W_0(z)}{z} = y_1'(z)y_0(z) - y_1(z)y_0'(z)$$

*which is precisely how the Wronskian should be defined.*

*Proof.* Some simple calculations leads to

$$y_1'(z)y_0(z) = y_0(z)y_0'(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt + \frac{W_0(z)}{z}$$

and

$$y_1(z)y_0'(z) = y_0(z)y_0'(z)\int_0^z \frac{W_0(t)}{ty_0^2(t)}dt,$$

thus the result follows. □

**Proposition 3.1.** $L_2(y) = 1$ *has a unique solution* $g(z) \in Q(R)[[z]]$ *beginning with* $z + O(z^2)$*. Moreover, the* $n$*-th coefficient of* $g(z)$ *has denominator dividing* $lcm(1, 2, \ldots, n)^2$.

Before we begin the proof, we remark that the operator $L_2$ has a symmetric square, denoted by $L_3$, which has certain properties and special characteristics . In Appendix B we explore the operator $L_3$ further and deduce key information about the coefficients of the power series solution of $L_3(y)$. This is a key step needed in proving the irrationality of $\zeta_2(3)$ and $\zeta_3(3)$ which we do not study here. (See Beukers' paper [4] for the full proof of the irrationality of $\zeta_2(3)$ and $\zeta_3(3)$).

*Proof.* Define

$$g(z) = y_1(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt.$$

Then

$$\begin{aligned} g'(z) = y_1'(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt &+ \cancel{y_1(z)\left(\frac{y_0(z)}{p(z)W_0(z)}\right)} \\ - y_0'(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt &- \cancel{y_0(z)\left(\frac{y_1(z)}{p(z)W_0(z)}\right)} \\ &= y_1'(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0'(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt \end{aligned}$$

and

$$\begin{aligned} g''(z) = y_1''(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt &+ y_1'(z)\left(\frac{y_0(z)}{p(z)W_0(z)}\right) \\ &- y_0''(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt - y_0'(z)\left(\frac{y_1(z)}{p(z)W_0(z)}\right). \end{aligned}$$

Substitution of $g(z)$ and its derivatives into $L_2(y)$ shows

$$\begin{aligned} L_2(g(z)) &= zp(z)\left(y_1'(z)\left(\frac{y_0(z)}{p(z)W_0(z)}\right) - y_0'(z)\left(\frac{y_1(z)}{p(z)W_0(z)}\right)\right) \\ &\quad + zp(z)\left[y_1''(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0''(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt\right] \\ &\quad + q(z)\left[y_1'(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0'(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt\right] \\ &\quad + r(z)\left[y_1(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0(z)\int_0^z \frac{y_1(t)}{p(t)W_0(t)}dt\right] \\ &= \frac{zp(z)}{p(z)W_0(z)}\left(y_1'(z)y_0(z) - y_0'(z)y_1(z)\right) = 1. \end{aligned}$$

The solution $g(z)$ is unique since the space of solutions in $Q(R)[[z]]$ has dimension one, not to mention one can observe this directly via the construction of $g(z)$ (it is a function of the unique solution, $y_0$).

To deduce information about the coefficients of the power series solution $g(z)$, we make use of the formulation $y_1(z) = y_0(z)\log(z) + \tilde{y}_0(z)$ alongside the identities proven in Lemma 3.1. A few calculations shows

$$\begin{aligned} g(z) &= (y_0(z)\log(z) + \tilde{y}_0(z))\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0(z)\int_0^z \frac{y_0(t)\log(t) + \tilde{y}_0(t)}{p(t)W_0(t)}dt \\ &= (y_0(z)\log(z) + \tilde{y}_0(z))\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0(z)\int_0^z \frac{\tilde{y}_0(t)}{p(t)W_0(t)}dt \\ &\quad - y_0(z)\left(\log(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - \int_0^z \frac{1}{x}\int_0^x \frac{y_0(t)}{p(t)W_0(t)}dt\right) \\ &= \tilde{y}_0(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt - y_0(z)\int_0^z \frac{\tilde{y}_0(t)}{p(t)W_0(t)}dt + y_0(z)\int_0^z \frac{1}{x}\int_0^x \frac{y_0(t)}{p(t)W_0(t)}dt \end{aligned}$$

which makes it easier to express $g(z)$ as a power series. We let

$$\begin{aligned} \tilde{y}_0(z) &= a_1z + a_2z^2 + \cdots, \\ y_0(z) &= 1 + b_1z + b_2z^2 + \cdots, \\ p(z) &= 1 + p_1z + p_2z^2 + \cdots, \\ W_0(z) &= 1 + d_1z + d_2z^2 + \cdots, \qquad \text{with } a_i, b_i, p_i, d_i \in R. \end{aligned}$$

Then,

$$\begin{aligned} \tilde{y}_0(z)\int_0^z \frac{y_0(t)}{p(t)W_0(t)}dt &= \tilde{y}_0(z)\int_0^z \frac{1 + b_1t + b_2t^2 + \cdots}{(1 + p_1t + \cdots)(1 + d_1t + \cdots)}dt \\ &= \tilde{y}_0(z)\int_0^z (1 + k_1t + k_2t^2 + \cdots)dt \qquad \text{with } k_i \in Q(R) \\ &= \tilde{y}_0(z)\left(z + \frac{k_1z^2}{2} + \frac{k_2z^3}{3} + \cdots\right). \end{aligned}$$

Recalling that the denominator of the $n$-th coefficient of $\tilde{y}_0(z)$ divides $\operatorname{lcm}(1, 2, \ldots, n)$, we can clearly see that in taking the above product, the denominator of the $n$-th coefficient of $\tilde{y}_0 \int (y_0/pW_0)$ divides $\operatorname{lcm}(1, 2, \ldots, n)^2$.

Similarly, we deduce information about the denominators of the $n$-th coefficient for the remaining two terms of the power series solution of $y_1(z)$.

$$\begin{aligned} y_0(z)\int_0^z \frac{\tilde{y}_0(t)}{p(t)W_0(t)}dt &= (1+b_1z+\cdots)\int_0^z \frac{a_1t+a_2t^2+\cdots}{(1+p_1t+\cdots)(1+d_1t+\cdots)}dt \\ &= (1+b_1z+\cdots)\int_0^z (l_1t+l_2t^2+\cdots)dt \quad \text{with } l_i\in Q(R) \\ &= (1+b_1z+\cdots)\left(l_1\frac{z^2}{2}+l_2\frac{z^3}{3}+\cdots\right). \end{aligned}$$

In this case we observe that each coefficient $l_i$ has denominator dividing $\mathrm{lcm}(1,2,\ldots,n)$, and again, when taking the above product, we clearly see that the denominator of the $n$-th coefficient of $y_0\int(\tilde{y}_0/pW_0)$ divides $\mathrm{lcm}(1,2,\ldots,n)^2$.

It is evident that the $n$-th coefficient of the final term will divide $\mathrm{lcm}(1,2,\ldots,n)^2$ due to the double integral. Some calculations shows

$$\begin{aligned} &y_0(z)\int_0^z \frac{1}{x}\int_0^x \frac{y_0(t)}{p(t)W_0(t)}dtdx \\ &= (1+b_1z+\cdots)\int_0^z \frac{1}{x}\int_0^x \frac{(1+b_1t+\cdots)}{(1+p_1t+\cdots)(1+d_1t+\cdots)}dtdx \\ &= (1+b_1z+\cdots)\int_0^z \frac{1}{x}\int_0^x (1+m_1t+m_2t^2+\cdots)dtdx \quad \text{with } m_i\in Q(R) \\ &= (1+b_1z+\cdots)\int_0^z \frac{1}{x}\left(x+\frac{m_1x^2}{2}+\frac{m_2x^3}{3}+\cdots\right)dx \\ &= (1+b_1z+\cdots)\left(z+\frac{m_1z^2}{2\cdot 2}+\frac{m_2z^3}{3\cdot 3}+\cdots\right) \end{aligned}$$

which confirms that the $n$-th coefficient of $y_0\int 1/x\int y_0/pW_0$ has denominator dividing $\mathrm{lcm}(1,2,\cdots,n)^2$, hence the denominator of the $n$-th coefficient of $g(z)$ divides $\mathrm{lcm}(1,2,\ldots,n)^2$. As a final remark, one easily observes that $g(z)=z+O(z^2)$. □

## 4. An Identity for $\Theta(x)$

Let $\mathbb{Q}(x)$ be the field of rational functions with a discrete valuation such that $|x|>1$. Denote its completion with respect to that valuation by $K$. Then $K$ is the field of formal Laurent series in $1/x$. Throughout this section we assume all calculations are over the field $K$.

We begin this section with a well-known definition, taken from [9].

**Definition 4.1.** *The Bernoulli numbers $B_n$ are defined via the generating function*

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_nx^n}{n!}.$$

**Lemma 4.1.** *$B_1=-1/2$ and $B_{2n+1}=0$ for all $n\geq 1$.*

*Proof.* We follow the proof on `PlanetMath.Org` [1], although of course we've checked the details. We obtain $B_1=-1/2$ by writing the first two terms of the Taylor expansion in Definition 4.1. Adding $x/2$ to both sides gives

$$\sum_{n\geq 0, n\neq 1} \frac{B_n x^n}{n!} = \frac{x}{e^x-1} + \frac{x}{2} = \frac{x}{2}\cdot\frac{e^x+1}{e^x-1} = \frac{x}{2}\cdot\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \eta(x).$$

The function $\eta(x)$ is clearly even, so all terms $B_n x^n/n!$ with $n$ odd and $n \neq 1$ are zero. □

**Lemma 4.2.** *For any $n > 1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} B_k = B_n.$$

*When $n = 1$ we have $B_0 + B_1 = 1 + B_1$.*

*Proof.* We expand Definition 4.1 using Taylor polynomials and equate coefficients of $x^n$. Beukers takes a different approach in [4], however, I feel that the proof below is more transparent.

Expanding Definition 4.1 gives

$$x = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\cdot\left(B_0 + B_1 x + B_2\frac{x^2}{2!} + \cdots\right).$$

Equating the constant terms, we see that $B_0 = 1$. Therefore when $n = 1$ we have the relation $B_0 + B_1 = 1 + B_1$. Comparing the coefficients of $x^n$ for $n > 1$ we find

$$\frac{B_0}{0!}\cdot\frac{1}{n!} + \frac{B_1}{1!}\cdot\frac{1}{(n-1)!} + \frac{B_2}{2!}\cdot\frac{1}{(n-2)!} + \cdots + \frac{B_{n-1}}{(n-1)!}\cdot\frac{1}{1!} = 0.$$

Multiplying by $n!$ we obtain

$$\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$$

and adding $B_n$ to both sides gives us the desired results. □

**Definition 4.2.** *We define the following infinite Laurent series as Beukers, [4] states in the introduction .*

$$\Theta(x) = \sum_{n\geq 0} t_n\,(-1/x)^{n+1}$$

$$R(x) = \sum_{n\geq 0} B_n\,(-1/x)^{n+1}$$

$$T(x) = \sum_{n\geq 0}(n+1)B_n\,(-1/x)^{n+2}$$

*where $B_n$ are the Bernoulli numbers and $t_n = (2^{n+1}-2)B_n$.*

**Lemma 4.3.** *$T(x) = R'(x)$ and $\Theta(x) = R(x/2) - 2R(x)$.*

*Proof.* The first assertion follows directly by term-by-term differentiation of $R(x)$.

To prove the second assertion, we make a few simple calculations.

$$\begin{aligned} R(x/2) - 2R(x) &= \sum_{n\geq 0} B_n \left(-2/x\right)^{n+1} - 2\sum_{n\geq 0} B_n \left(-1/x\right)^{n+1} \\ &= \sum_{n\geq 0} \left(2^{n+1} B_n \left(-1/x\right)^{n+1} - 2B_n \left(-1/x\right)^{n+1}\right) \\ &= \sum_{n\geq 0} t_n \left(-1/x\right)^{n+1} = \Theta(x). \end{aligned}$$

□

**Lemma 4.4.** *Given $A(x) \in K$, suppose there exists $\lambda \neq 0$, where $\lambda \in \mathbb{Q}$ such that $A(x + \lambda) = A(x)$. Then $A(x)$ is constant.*

*Proof.* Suppose $A(x)$ is an infinite Laurent series given by

$$A(x) = a_{-m}x^m + a_{(-m+1)}x^{m-1} + \cdots + a_0 + a_1 x^{-1} + \cdots$$

Shifting $A$ by $-a_0$, we can assume without loss of generality that $a_0 = 0$. Therefore, want to show that $A \equiv 0$. In other words, we prove that $a_i = 0$ for all $i$. By the Binomial Theorem, for $n > 0$,

$$(x + \lambda)^n - x^n = n\lambda x^{n-1} + \text{l.o.t.}$$

and

$$\begin{aligned} (x + \lambda)^{-n} - x^{-n} &= x^{-n} \left((1 + \lambda/x)^{-n} - 1\right) \\ &= x^{-n} \left(-n\lambda/x + \text{l.o.t.}\right) \\ &= -n\lambda x^{-n-1} + \text{l.o.t.} \end{aligned}$$

where *lower order terms* has been abbreviated to *l.o.t.*

For a contradiction, we suppose $A(x)$ is non-zero. If the highest order term of $A(x)$ is $a_n x^n$ with positive exponent $n$, then we see that

$$\begin{aligned} A(x + \lambda) - A(x) &= na_n\lambda x^{n-1} + \text{l.o.t.} + (n-1)a_{n-1}\lambda x^{n-2} + \text{l.o.t.} \\ &= na_n\lambda x^{n-1} + \text{l.o.t.} \end{aligned}$$

Since $A(x + \lambda) - A(x) = 0$, then $na_n\lambda$ must equal 0, implying that $a_n = 0$ which contradicts the assumption that $a_n x^n$ is the highest order term. Thus the highest order term is $a_n x^n$ with $n < 0$. A similar calculation shows that $na_n\lambda = 0$, again giving a contradiction. Hence $A(x)$ is constant. □

**Lemma 4.5.** *For all $x \in \mathbb{R}$, we have*

*i.* $R(x + 1) - R(x) = 1/x^2$
*ii.* $R(x) + R(-x) = -1/x^2$
*iii.* $R(x) + R(1 - x) = 0$
*iv.* $R(x) + R\left(x + \frac{1}{2}\right) = 4R(2x)$.

*Proof.* We begin with the proof of statement *i.* which is only outlined in [4].

$$\begin{aligned}
R(x+1) &= \sum_{k\geq 0} B_k \left(\frac{-1}{x+1}\right)^{k+1} = \sum_{k\geq 0} B_k \left(\frac{-1}{x}\right)^{k+1} \cdot \left(\frac{1}{1+1/x}\right)^{k+1} \\
&= \sum_{k\geq 0} B_k \left(\frac{-1}{x}\right)^{k+1} \left(\sum_{n\geq 0} \binom{n+k}{k} \left(\frac{-1}{x}\right)^n\right) && \text{using equation(A.1)} \\
&= \sum_{k\geq 0}\sum_{n\geq 0} B_k \binom{n+k}{k} \left(\frac{-1}{x}\right)^{n+k+1} \\
&= \sum_{k\geq 0}\sum_{j=k}^{\infty} B_k \binom{j}{k} \left(\frac{-1}{x}\right)^{j+1} && \text{change variables and let } j=n+k \\
&= \sum_{j\geq 0}\sum_{k=0}^{j} B_k \binom{j}{k} \left(\frac{-1}{x}\right)^{j+1} && \text{interchange the summations} \\
&= \sum_{j>1} B_j \left(\frac{-1}{x}\right)^{j+1} && \text{by Lemma 4.2} \\
&\qquad + \underbrace{B_0 \left(\frac{-1}{x}\right)}_{k=0,j=0} + \underbrace{B_0 \left(\frac{-1}{x}\right)^2}_{k=0,j=1} + \underbrace{B_1 \left(\frac{-1}{x}\right)^2}_{k=1,j=1} \\
&= \sum_{j\geq 0} B_j \left(\frac{-1}{x}\right)^{j+1} + B_0 \left(\frac{-1}{x}\right)^2
\end{aligned}$$

where in the fourth line, we can also take the sum over $j \geq 0$ because if $j < k$, then the binomial coefficients are identically zero.

We now prove statement *ii.* Since $B_1 = -1/2$, we note

$$R(x) = \sum_{k\geq 0} B_k \left(-\frac{1}{x}\right)^{k+1} = -\frac{1}{2x^2} + \sum_{k>0,k\neq 1} B_k \left(-\frac{1}{x}\right)^{k+1}$$

and

$$R(-x) = -\frac{1}{2x^2} + \sum_{k>o,k\neq 1} B_k \left(\frac{1}{x}\right)^{k+1}.$$

Adding these equations and using the fact that all odd Bernoulli numbers are zero (Lemma 4.1), we find that $R(x) + R(-x) = -1/x^2$ as desired.

Statement *iii.* can be verified using statements *i.* and *ii.*

$$\begin{aligned}
R(x) + R(1-x) &= -1/x^2 - R(-x) + R(1-x) && \text{from } ii. \\
&= -1/x^2 - R(y) + R(1+y) && \text{let } y=-x \\
&= -1/x^2 + 1/x^2 = 0 && \text{apply } i.
\end{aligned}$$

To prove the last statement, we write $A(x) = 4R(2x) - R(x) - R(x+1/2)$. It is easy to see that the constant term of $A(x)$ is zero. We deduce from statements *i.* and *iii.*,

$$\begin{aligned} A\left(x+1/2\right)-A(x) &= 4R(2x+1)-4R(2x)-\cancel{R\left(x+1/2\right)} \\ &\quad +\cancel{R\left(x+1/2\right)}+R(x)-R(x-1) \\ &= 4/(2x)^2-1/x^2=0 \end{aligned}$$

and so by Lemma 4.3, we have $A(x)\equiv 0$. □

**Lemma 4.6.** *For all* $x\in\mathbb{R}$*, we have*

*i.* $T(x+1)-T(x)=-2/x^3$
*ii.* $T(x)-T(1-x)=0$
*iii.* $\Theta(x+1)+\Theta(x)=-2/x^2$
*iv.* $\Theta(x)=\frac{1}{2}\left(R\left(\frac{x}{2}\right)-R\left(\frac{x}{2}+\frac{1}{2}\right)\right)$

*Proof.* Proving the first statement requires the use of Lemma 4.3 and statement *i.* from Lemma 4.5 which simply gives

$$T(x+1)-T(x)=R'(x+1)-R'(x)=-2/x^3.$$

Part *ii.* can be shown using a similar method, but using statement *ii.* from Lemma 4.5.

$$T(x)-T(1-x)=R'(x)+R'(1-x)=0.$$

The proof of statement *iii.* uses Lemma 4.3 which says $\Theta(x)=R(x/2)-2R(x)$ and Lemma 4.5 where we let $y=x/2$ in the statement $R(y)+R(y+1/2)=4R(2y)$.

$$\begin{aligned} \Theta(x+1)+\Theta(x) &= R\left(\frac{x}{2}+\frac{1}{2}\right)-2R(x+1)+R\left(\frac{x}{2}\right)-2R(x) \\ &= 4R(x)-2R(x+1)-2R(x) \\ &= -2/x^2. \end{aligned}$$

Proving the last statement is similar to that shown above, namely using Lemma 4.3 and statement *iii.* from Lemma 4.5. We remark here that our proof shows that the statement from [4] needs to be adjusted by a factor of 1/2.

$$\begin{aligned} \Theta(x) &= R\left(x/2\right)-2R(x) \\ &= R\left(x/2\right)-\frac{1}{2}R\left(x/2\right)-\frac{1}{2}R\left(x/2+1/2\right) \\ &= \frac{1}{2}\left(R\left(x/2\right)-R\left(x/2+1/2\right)\right). \end{aligned}$$

□

We now take a look at the main Proposition for this section. The Proposition has been modified from Beukers' paper [4] and we give a full detailed proof here. Before we state Proposition 4.1, we introduce a few definitions which we shall constantly refer back to in the remaining sections.

**Definition 4.3.**

$$\begin{bmatrix} n \\ x \end{bmatrix}=\frac{n!}{x(x+1)\cdots(x+n)}.$$

**Definition 4.4.** *The forward difference operator* $\Delta_n$ *is defined by*

$$\Delta_n(g(n)) := g(n+1) - g(n).$$

*On summation, we have a telescoping series*

$$\sum_{n=0}^{\infty} \Delta_n(g(n)) = \lim_{k\to\infty} g(k) - g(0).$$

**Proposition 4.1.** *Let* $\Theta(x) \in K$ *be the Taylor series in* $1/x$ *as stated in Definition 4.2. Then*

$$\Theta(x) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.$$

*Proof.* We let

$$S(x) = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.$$

Some straightforward calculations shows

$$\begin{aligned}
&\begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} + \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ -x \end{bmatrix} \\
&= \frac{n!}{x(x+1)\cdots(x+n)} \cdot \frac{n!}{(1-x)(2-x)\cdots(1+n-x)} \\
&\qquad + \frac{n!}{(x+1)\cdots(x+1+n)} \cdot \frac{n!}{(-x)(1-x)\cdots(n-x)} \\
&= \frac{(x+1+n)(n!)^2 - (1+n-x)(n!)^2}{x(x+1)\cdots(x+1+n)(1-x)\cdots(1+n-x)} \\
&= \frac{2x(n!)^2}{x(x+1)\cdots(x+1+n)(1-x)\cdots(1+n-x)} \\
&= \frac{n!}{(x+1)\cdots(x+1+n)} \cdot \frac{n!}{(1-x)\cdots(1+n-x)} = 2\begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.
\end{aligned}$$

We notice the following which will make calculations in the remainder of the proof more elegant. Before we progress, we note that

$$\begin{bmatrix} n+1 \\ x+1 \end{bmatrix} = \frac{(n+1)n!}{(x+1)\cdots(n+2+x)} = \frac{n+1}{n+2+x} \cdot \begin{bmatrix} n \\ x+1 \end{bmatrix}$$

and similarly

$$\begin{bmatrix} n+1 \\ 1-x \end{bmatrix} = \frac{(n+1)n!}{(1-x)\cdots(n+2-x)} = \frac{n+1}{n+2-x} \cdot \begin{bmatrix} n \\ 1-x \end{bmatrix}.$$

Then,

$$\begin{aligned}
&\Delta_n\left(\frac{(n+1-x)(n+1+x)}{x^2}\begin{bmatrix} n \\ 1-x \end{bmatrix}\begin{bmatrix} n \\ 1+x \end{bmatrix}\right)\\
&\quad=\left(\frac{(n+2-x)(n+2+x)}{x^2}\begin{bmatrix} n+1 \\ 1-x \end{bmatrix}\begin{bmatrix} n+1 \\ 1+x \end{bmatrix}\right)\\
&\qquad\qquad-\left(\frac{(n+1-x)(n+1+x)}{x^2}\begin{bmatrix} n \\ 1-x \end{bmatrix}\begin{bmatrix} n \\ 1+x \end{bmatrix}\right)\\
&=\left(\frac{\cancel{(n+2-x)(n+2+x)}(n+1)^2}{\cancel{(n+2-x)(n+2+x)}\cdot x^2}-\frac{(n+1)^2-x^2}{x^2}\right)\begin{bmatrix} n \\ 1-x \end{bmatrix}\begin{bmatrix} n \\ 1+x \end{bmatrix}\\
&=\begin{bmatrix} n \\ x+1 \end{bmatrix}\begin{bmatrix} n \\ 1-x \end{bmatrix}.
\end{aligned}$$

So far we have shown

$$\begin{aligned}
\begin{bmatrix} n \\ x \end{bmatrix}\begin{bmatrix} n \\ 1-x \end{bmatrix}+\begin{bmatrix} n \\ x+1 \end{bmatrix}\begin{bmatrix} n \\ -x \end{bmatrix}&=2\begin{bmatrix} n \\ x+1 \end{bmatrix}\begin{bmatrix} n \\ 1-x \end{bmatrix}\\
&=2\Delta_n\left(\frac{(n+1-x)(n+1+x)}{x^2}\begin{bmatrix} n \\ 1-x \end{bmatrix}\begin{bmatrix} n \\ 1+x \end{bmatrix}\right).
\end{aligned}$$

Summation over $n \geq 0$ gives

$$\begin{aligned}
S(x)+S(x+1)&=2\sum_{n=0}^{\infty}\Delta_n\left(\frac{(n+1-x)(n+1+x)}{x^2}\begin{bmatrix} n \\ 1-x \end{bmatrix}\begin{bmatrix} n \\ 1+x \end{bmatrix}\right)\\
&=2\lim_{k\to\infty}\frac{(k+1-x)(k+1+x)}{x^2}\begin{bmatrix} k \\ 1-x \end{bmatrix}\begin{bmatrix} k \\ 1+x \end{bmatrix}\\
&\qquad-2\left(\frac{(1-x)(1+x)}{x^2}\frac{0!}{(1-x)}\frac{0!}{(1+x)}\right)\\
&=-2/x^2.
\end{aligned}$$

In order to complete the proof, we need to show

$$\lim_{k\to\infty}(k+1-x)(k+1+x)\begin{bmatrix} k \\ 1-x \end{bmatrix}\begin{bmatrix} k \\ 1+x \end{bmatrix}=0$$

which is equivalent to evaluating

$$\lim_{k\to\infty}\frac{(k!)^2}{(1-x^2)(4-x^2)\cdots(k^2-x^2)}.$$

As $k \to \infty$, we have $|k!|_p \to 0$. Moreover, since $|x|_p > 1$, using the ultrametric inequality, we have $|i^2 - x^2|_p > 1$ for all $i$.

Therefore, $S(x+1) + S(x) = -2/x^2$. Now $S(x+1) - \Theta(x+1) + S(x) - \Theta(x) = 0$ hence $S(x) - \Theta(x)$ is periodic with period 1. By Lemma 4.3, we have $S(x) - \Theta(x) \equiv 0$. □

As a final remark, please see Appendix C, where we have constructed identities with a similar structure for $R(x)$ and $T(x)$.

## 5. $p$-adic Hurwitz Series

From here on we let $p$ be a prime. Let $F$ be a positive integer and let $a$ be an integer such that $F$ does not divide $a$.

**Definition 5.1.** *Define the Hurwitz zeta function as follows*

$$H(s,a,F) = \sum_{n=0}^{\infty} \frac{1}{(a+nF)^s}.$$

The series converges for all $s \in \mathbb{C}$ such that $Re(s) > 1$. The Hurwitz zeta function can be continued analytically to the whole complex $s$-plane, except for a pole at $s = 1$.

**Definition 5.2.** *Define the generalised Bernoulli Numbers $B_n(a,F)$ as Murty and Reece do in* [9]*, by the following*

$$\frac{te^{at}}{e^{Ft}-1} = \sum_{n\geq 0} \frac{B_n(a,F)}{n!} t^n.$$

**Proposition 5.1.** *Let $n$ be an integer such that $n \geq 1$. Then for any $F, a, n$ we have*

$$H(1-n,a,F) = -\frac{B_n(a,F)}{n}.$$

*Proof.* We won't prove the proposition here since there are extensive calculations involved which can go deep the theory of complex analysis, thus sidetracking us. We shall take this as given from Beukers' paper [4] but those interested should consider reading [9] which looks at relations between the Hurwitz zeta function and the Bernoulli numbers. □

**Lemma 5.1.** *For any positive integer $n$ we have*

$$B_n(a,F) = \frac{a^n}{F} \sum_{j=0}^{n} \binom{n}{j} B_j \left(\frac{F}{a}\right)^j.$$

*Proof.*

$$\begin{aligned}
\frac{te^{at}}{e^{Ft}-1} &= \frac{e^{at}}{F} \cdot \frac{Ft}{e^{Ft}-1} = \frac{e^{at}}{F} \sum_{j\geq 0} B_j \frac{(Ft)^j}{j!} \\
&= \frac{1}{F} \left( \sum_{k\geq 0} \frac{(at)^k}{k!} \right) \left( \sum_{j \geq 0} B_j \frac{(Ft)^j}{j!} \right) \\
&= \frac{1}{F} \sum_{n\geq 0} \sum_{k+j=n} \left( \frac{a^k}{k!} \frac{B_j}{j!} F^j \right) t^n \\
&= \frac{1}{F} \sum_{n\geq 0} \sum_{j=0}^{n} \left( \frac{a^n}{(n-j)!} \frac{B_j}{j!} \left(\frac{F}{a}\right)^j \right) t^n \\
&= \frac{1}{F} \sum_{n\geq 0} \left( \sum_{j=0}^{n} \binom{n}{j} B_j \left(\frac{F}{a}\right)^j \right) \frac{a^n}{n!} t^n.
\end{aligned}$$

We get the desired results by comparing coefficients. □

From here on, we shall assume $p$ is a prime such that $p \mid F$ and $p \nmid a$. We therefore notice that $|B_j(F/a)^j|_p \to 0$ as $j \to \infty$.

**Definition 5.3.** *The Teichmüller character, $\omega : \mathbb{Z} \mapsto \mathbb{Z}_p$ is defined as follows*

$$\omega(m) = \begin{cases} 0 \text{ when } p \mid m; \\ m \mod p \text{ when } p \text{ odd and } \gcd(p,m) = 1; \\ (-1)^{\frac{m-1}{2}} \text{ when } p = 2 \text{ and } m \text{ odd.} \end{cases}$$

Notice that when $p$ is odd and $\gcd(p, m)$=1, we can apply Fermat's Little Theorem to get $\omega(m)^{p-1} = 1$.

**Definition 5.4.** *For all $x \in \mathbb{Z}$ such that $p \nmid x$ we can define*

$$< x >= \omega(x)^{-1}x.$$

**Definition 5.5.** *We define for all $s \in \mathbb{Z}_p$ the p-adic Hurwitz zeta function,*

$$H_p(s, a, F) = \frac{1}{F(s-1)} < a >^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} B_n \left(\frac{F}{a}\right)^n.$$

When $s = 2$, we can make an explicit calculation.

$$\begin{aligned} H_p(2, a, F) &= \frac{1}{F} < a >^{-1} \sum_{n=0}^{\infty} \binom{-1}{n} B_n \left(\frac{F}{a}\right)^n \\ &= \frac{1}{aF}\omega(a) \sum_{n=0}^{\infty} (-1)^n B_n \left(\frac{F}{a}\right)^n \\ &= -\frac{\omega(a)}{F^2} \sum_{n=0}^{\infty} B_n \left(-\frac{F}{a}\right)^{n+1} = -\frac{\omega(a)}{F^2} R\left(\frac{a}{F}\right). \end{aligned}$$

Likewise for $s = 3$, we have,

$$\begin{aligned} H_p(3, a, F) &= \frac{1}{2F} < a >^{-2} \sum_{n=0}^{\infty} \binom{-2}{n} B_n \left(\frac{F}{a}\right)^n \\ &= (-1)^2 \cdot \frac{1}{2a^2F}\omega(a)^2 \sum_{n=0}^{\infty} (-1)^n (n+1) B_n \left(\frac{F}{a}\right)^n \\ &= \frac{\omega(a)^2}{2F^3} \sum_{n=0}^{\infty} (n+1) B_n \left(-\frac{F}{a}\right)^{n+2} = \frac{\omega(a)^2}{2F^3} T\left(\frac{a}{F}\right). \end{aligned}$$

where we defined the Laurent Series $R(x)$ and $T(x)$ in 4.2.

**Definition 5.6.** *Let $\phi : \mathbb{Z} \mapsto \bar{\mathbb{Q}}$ be a periodic function with period $f$. Let $F = lcm(f, p)$ if $p$ is odd and $F = lcm(f, 4)$ if $p = 2$. We define the p-adic Kubota-Leopoldt L-series associated to $\phi$ as*

$$L_p(s, \phi) = \sum_{\substack{a=1 \\ p \nmid a}}^{F} \phi(a) H_p(s, a, F).$$

**Proposition 5.2.** *The value of $L_p(s, \phi)$ does not change if we choose a multiple period, for example $mF$ with $m \in \mathbb{Z}^+$.*

*Proof.* It is enough to show for all $n \in \mathbb{Z}$

$$\sum_{\substack{a=1\\p\nmid a}}^{F} \phi(a)H(1-n,a,F) = \sum_{\substack{a=1\\p\nmid a}}^{mF} \phi(a)H(1-n,a,mF).$$

First we calculate using the periodicity of $\phi$,

$$\frac{e^{Ft}-1}{t}\sum_{\substack{a=1\\p\nmid a}}^{mF} \frac{t\phi(a)e^{at}}{e^{mFt}-1} = \frac{e^{Ft}-1}{e^{mFt}-1}\sum_{\substack{a=1\\p\nmid a}}^{mF} \phi(a)e^{at}$$

$$= \frac{t}{e^{mFt}-1}\left(\phi(1)e^{t}+\cdots+\phi(F)e^{Ft}+\phi(F+1)e^{(F+1)t}+\cdots+\phi(mF)e^{mFt}\right)$$

$$= \frac{e^{Ft}-1}{e^{mFt}-1}\Big[\phi(1)\left(e^{t}+e^{(F+1)t}+\cdots+e^{((m-1)F+1)t}\right)+\cdots$$
$$\cdots+\phi(F)\left(e^{Ft}+e^{2Ft}+\cdots+e^{mFt}\right)\Big]$$

$$= \frac{e^{Ft}-1}{e^{mFt}-1}\Big[\phi(1)e^{t}\left(1+e^{Ft}+\cdots+e^{(m-1)Ft}\right)+\cdots$$
$$\cdots+\phi(F)e^{Ft}\left(1+e^{Ft}+\cdots+e^{(m-1)Ft}\right)\Big]$$

$$= \frac{e^{Ft}-1}{e^{mFt}-1}\cdot\left(1+e^{Ft}+\cdots+e^{(m-1)Ft}\right)\left[\phi(1)e^{t}+\cdots+\phi(F)e^{Ft}\right]$$

$$= \sum_{\substack{a=1\\p\nmid a}}^{F} \phi(a)e^{at}$$

where we use

$$(e^{Ft}-1)\cdot\left(1+e^{Ft}+\cdots+e^{(m-1)Ft}\right)$$
$$= \cancel{e^{Ft}}+\cdots+\cancel{e^{(m-1)Ft}}+e^{mFt}-(1+\cancel{e^{Ft}}+\cdots+\cancel{e^{(m-1)Ft}})$$
$$= e^{mFt}-1$$

and so we have

$$\sum_{\substack{a=1\\p\nmid a}}^{mF} \frac{t\phi(a)e^{at}}{e^{mFt}-1} = \sum_{\substack{a=1\\p\nmid a}}^{F} \frac{t\phi(a)e^{at}}{e^{Ft}-1}.$$

To finish the proof, we see that

$$\sum_{\substack{a=1\\p\nmid a}}^{mF} \frac{t\phi(a)e^{at}}{e^{mFt}-1} = \sum_{\substack{a=1\\p\nmid a}}^{F} \frac{t\phi(a)e^{at}}{e^{Ft}-1}$$

$$\sum_{\substack{a=1\\p\nmid a}}^{mF} \phi(a)\sum_{n\geq 0}\frac{B_n(a,mF)}{n!}t^n = \sum_{\substack{a=1\\p\nmid a}}^{F} \phi(a)\sum_{n\geq 0}\frac{B_n(a,F)}{n!}t^n$$

$$\sum_{\substack{a=1\\p\nmid a}}^{mF} \phi(a)\sum_{n\geq 0}\frac{-n\cdot H(1-n,a,mF)}{n!}t^n = \sum_{\substack{a=1\\p\nmid a}}^{F} \phi(a)\sum_{n\geq 0}\frac{-n\cdot H(1-n,a,F)}{n!}t^n$$

where the last line follows from Proposition 5.1. For each $n \geq 0$ we equate coefficients of $t^n$ to conclude

$$\sum_{\substack{a=1 \\ p\nmid a}}^{mF} \phi(a) H(1-n, a, mF) = \sum_{\substack{a=1 \\ p\nmid a}}^{F} \phi(a) H(1-n, a, F).$$

□

The above proposition was stated as a remark in Beukers' paper [4]. However, the proposition plays a vital role in proving the irrationality of $\zeta_2(2)$ and $\zeta_3(2)$ (seen shortly in Chapter 6), and so we have provided our complete proof here.

When $\phi(n) = 1$ for all $n \in \mathbb{Z}$, the period of $\phi$ is clearly 1. If $p$ is odd, then we have $F = \text{lcm}(1, p) = p$. Therefore we can define the $p$-adic zeta-function for odd $p$ as

$$L_p(s, 1) = \sum_{a=1}^{p-1} H_p(s, a, p) = \zeta_p(s). \tag{5.1}$$

If $p = 2$, then we have $F = \text{lcm}(1, 4) = 4$ and can therefore define the 2-adic zeta function as

$$\zeta_2(s) = \sum_{\substack{a=1 \\ p\nmid a}}^{4} H_2(s, a, 4) = H_2(s, 1, 4) + H_2(s, 3, 4). \tag{5.2}$$

## 6. Some $p$-adic Identities

In this section, we relate the $p$-adic values of $R(x), T(x)$ and $\Theta(x)$ with some $p$-adic $L$-series. We continue to assume $a/F \in \mathbb{Q}$ with $p \mid F$. On substitution of $x = a/F$ into the Laurent Series of $R(x), T(x)$ and $\Theta(x)$ , we can easily verify their $p$-adic convergence . We shall denote the $p$-adic values of $R(x), T(x)$ and $\Theta(x)$ by $R_p(x), T_p(x)$ and $\Theta_p(a/F)$ respectively.

We begin with a few fundamental definitions. Definitions 6.1, 6.2 and Lemma 6.1 have been taken directly from Mollin's book, [3] (Chapter 7; page 247).

**Definition 6.1.** *Fix $d \in \mathbb{N}$. Let $\chi : \mathbb{N} \mapsto \mathbb{C}$ be a map such that*

1. *$\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$*
2. *$m \equiv n \pmod d$, then $\chi(m) = \chi(n)$*
3. *$\chi(n) = 0$ iff $gcd(n, d) > 1$.*

*Then $\chi$ is a Dirichlet Character modulo d. We say that $\chi$ is primitive if $\chi$ has period exactly equal to d. We say that $\chi$ is even if $\chi(-1) = 1$.*

We remark that if $\gcd(n, d) = 1$, then $\chi(n)^{\phi(d)} = \chi(n^{\phi(d)}) = \chi(1)$ (where $\phi(d)$ denotes the Euler totient). Moreover $\chi(1) = \chi(1^2) = \chi(1)^2$ and $\chi(1) \neq 0$ which means $\chi(1) = 1$. Dirichlet characters are completely multiplicative.

**Lemma 6.1.** *Given $d > 1$, there exists exactly $\phi(d)$ distinct Dirichlet characters modulo d.*

**Definition 6.2.** *Given $d > 1$ and $\chi$ a Dirichlet character modulo $d$. Let $s \in \mathbb{C}$. Then,*

$$\sum_{n=1} \chi(n) \cdot n^{-s} = L(s, \chi)$$

*is a Dirichlet L-function.*

The remainder of this chapter will focus on proving the identities below.

**Proposition 6.1.** *Let $\chi_d$ be the primitive even Dirichlet character modulo $d$. Then,*

*i.* $\Theta_2(1/2) = -8\zeta_2(2)$
*ii.* $\Theta_2(1/6) = -40\zeta_2(2)$
*iii.* $\Theta_2(1/4) = -16L_2(2, \chi_8)$
*iv.* $\Theta_3(1/3) = -(27/2)\zeta_3(2)$
*v.* $\Theta_3(1/6) = -36L_3(2, \chi_{12})$.

*Proof.* From Section 5 we have already shown

$$R_p(a/F) = -F^2\omega(a)^{-1}H_p(2, a, F). \tag{6.1}$$

Using Lemma 4.6 from Section 4 we deduce

$$\begin{aligned}
\Theta_p(a/F) &= \frac{1}{2}\left(R_p\left(\frac{a}{2F}\right) - R_p\left(\frac{a+F}{2F}\right)\right) \\
&= \frac{1}{2}\left(-4F^2\omega(a)^{-1}H_p(2, a, 2F) + 4F^2\omega(a+F)^{-1}H_p(2, a+F, 2F)\right) \\
&= -2F^2\left(\omega(a)^{-1}H_p(2, a, 2F) - \omega(a+F)^{-1}H_p(2, a+F, 2F)\right).
\end{aligned}$$

To prove statement *i.*, we recall that $\omega(m) = (-1)^{(m-1)/2}$ for odd $m$ and when $p = 2$. This will be useful in proving statements *ii.* and *iii.* Therefore,

$$\begin{aligned}
\Theta_2(1/2) &= -2 \cdot 4\left(\omega(1)^{-1}H_2(2, 1, 4) - \omega(3)^{-1}H_2(2, 3, 4)\right) \\
&= -8\left(H_2(2, 1, 4) + H_2(2, 3, 4)\right) \\
&= -8\zeta_2(2).
\end{aligned}$$

Statement *i.* is the only part proved in Beukers' paper. We supply the proof for the other identities. Statement *ii.* is the most subtle since it requires the periodicity property of $L$-functions established in Proposition 5.2. We take $m = 3$, $\phi = 1$, and $F = 4$. Then

$$\zeta_2(2) = \sum_{\substack{a=1 \\ p\nmid a}}^{11} H_2(2, a, 12).$$

From (6.1) we have,

$$-144\zeta_2(2) = R_2(1/12) - R_2(1/4) + R_2(5/12) - R_2(7/12) + R_2(3/4) - R_2(11/12).$$

Now

$$R_2(1/4) - R_2(3/4) = 16\zeta_2(2),$$

from the above. Moreover,

$$\Theta_2(1/6) = \frac{1}{2}(R_2(1/12) - R_2(7/12)) = \frac{1}{2}(R_2(5/12) - R_2(11/12)),$$

from statement *iv.* of Proposition 4.6 and statement *iii.* of Lemma 4.5. Hence,

$$-144\zeta_2(2) = 4\Theta_2(1/6) + 16\zeta_2(2).$$

Rearranging, we obtain *ii.*

To prove statement *iii.*, we first determine the primitive even Dirichlet character modulo 8, denoted by $\chi_8$. Of course, $\chi_8(1) = 1$, as $\chi_8$ is a group homomorphism. Therefore we notice $\chi_8(7) = \chi_8(-1) = 1$, since $\chi_8$ is even and $\chi_8(0) = \chi_8(2) = \chi_8(4) = \chi_8(6) = 0$.

Now observe that $3^2 \equiv 5^2 \equiv 1 \pmod 8$, so

$$\chi_8(3^2) = \chi_8(3)^2 = 1,$$

so $\chi_8(3) = \pm 1$ and similarly $\chi_8(5) = \pm 1$. Moreover, $5 \equiv -3 \pmod 8$, so $\chi_8(3) = \chi_8(5)$. If they're both 1, then the $\chi_8$ is not primitive, giving a contradiction. So $\chi_8(3) = \chi_8(5) = -1$. Now we ready to prove *iii.* We have,

$$\begin{aligned}\Theta_2(1/4) &= \frac{1}{2}\left(R_2(1/8) - R_2(5/8)\right) && \text{Proposition 4.6}\\ &= -32\left(H_2(2,1,8) - H_2(2,5,8)\right) && \text{using (6.1).}\end{aligned}$$

Applying $R(x) = -R(1-x)$ from Proposition 4.5, we have

$$\begin{aligned}\Theta_2(1/4) &= \frac{1}{2}\left(-R_2(7/8) + R_2(3/8)\right)\\ &= -32\left(H_2(2,7,8) - H_2(2,3,8)\right).\end{aligned}$$

Averaging the last two equations, we obtain

$$\begin{aligned}\Theta_2(1/4) &= -16\left(H_2(2,1,8) - H_2(2,3,8) - H_2(2,5,8) + H_2(2,7,8)\right)\\ &= -16L_2(2,\chi_8).\end{aligned}$$

Let's now prove statement *iv.* From Proposition 4.5 we have,

$$R_3(1/6) + R_3(2/3) = 4R_3(1/3), \qquad R_3(1/3) = -R_3(2/3).$$

Therefore $R_3(1/6) = 5R_3(1/3)$ and $R_3(5/6) = -5R_3(1/3)$. Now,

$$\Theta_3(1/3) = \frac{1}{2}\left(R_3(1/6) - R_3(2/3)\right) = 3R_3(1/3).$$

But,

$$\zeta_3(2) = H_3(2,1,3) + H_3(2,2,3).$$

Applying (6.1) gives

$$\zeta_3(2) = -\frac{1}{9}R_3(1/3) + \frac{1}{9}R_3(2/3) = -\frac{2}{9}R_3(1/3)$$

which proves statement *iv.* It remains to prove statement *v.* We can determine $\chi_{12}$ as before:

$$\chi_{12}(1) = \chi_{12}(11) = 1, \qquad \chi_{12}(5) = \chi_{12}(7) = -1.$$

Therefore,

$$\begin{aligned}L_3(2,\chi_{12}) &= H_3(2,1,12) - H_3(2,5,12) - H_3(2,7,12) + H_2(2,11,12)\\ &= -\frac{1}{144}\left(R_3(1/12) + R_3(5/12) - R_3(7/12) - R_3(11/12)\right)\\ &= -\frac{1}{72}\left(R_3(1/12) - R_3(7/12)\right)\\ &= -\frac{1}{36}\Theta_3(1/6).\end{aligned}$$

This completes the proof. □

## 7. Continued Fraction Approximations

The aim of this section is to find a continued fraction approximation for $\Theta(x)$. Some theory of continued fractions will be useful, so outlined below are some of the fundamentals of the theory required.

Let

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

be a continued fraction. The usual theory of continued fractions has $b_0 = b_1 = \cdots = 1$. This is not so for the continued fractions that we need to follow in Beukers' paper, [4]. We have not found a suitable reference, so we shall simply modify the usual theory (found in [6] for example) to suit our purposes. We define the convergents $V_n/U_n$ using the following inductive formulae:

$$U_0 = 1, \qquad V_0 = a_0, \qquad U_1 = a_1, \qquad V_1 = a_1a_0 + b_0.$$

and

$$V_n = a_nV_{n-1} + b_{n-1}V_{n-2}, \qquad U_n = a_nU_{n-1} + b_{n-1}U_{n-2}.$$

**Lemma 7.1.** *For all $n \geq 0$,*

$$\frac{V_n}{U_n} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{\ddots + \cfrac{\ddots}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}.$$

*Proof.* This is clear for $n = 0$, $n = 1$. Suppose it is true for all $n \leq k$. We define sequences $a'_n$ and $b'_n$ by

$$a'_n = a_n, \qquad b'_n = b_n \qquad \text{for } n \leq k-1$$

and $a'_k = a_k + b_k/a_{k+1}$. Let $V'_n$ and $U'_n$ be the corresponding convergents. Then by the inductive hypothesis,

$$\frac{V'_k}{U'_k} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{\ddots + \cfrac{\ddots}{a_k + \cfrac{b_k}{a_{k+1}}}}}.$$

But

$$\begin{aligned}\frac{V'_k}{U'_k} &= \frac{(a_k + (b_k/a_{k+1}))V_{k-1} + b_{k-1}V_{k-2}}{(a_k + (b_k/a_{k+1}))U_{k-1} + b_{k-1}U_{k-2}} \\ &= \frac{(a_{k+1}a_k + b_k)V_{k-1} + a_{k+1}b_{k-1}V_{k-2}}{(a_{k+1}a_k + b_k)U_{k-1} + a_{k+1}b_{k-1}U_{k-2}} \\ &= \frac{a_{k+1}(a_kV_{k-1} + b_{k-1}V_{k-2}) + b_kV_{k-1}}{a_{k+1}(a_kU_{k-1} + b_{k-1}U_{k-2}) + b_kU_{k-1}} \\ &= \frac{a_{k+1}V_k + b_kV_{k-1}}{a_{k+1}U_k + b_kU_{k-1}} \\ &= \frac{V_{k+1}}{U_{k+1}}.\end{aligned}$$

□

Let $a_n = x^2 - x + 2n^2 - 2n + 1$ and $b_n = -n^4$. Then,

$$\begin{aligned}U_{n+1} &= a_{n+1}U_n + b_nU_{n-1} \\ &= (x^2 - x + 2(n+1)^2 - 2(n+1) + 1)U_n - n^4U_{n-1} \\ &= (x^2 - x + 2n^2 + 4n + 2 - 2n - 2 + 1)U_n - n^4U_{n-1} \\ &= (x^2 - x + 2n^2 + 2n + 1)U_n - n^4U_{n-1}.\end{aligned}$$

Substituting $U_n = (n!)^2u_n$ results in

$$\begin{aligned}((n+1)!)^2u_{n+1} &= (x^2 - x + 2n^2 + 2n + 1)(n!)^2u_n - n^4((n-1)!)^2u_{n-1} \\ n^2(n+1)^2u_{n+1} &= (x^2 - x + 2n^2 + 2n + 1)n^2u_n - n^4u_{n-1}\end{aligned}$$

$$(n+1)^2u_{n+1} = (x^2 - x + 2n^2 + 2n + 1)u_n - n^2u_{n-1}. \tag{7.1}$$

It is easily verified that $V_n$ and $v_n$ satisfy the same recurrence relation as $U_n$ and $u_n$ respectively, although their terms differ when $n = 0, 1$.

We calculate some solutions $p_n(x)$ and $q_n(x)$ for $n \geq 0$ below. Notice that the sequence $p_n(x)/q_n(x)$ are the convergents of our continued fraction.

$$\begin{aligned}
p_0(x) &= 0 \\
p_1(x) &= 1 \\
p_2(x) &= (x^2 - x + 5)/4 \\
p_3(x) &= (x^4 - 2x^3 + 19x^2 - 18x + 49)/36 \\
\cdots &= \cdots
\end{aligned}$$

and

$$\begin{aligned}
q_0(x) &= 1 \\
q_1(x) &= x^2 - x + 1 \\
q_2(x) &= (x^4 - 2x^3 + 7x^2 - 6x + 4)/4 \\
q_3(x) &= (x^6 - 3x^5 + 22x^4 - 39x^3 + 85x^2 - 66x + 36)/36 \\
\cdots &= \cdots
\end{aligned}$$

We now deviate slightly and look at some theory on Hypergeometric Series. Our studies on the topic will allow us to write the convergents $p_n$ and $q_n$ in a different format, thus enabling us to deduce further properties about our continued fraction approximations. We have taken the basics directly from Bailey's book, [2] (chapter 1, pages 1-2).

**Definition 7.1.** *The series, with $c$ a non-negative integer,*

$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + h.o.t.$$

*is called the hypergeometric series, where we abbreviate higher order terms to h.o.t. We denote the hypergeometric series by $F(a, b, c; z)$. Notice that we can also write*

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$$

*where $(\cdot)_n$ is defined in Definition 2.1.*

The hypergeometric series converges when $|z| < 1$, see [2] (page 2) for further details.

**Lemma 7.2.** *Consider the generating function*

$$y_0(z) = \sum_{n=0}^{\infty} q_n(x) z^n.$$

*Then $y_0(z)$ is a power series solution to the second order linear differential equation*

$$(7.2) \quad L_2(y) = z(z-1)^2 y'' + (3z-1)(z-1)y' + (z - 1 + x(1-x))y = 0$$

*where differentiation is with respect to $z$. The solution $y_0(z)$ is unique up to a scalar factor.*

*Proof.* First we calculate $y'(z)$ and $y''(z)$ of the generating function.

$$\begin{aligned} y_0'(z) &= \sum_{n=1}^{\infty} nq_n(x)z^{n-1} \\ y_0''(z) &= \sum_{n=2}^{\infty} n(n-1)q_n(x)z^{n-2}. \end{aligned}$$

Substituting $y_0(z)$, $y_0'(z)$ and $y_0''(z)$ into $L_2(y)$ gives

$$\begin{aligned} L_2(y) &= z(z-1)^2y'' + (3z-1)(z-1)y' + (z-1+x(1-x))y \\ &= (z-1)^2\left(\sum_{n=2}^{\infty} n(n-1)q_n(x)z^{n-1}\right) + (3z^2-4z+1)\left(\sum_{n=1}^{\infty} nq_n(x)z^{n-1}\right) \\ &\quad + z\left(\sum_{n=0}^{\infty} q_n(x)z^n\right) + (-x^2+x-1)\left(\sum_{n=0}^{\infty} q_n(x)z^n\right). \end{aligned}$$

We rearrange equation (7.1) (which $q_n(x)$ satisfies for all $n \in \mathbb{N}, n \geq 1$) and substitute into $L_2(y)$.

$$(-x^2+x-1)q_n(x) = (2n^2+2n)q_n(x) - (n+1)^2q_{n+1}(x) - n^2q_{n-1}(x).$$

Recall that $q_0(x) = 1$ and $q_1(x) = x^2 - x + 1$ as we shall be using them in calculations.

$$\begin{aligned} L_2(y) &= (z-1)^2\sum_{n=2}^{\infty}(n^2-n)q_n(x)z^{n-1} + (3z^2-4z+1)\sum_{n=1}^{\infty} nq_n(x)z^{n-1} \\ &\quad + \sum_{n=0}^{\infty} q_n(x)z^{n+1} + \sum_{n=1}^{\infty}(2n^2+2n)q_n(x)z^n - \sum_{n=1}^{\infty}(n+1)^2q_{n+1}(x)z^n \\ &\quad - \sum_{n=1}^{\infty} n^2q_{n-1}(x)z^n + (-x^2+x-1)q_0(x) \\ &= \sum_{n=2}^{\infty}(n^2-n)q_n(x)(z^{n+1}-2z^n+z^{n-1}) + \sum_{n=2}^{\infty} nq_n(x)(3z^{n+1}-4z^n+z^{n-1}) \\ &\quad + q_1(x)(3z^2-4z+1) + \sum_{n=2}^{\infty} q_n(x)z^{n+1} + q_0(x)z + q_1(x)z^2 \\ &\quad + \sum_{n=2}^{\infty}(2n^2+2n)q_n(x)z^n - \sum_{n=2}^{\infty} n^2q_n(x)z^{n-1} - \sum_{n=2}^{\infty}(n+1)^2q_n(x)z^{n+1} \\ &\quad + 4q_1(x)z - q_0(x)z - 4q_1(x)z^2 + (-x^2+x-1)q_0(x) \end{aligned}$$

$$\begin{aligned}L_2(y) = &\sum_{n=2}^{\infty}(n^2 - n + 3n + 1 - n^2 - 2n - 1)q_n(x)z^{n+1}\\ &+ \sum_{n=2}^{\infty}(-2n^2 + 2n - 4n + 2n^2 + 2n)q_n(x)z^n\\ &+ \sum_{n=2}^{\infty}(n^2 - n + n - n^2)q_n(x)z^{n-1}\\ &+ q_1(x)(3z^2 - 4z + 1) + z + q_1(x)z^2 + 4q_1(x)z - z\\ &- 4q_1(x)z^2 - x^2 + x - 1\\ = 0.&\end{aligned}$$

The solution $y_0(z)$ is unique (up to scalar factor) because power series solutions are unique. □

**Lemma 7.3.**

$$y_0(z) = (1-z)^{x-1}{}_1F_2(x, x, 1; z),$$

*where* ${}_1F_2$ *denotes the hypergeometric function.*

*Proof.* By the above, $y_0(z)$ is the unique power series solution to the second order linear differential equation (7.2) with $y_0(0) = 1$. Let $y(z)$ be any solution to (7.2) and write $w(z) = y(z)/(1-z)^{x-1}$. The hypergeometric function ${}_1F_2(a, b, c; z)$ is a solution to Euler's hypergeometric differential equation (see pages 1-2 in [2]),

$$z(1-z)w'' + (c - (a+b+1)z)\, w' - abw = 0.$$

Our first step is to check that $w(z) = y(z)/(1-z)^{x-1}$ satisfies the same differential equation with $a = b = x$, and $c = 1$. We calculate

$$y'(z) = (1-z)^{x-1}w'(z) + (1-x)(1-z)^{x-2}w(z)$$

and

$$y''(z) = (1-z)^{x-1}w''(z) + 2(1-x)(1-z)^{x-2}w'(z) + (1-x)(2-x)(1-z)^{x-3}w(z).$$

Substituting in (7.2), we find that

$$\begin{aligned}&z(1-z)^2\left((1-z)^{x-1}w''(z) + 2(1-x)(1-z)^{x-2}w'(z)\right)\\ &\quad + z(1-z)^2\left((1-x)(2-x)(1-z)^{x-3}w(z)\right)\\ &\qquad + (3z-1)(z-1)\left((1-z)^{x-1}w'(z) + (1-x)(1-z)^{x-2}w(z)\right)\\ &\qquad\quad + (z - 1 + x(1-x))(1-z)^{x-1}w(z) = 0.\end{aligned}$$

Rearranging, we can simplify to

$$\begin{aligned}&z(1-z)^{x+1}w''(z) + \left(2z(1-x)(1-z)^x - (3z-1)(1-z)^x\right)w'(z)\\ &\quad + \big(z(1-x)(2-x)(1-z)^{x-1} - (3z-1)(1-x)(1-z)^{x-1}\\ &\qquad -(1-z)^x + x(1-x)(1-z)^{x-1}\big)\, w(z) = 0.\end{aligned}$$

Dividing by $(1-z)^x$ we obtain,

$$z(1-z)w''(z) + (1 - z(1+2x))\, w'(z) + \left(-1 + \frac{(1-x)(z(2-x) - (3z-1) + x)}{1-z}\right) w(z) = 0.$$

Simplifying we find

$$z(1-z)w''(z) + (1 - (1+2x)z)\, w' - x^2 w(z) = 0.$$

Thus $w(z)$ indeed satisfies Euler's hypergeometric differential equation with parameters $a = b = x$ and $c = 1$. The hypergeometric function ${}_2F_1(a,b,c,z)$ has the power series expansion

$${}_2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Hence $(1-z)^{x-1}{}_2F_1(x,x,1,z)$ is also a solution to (7.2) whose power series expansion begins with 1. By the uniqueness of $y_0$ we find that $y_0 = (1-z)^{x-1}{}_2F_1(x,x,1,z)$. □

**Lemma 7.4.**

$$q_n(x) = \sum_{j=0}^{n} \frac{(1-x)_{n-j}(x)_j^2}{(n-j)!(j!)^2}.$$

*Proof.* First we observe that in taking $\beta = 1$ in Definition 2.1, we have

$$(1)_n = 1 \cdot 2 \cdots n = n!$$

which we substitute into $y_0$ which gives us

$$\begin{aligned}
y_0(z) &= (1-z)^{x-1}{}_2F_1(x,x,1,z) \\
&= \left(\sum_{n=0}^{\infty} \binom{x-1}{n} (-z)^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(x)_n (x)_n}{n!(1)_n} \cdot z^n\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{(x-1)!}{(x-n-1)!n!} \cdot (-z)^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(x)_n^2}{(n!)^2} \cdot z^n\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{(x-1)(x-2)\cdots(x-n)}{n!} \cdot (-1)^n \cdot z^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(x)_n^2}{(n!)^2} \cdot z^n\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{(1-x)_n}{n!} z^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(x)_n^2}{(n!)^2} \cdot z^n\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{(1-x)_{n-j}}{(n-j)!} \cdot \frac{(x)_j^2}{(j!)^2}\right) z^n.
\end{aligned}$$

The result can be easily seen by comparing coefficients.

□

**Lemma 7.5.** *Let*

$$y_1(z) = \sum_{n=0}^{\infty} p_n(x) z^n$$

*be the generating function for* $p_n(x)$*. Then* $L_2(y_1)$*=1.*

*Proof.* We follow similar steps to those used in the proof of Lemma 7.2. The calculations are almost identical (since both $p_n$ and $q_n$ satisfy the same recurrence relation) hence we have omitted many of the steps. Recall that $p_0(x) = 0$ and $p_1(x) = 1$ which is crucial and needs to be incorporated in order to adapted the proof of Lemma 7.2. Therefore we find,

$$\begin{aligned} L_2(y_1) = p_1(x)(3z^2 - 4z + 1) + \cancel{p_0(x)z} + p_1(x)z^2 + 4p_1(x)z \\ - \cancel{p_0(x)z} - 4p_1(x)z^2 + \cancel{p_0(x)(-x^2 + x - 1)} = 1. \end{aligned}$$

□

We must now show that the convergents $p_n(x)/q_n(x)$ do indeed approximate $\Theta(x)$ in $K$. For each $n$, we define $\Theta(n, x)$ as follows.

**Definition 7.2.**

$$\Theta(n, x) = (-1)^n \sum_{k=0}^{\infty} \binom{k}{n} \left[ \begin{array}{c} k \\ x \end{array} \right] \left[ \begin{array}{c} k \\ 1-x \end{array} \right].$$

**Proposition 7.1.** *For each $n$ we have*

$$p_n(x) + q_n(x)\Theta(x) = \Theta(n, x) = O(1/x^{2n+2})$$

*as a Laurent series in $1/x$.*

*Proof.* Let

$$F(k, n) = (-1)^n \binom{k}{n} \left[ \begin{array}{c} k \\ x \end{array} \right] \left[ \begin{array}{c} k \\ 1-x \end{array} \right].$$

Then,

$$\begin{aligned} & n^2 F(k, n-1) - (-x + x^2 + 2n^2 + 2n + 1)F(k, n) + (n+1)^2 F(k, n+1) \\ & = n^2(-1)^{n-1} \binom{k}{n-1} \left[ \begin{array}{c} k \\ x \end{array} \right] \left[ \begin{array}{c} k \\ 1-x \end{array} \right] \\ & \quad - (-x + x^2 + 2n^2 + 2n + 1)(-1)^n \binom{k}{n} \left[ \begin{array}{c} k \\ x \end{array} \right] \left[ \begin{array}{c} k \\ 1-x \end{array} \right] \\ & \quad + (n+1)^2(-1)^{n+1} \binom{k}{n+1} \left[ \begin{array}{c} k \\ x \end{array} \right] \left[ \begin{array}{c} k \\ 1-x \end{array} \right] \\ & = F(k, n) \left( -n^2 \cdot \frac{n}{k-n+1} + (x - x^2 - 2n^2 - 2n - 1) - (n+1)^2 \cdot \frac{k-n}{n+1} \right) \\ & = F(k, n) \left( \frac{-n^3 + (k-n+1)(x - x^2 - 2n^2 - 2n - 1 - (n+1)(k-n))}{k-n+1} \right). \end{aligned}$$

Observe,

$$
\begin{aligned}
&-n^3+(k-n+1)(x-x^2-2n^2-2n-1-(n+1)(k-n))\\
=&-n^3+(k-n+1)(xk+x-x^2+k^2+k-xk)\\
&\quad+(-k^2-k-n^2-n-1-nk-k)\\
=&-n^3+(k-n+1)(x+k)(k+1-x)-(k+1)^3\\
&\quad+(k+1)(-n^2-n-nk)-n(-k^2-k-n^2-n-1-nk-k)\\
=&-\cancel{n^3}+(k-n+1)(x+k)(k+1-x)\\
&\quad-(k+1)^3-n(k+1)(n+1+k)+\cancel{n^3}+n(k^2+2k+1+n+nk)\\
=&(k-n+1)(x+k)(k+1-x)-(k+1)^3\\
&\quad-n(k+1)(n+1+k)+n((k+1)^2+n(1+k))\\
=&(k-n+1)(x+k)(k+1-x)-(k+1)^3-\cancel{n(k+1)(n+1+k)}\\
&\quad+\cancel{n(k+1)(k+1+n)}\\
=&(x+k)(k+1-x)(k-n+1)-(k+1)^3.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&n^2F(k,n-1)-(-x+x^2+2n^2+2n+1)F(k,n)+(n+1)^2F(k,n+1)\\
&=F(k,n)\left((x+k)(k+1-x)-\frac{(k+1)^3}{k-n+1}\right)\\
&=F(k,n)(x+k)(k+1-x)\\
&\qquad-(-1)^n\binom{k+1}{n}\begin{bmatrix}k+1\\x\end{bmatrix}\begin{bmatrix}k+1\\1-x\end{bmatrix}(x+k+1)(k+2-x)\\
&=F(k,n)(x+k)(k+1-x)-F(k+1,n)(x+k+1)(k+2-x)\\
&=-\Delta_k\left(F(k,n)(x+k)(k+1-x)\right).
\end{aligned}
$$

where $\Delta_k$ is the forward difference operator as defined in Section 4.

We have just shown the following

$$
\begin{aligned}
n^2F(k,n-1)-(-x+x^2+2n^2+2n+1)F(k,n)+(n+1)^2F(k,n+1)&\\
=-\Delta_k\left(F(k,n)(x+k)(k+1-x)\right).&
\end{aligned}
$$

For $n \geq 1$, summation over $k$ results in

$$
\begin{aligned}
& n^2\Theta(n-1,x) - (-x+x^2+2n^2+2n+1)\Theta(n,x) + (n+1)^2\Theta(n+1,x) \\
=& -\sum_{k=0}^{\infty} \Delta_k \left(F(k,n)(x+k)(k+1-x)\right) \\
=& F(0,n)(x)(1-x) - \lim_{k\to\infty} F(k,n)(x+k)(k+1-x) \\
=& (-1)^n \binom{0}{n} \begin{bmatrix} 0 \\ x \end{bmatrix} \begin{bmatrix} 0 \\ 1-x \end{bmatrix} (x)(1-x) \\
& \qquad - \lim_{k\to\infty} (-1)^n \binom{k}{n} \begin{bmatrix} k \\ x \end{bmatrix} \begin{bmatrix} k \\ 1-x \end{bmatrix} (x+k)(k+1-x) \\
=& 0 - \frac{(-1)^n}{n!} \lim_{k\to\infty} \frac{k!}{(k-n)!} \cdot \frac{k!}{x\cdots(x+k-1)} \cdot \frac{k!}{(1-x)\cdots(k-x)} \\
=& 0
\end{aligned}
$$

where the limit has been taken with discrete valuation, and so tends to zero as we saw in Section 4. Definition A.1 has been used to set the first term to zero.

For $n = 0$, we have

$$-(-x+x^2+1)F(k,0) + F(k,1) = -\Delta_k \left(F(k,0)(x+k)(k+1-x)\right).$$

Summation over $k$ gives,

$$
\begin{aligned}
-(-x+x^2+1)\Theta(0,x) + \Theta(1,x) &= F(0,0)(x)(1-x) \\
&= \binom{0}{0} \frac{1}{x(1-x)} \cdot x(1-x) = 1.
\end{aligned}
$$

Putting together all of the above, and simply noticing that $\Theta(0,x) = \Theta(x)$, we can conclude that for all $n \geq 0$,

$$p_n(x) + q_n(x)\Theta(x) = \Theta(n,x).$$

□

## 8. Criterion for Irrationality

In this final chapter, we state our main Theorem which sets a criterion for the irrationality of the $p$-adic number $\Theta_p(a/F)$, where $p$ is a prime such that $p \mid F$ but $p \nmid a$.

**Proposition 8.1.** *Let $p_n$ and $q_n$ be sequences of rational integers as defined in chapter 7. Let $\mu_n(F)$ be defined as in Lemma 2.2. Then,*

*i. For every $n$, the number $q_n(a/F)$ is rational with denominator dividing $\mu_n(F)^2$.*

*ii. For every $n$, the number $p_n(a/F)$ is rational with denominator dividing $lcm(1,\ldots,n)^2\mu_n(F)^2$.*

*iii. For every $\epsilon > 0$, we have $|q_n(a/F)|$, $|p_n(a/F)| < e^{\epsilon n}$ for sufficiently large $n$.*

*iv. Suppose $p^r || F$ with $r > 0$ and $p \nmid a$. Then for every $n$ we have,*

$$\left|p_n(a/F) + \Theta(a/F)q_n(a/F)\right|_p \leq p^2 n^2 p^{-2n(r+1/(p-1))}.$$

*Proof.* To prove assertion *i.*, we recall

$$q_n(a/F) = \sum_{j=0}^{n} \frac{(1-a/F)_{n-j}(a/F)_j^2}{(n-j)!(j!)^2}.$$

Looking at each term individually (i.e. fix $j$), we apply Lemma 2.2 and notice that $(1-a/F)_{n-j}/(n-j)!$ has denominator dividing $\mu_{n-j}(F)$ and $(a/F)_j^2/(j!)^2$ has denominator dividing $\mu_j(F)^2$. Therefore, every term has denominator which divides

$$\begin{aligned}\mu_{n-j}(F)\cdot\mu_j(F)^2 &= F^{n-j}\prod_{q|F} q^{\left[\frac{n-j}{q-1}\right]}\cdot F^j\prod_{q|F} q^{\left[\frac{j}{q-1}\right]}\cdot\mu_j(F)\\ &= F^n\prod_{q|F} q^{\left[\frac{n-j}{q-1}\right]+\left[\frac{j}{q-1}\right]}\cdot\mu_j(F)\\ &\leq F^n\prod_{q|F} q^{\left[\frac{n}{q-1}\right]}\cdot\mu_j(F)\\ &= \mu_n(F)\cdot\mu_j(F)\leq\mu_n(F)^2\end{aligned}$$

where it remains to justify the third line. If we let $n-j=\alpha(q-1)+r$ and $j=\beta(q-1)+s$, where $\alpha$ and $\beta$ are integers and $0\leq r,s<q-1$. Adding together we get

$$n=(\alpha+\beta)(q-1)+r+s \qquad \text{with } 0\leq r+s<2(q-1).$$

Therefore

$$\left[\frac{n}{q-1}\right]=\begin{cases}\alpha+\beta & \text{if } 0\leq r+s<(q-1)\\ \alpha+\beta+1 & \text{if } q-1\leq r+s<2(q-1)\end{cases}$$

justifies our inequality since

$$\left[\frac{n-j}{q-1}\right]+\left[\frac{j}{q-1}\right]=\alpha+\beta\leq\left[\frac{n}{q-1}\right].$$

Notice that when $j=n$ we have the term $(a/F)_n^2/(n!)^2$ which has denominator dividing $\mu_n(F)^2$. Since all of the other terms have denominators which divide something less than $\mu_n(F)^2$, summation over $j$ results in the denominator of $q_n(x)$ dividing $\mu_n(F)^2$, thus completing our proof.

In order to prove assertion *ii.*, we notice that since $p_n(a/F)$ satisfies the same difference equation (although with different initial conditions) as $q_n(a/F)$, it readily follows that $p_n(x)$ has denominator dividing $\mu_n(F)^2$. We shall require Proposition 3.1 to show that the denominator of $p_n(x)$ divides $\text{lcm}(1,2,\ldots,n)^2$. We know that the generating function $y_0(z)$ is a solution to the second order linear differential equation $L_2(y)=z(z-1)^2y''+(3z-1)(z-1)y'+(z-1+a/F(1-a/F))y=0$. Note however that the coefficient of $y(z)$ does not belong to the ring of integers. Therefore, we must make some adjustments to the differential equation in order to apply Proposition 3.1. We define

$$\lambda=\prod_{q|F} q^{\frac{1}{q-1}}$$

and make the substitution $z \mapsto F^2\lambda^2 z$ into $L_2(y) = 0$ to give

$$F^2\lambda^2 z(F^2\lambda^2 z - 1)^2 y''(F^2\lambda^2 z) + (3F^2\lambda^2 z - 1)(F^2\lambda^2 z - 1)y'(F^2\lambda^2 z) + \frac{1}{F^2\lambda^2} \cdot (F^4\lambda^4 z - F^2\lambda^2 + aF\lambda^2 - a^2\lambda^2)y(F^2\lambda^2 z) = 0.$$

Notice that in the above we have made the following substitutions

$$\begin{aligned} y(z) &\mapsto y(F^2\lambda^2 z) \\ y'(z) &\mapsto F^2\lambda^2 y'(F^2\lambda^2 z) \\ y''(z) &\mapsto F^4\lambda^4 y''(F^2\lambda^2 z) \end{aligned}$$

which we can use again in order to state the differential equation in terms of $y(z)$ and its derivatives.

$$z(F^2\lambda^2 z - 1)^2 y''(z) + (3F^2\lambda^2 z - 1)(F^2\lambda^2 z - 1)y'(z) + (F^4\lambda^4 z - F^2\lambda^2 + aF\lambda^2 - a^2\lambda^2)y(z) = 0.$$

If we take the ring $R = \mathbb{Z}[\lambda]$, then it is easy to see that the conditions stated in Proposition 3.1 are now satisfied since $y_0(F^2\lambda^2 z) \in R[[z]]$ remains the unique power series solution of $L_2(y) = 0$. We still have $y_0(0) = 1$ and letting $p(z) = (F^2\lambda^2 z - 1)^2$, we easily see that $p(0) = 1$. We use similar calculations to substitute the transformations stated above into $L_2(y) = 1$ and arrive at

$$L_2(y) = z(F^2\lambda^2 z - 1)^2 y''(z) + (3F^2\lambda^2 z - 1)(F^2\lambda^2 z - 1)y'(z) + (F^4\lambda^4 z - F^2\lambda^2 + aF\lambda^2 - a^2\lambda^2)y(z) = F^2\lambda^2.$$

Dividing through by $F^2\lambda^2$, we see that $y_1(F^2\lambda^2 z) \in Q(R)[[z]]$ is the unique solution of $L_2(y) = 1$. Hence we apply Proposition 3.1 in order to deduce that the denominator of $p_n(a/F)$ divides $\text{lcm}(1, 2, \ldots, n)^2$.

To prove assertion *iii.*, we need to show that $y_0$ and $y_1$ have radius of convergence equal to 1 under the Archimedean norm. Since
$y_0(z) = (1 - z)^{x-1} {}_1F_2(x, x, 1, z)$ we note that the hypergeometric function converges for $|z| < 1$ and it is relatively easy to see that $(1 - z)^{x-1}$ also has radius of convergence equal to 1.

It isn't very clear how Beukers deduced that the radius of convergence for $y_1$ is also equal to 1 in [4]. For now, we shall assume that the statement holds true in order to complete the proof.

Therefore $y_0(z)$ converges for all $|z| < 1$. Since $e^{-\epsilon} < 1$ we can make the substitution, $z = e^{-\epsilon}$ to get $|q_n e^{-\epsilon n}| < 1$ and so $|q_n| < e^{\epsilon n}$. A similar calculation shows $|p_n| < e^{\epsilon n}$.

To prove assertion *iv.* we use Proposition 7.1 and the ultrametric inequality.

$$\begin{aligned} |p_n(a/F) + \Theta(a/F)q_n(a/F)|_p &= |\Theta(n, a/F)|_p \\ &= \left| \sum_{k=0}^{\infty} \binom{k}{n} \begin{bmatrix} k \\ a/F \end{bmatrix} \begin{bmatrix} k \\ 1 - a/F \end{bmatrix} \right|_p \\ &\leq \max_{n \leq k \leq \infty} \left\{ \binom{k}{n} \begin{bmatrix} k \\ a/F \end{bmatrix} \begin{bmatrix} k \\ 1 - a/F \end{bmatrix} \right\}. \end{aligned}$$

The binomial coefficients are well defined for $k \geq n$ although when $k < n$ they are identically zero. We say $k = \alpha + n$ with $\alpha$ a positive integer. Therefore we can bound the binomial coefficients, namely by

$$\left|\binom{k}{n}\right|_p = \left|\binom{\alpha+n}{n}\right|_p = \left|\frac{(\alpha+n)\cdots(n+1)n!}{n!}\right|_p \leq \left|\frac{n!}{n!}\right|_p \leq 1.$$

It is obvious that we have an upper bound for the binomial coefficients when $k = n$. Furthermore, we have bounds on the following

$$\begin{aligned}
&\left|\begin{bmatrix} k \\ a/F \end{bmatrix}\begin{bmatrix} k \\ 1-a/F \end{bmatrix}\right|_p \\
&= \left|\frac{k!}{a/F\cdots(a/F+k)}\cdot\frac{k!}{(1-a/F)\cdots(k+1-a/F)}\right|_p \\
&= \left|\frac{k!}{(a/F)_k(a/F+k)}\cdot\frac{k!}{(1-a/F)_k(k+1-a/F)}\right|_p \\
&< \left|\frac{F}{(a+Fk)}\cdot\frac{F}{(Fk+F-a)}\right|_p \cdot p^{-2\left(k\left(r-\frac{1}{p-1}\right)+\frac{\log k}{\log p}+1\right)} \qquad \text{by Lemma 2.2} \\
&< p^{-2\left(k\left(r+\frac{1}{p-1}\right)-\frac{\log k}{\log p}-1\right)} \\
&< p^2k^2p^{-2k\left(r+\frac{1}{p-1}\right)}.
\end{aligned}$$

We have the factor of $k^2$ from the manipulation of a few equations.

$$p^{\frac{\log k}{\log p}} = \exp\left\{\log p^{\frac{\log k}{\log p}}\right\} = \exp\left\{\frac{\log k}{\log p}\cdot\log p\right\} = k.$$

Again, we have a maximum bound when $n = k$, therefore

$$|\Theta_p(a/F)|_p < p^2n^2p^{-2n\left(r+\frac{1}{p-1}\right)}.$$

□

To conclude, we state the main theorem alongside the some key results and conclusions.

**Theorem 8.1.** *Let $a \in \mathbb{Z}$ and $F \in \mathbb{N}$ such that $p|F$ and $p \nmid a$. Define $r$ by $|F|_p = p^{-r}$. Suppose*

$$\log F + \sum_{q|F}\frac{\log q}{q-1} + 1 < 2r\log p + \frac{2\log p}{p-1}.$$

*Then $\Theta_p(a/F)$ is irrational.*

*Proof.* Firstly, we want to show that $(p_n(a/F) + q_n(a/F)\Theta_p(a/F))$ is non-zero for infinitely many $n$ in order to apply Lemma 2.1. It suffices to show that consecutive terms cannot both be zero. Suppose for a contradiction that

$$p_n(x) + q_n(x)\Theta_p(a/F) = 0 = p_{n+1}(x) + q_{n+1}(x)\Theta_p(a/F).$$

Then the following is also true

$$q_{n+1}(x)p_n(x)+q_{n+1}(x)q_n(x)\Theta_p(a/F) = 0 = q_n(x)p_{n+1}(x)+q_n(x)q_{n+1}(x)\Theta_p(a/F).$$

Subtracting the two equations gives

$$p_{n+1}(x)q_n(x) - p_n(x)q_{n+1}(x) = 0$$

and thus we want to show that $p_{n+1}(x)q_n(x) - p_n(x)q_{n+1}(x)$ is non-zero for infinitely many $n$. To that end we prove the following statement by induction

$$p_{n+1}(x)q_n(x) - p_n(x)q_{n+1}(x) = 1/(n+1)^2. \tag{8.1}$$

For $n = 0$, we see below that equation (8.1) is true.

$$p_1(x)q_0(x) - p_0(x)q_1(x) = 1 \cdot 1 - 0 \cdot (x^2 - x + 1) = 1.$$

We assume equation (8.1) is true for $n = 0, 1, \ldots, k-1$.

$$p_k(x)q_{k-1}(x) - p_{k-1}(x)q_k(x) = 1/k^2.$$

We shall use the inductive hypothesis and the recurrence relations for $p_n$ and $q_n$ to show that equation (8.1) is true for $n = k$.

$$\begin{aligned}
&p_{k+1}(x)q_k(x) - p_k(x)q_{k+1}(x) \\
&= \frac{1}{(k+1)^2}\left((2k^2+2k+1-x+x^2)p_k(x) - k^2 p_{k-1}(x)\right) q_k(x) \\
&\qquad - \frac{1}{(k+1)^2}\left((2k^2+2k+1-x+x^2)q_k(x) - k^2 q_{k-1}(x)\right) p_k(x) \\
&= \frac{1}{(k+1)^2}\Big(\cancel{(2k^2+2k+1-x+x^2)}p_k(x)q_k(x) - k^2 p_{k-1}(x)q_k(x)\Big) \\
&\qquad - \frac{1}{(k+1)^2}\Big(\cancel{(2k^2+2k+1-x+x^2)}q_k(x)p_k(x) - k^2 q_{k-1}(x)p_k(x)\Big) \\
&= \frac{k^2}{(k+1)^2}\left(p_k(x)q_{k-1}(x) - p_{k-1}(x)q_k(x)\right) = \frac{1}{(k+1)^2}
\end{aligned}$$

which completes our proof.

Let $\epsilon > 0$. We use Proposition 8.1 to see that the common denominator of $q_n(a/F)$ and $p_n(a/F)$ divides $Q_n := \mathrm{lcm}(1, 2, \ldots, n)^2 \mu_n(F)^2$. We take a common denominator in order to take out common factors, so that we end up with approximations to $\Theta(x)$ of the form $c_n/d_n$, with $c_n$ and $d_n$ integers. Once we have the correct format for $c_n/d_n$ , we can apply Lemma 2.1.

It is straightforward to see the following inequality, which uses Proposition 8.1.

$$\begin{aligned}
\left|Q_n p_n(a/F) + Q_n q_n(a/F)\Theta_p(a/F)\right|_p &\le |Q_n|_p \cdot \left|p_n(a/F) + q_n(a/F)\Theta_p(a/F)\right|_p \\
&\le |Q_n|_p \cdot p^2 n^2 p^{-2n\left(r + \frac{1}{(p-1)}\right)} \\
&< |Q_n|_p \cdot p^{\epsilon n} p^{-2n\left(r + \frac{1}{(p-1)}\right)} \\
&< |Q_n|_p \cdot p^{n\left(\epsilon - 2r - \frac{2}{(p-1)}\right)}
\end{aligned}$$

where $p^2n^2 < p^{\epsilon n}$ holds for any $\epsilon > 0$ as long as $n$ is large enough. We also have

$$\begin{aligned}|Q_n|_p &= \left|\mathrm{lcm}(1,2,\ldots,n)^2\mu_n(F)^2\right|_p \\ &< p^{-2\left[\frac{\log n}{\log p}\right]} \cdot p^{2\left(-nr-\frac{n}{p-1}+\frac{\log n}{\log p}+1\right)} \\ &\leq p^{-2\frac{\log n}{\log p}+2} \cdot p^{2\left(-nr-\frac{n}{p-1}+\frac{\log n}{\log p}+1\right)} \\ &= p^{2\left(2-nr-\frac{n}{p-1}\right)} = p^{-2nr-\frac{2n}{p-1}+4} \\ &< p^{-2nr-\frac{2n}{p-1}+2\epsilon n}\end{aligned}$$

where we use $p^2 < p^{\epsilon n}$ which holds for any $n$ provided $\epsilon$ is small enough.

We now apply Lemma 2.1 with $\alpha = \Theta_p(a/F)$, $d_n = Q_nq_n(a/F)$ and $c_n = Q_np_n(a/F)$. We bound

$$\begin{aligned}|c_n + d_n\Theta_p(a/F)|_p &= |Q_n|_p\,|p_n(a/F) + q_n(a/F)\Theta_p(a/F)|_p \\ &< p^{-2nr-\frac{2n}{p-1}+2\epsilon n} \cdot p^{n\left(\epsilon-2r-\frac{2}{(p-1)}\right)} \\ &< \exp\left\{-4nr\log p - \frac{4n}{(p-1)}\log p + 4\epsilon n\log p\right\}.\end{aligned}$$

By Proposition 8.1, we also have the following bounds for $n$ large enough.

$$\begin{aligned}|d_n|, |c_n| &< e^{\epsilon n}\mathrm{lcm}(1,2,\ldots,n)^2\mu_n(F)^2 \\ &< e^{\epsilon n}e^{2(1+\epsilon)n} \cdot F^{2n}\prod_{q|F} q^{2\left[\frac{n}{q-1}\right]} \\ &\leq \exp\left\{3\epsilon n + 2n + \log\left(F^{2n}\prod_{q|F} q^{\left(\frac{2n}{q-1}\right)}\right)\right\} \\ &= \exp\left\{3\epsilon n + 2n + 2n\log F + 2n\sum_{q|F}\frac{\log q}{q-1}\right\} \\ &< \exp\left\{2n\left(2\epsilon + 1 + \log F + \sum_{q|F}\frac{\log q}{q-1}\right)\right\}\end{aligned}$$

where we estimate $\mathrm{lcm}(1,2,\ldots,n) < e^{(1+\epsilon)n}$, a result deduced from the prime number theorem. We also calculate

$$\begin{aligned}&\max(|c_n|, |d_n|) \cdot |c_n + d_n\Theta_p(a/F)|_p \\ &< \exp\left\{2n\left(2\epsilon + 1 + \log F + \sum_{q|F}\frac{\log q}{q-1}\right)\right\} \cdot \exp\left\{-4nr\log p - \frac{4n}{(p-1)}\log p + 3\epsilon n\log p\right\} \\ &= \exp\left\{2n\left(2\epsilon + 1 + \log F + \sum_{q|F}\frac{\log q}{q-1} - 2r\log p - \frac{2}{(p-1)}\log p + 2\epsilon\log p\right)\right\}\end{aligned}$$

where

$$\lim_{n\to\infty}\max(|c_n|, |d_n|) \cdot |c_n + d_n\Theta_p(a/F)|_p = 0$$

will certainly hold if we have

$$2\epsilon(1+\log p)+1+\log F+\sum_{q|F}\frac{\log q}{q-1}-2r\log p-\frac{2}{(p-1)}\log p<0$$

and this is certainly true by our assumption if we take $\epsilon$ small enough.

□

**Corollary 8.1.** *Let $\chi_8$ be the primitive even character modulo 8. Then $\zeta_2(2)$, $\zeta_3(2)$ and $L_2(2,\chi_8)$ are all irrational.*

*Proof.* First, we look at $\zeta_2(2)$. From chapter 6, we have

$$\zeta_2(2)=-\frac{1}{8}\Theta_2(1/2)=-\frac{1}{40}\Theta_2(1/6).$$

We want to apply Theorem 8.1. We take $p=2, a=1, F=2$. Then $|2|_2=2^{-1}$ and so $r=1$. Some calculations shows

$$\begin{aligned} & 1+\log F+\sum_{q|F}\frac{\log q}{q-1}<2r\log p+\frac{2}{(p-1)}\log p \\ \Rightarrow\quad & 1+\log 2+\sum_{q|2}\frac{\log q}{q-1}<2\log 2+2\log 2 \\ \Rightarrow\quad & 1+\log 2+\log 2<2\log 2+2\log 2 \\ \Rightarrow\quad & 1<2\log 2\approx 1.38629 \end{aligned}$$

and therefore the assumptions stated in Theorem 8.1 are all satisfied. Therefore on application, we find that $\zeta_2(2)$ is irrational. We remark that performing similar calculations for $\zeta_2(2)=-(1/40)\Theta_2(1/6)$ results in

$$1<3\log 2-\log 6-(1/2)\log 3\approx -0.261624$$

which violates the inequality of Theorem 8.1 and so in this case, we cannot apply the Theorem.

We now apply Theorem 8.1 for $\zeta_3(2)=-(2/27)\Theta_3(1/3)$. We take $p=3, a=1, F=3$. Then $|3|_3=3^{-1}$ and so $r=1$. Again, we make some simple calculations,

$$\begin{aligned} & 1+\log F+\sum_{q|F}\frac{\log q}{q-1}<2r\log p+\frac{2}{(p-1)}\log p \\ \Rightarrow\quad & 1+\log 3+\sum_{q|3}\frac{\log q}{q-1}<2\log 3+\log 3 \\ \Rightarrow\quad & 1+\log 3+\frac{1}{2}\log 3<3\log 3 \\ \Rightarrow\quad & 1<(3/2)\log 3\approx 1.647918 \end{aligned}$$

and so we see that the assumptions stated in Theorem 8.1 have all been satisfied, thus concluding the irrationality of $\zeta_3(2)$.

We calculate for $L_2(2,\chi_8) = -(1/16)\Theta_2(1/4)$. Taking $p=2, a=1, F=4$. Then $|4|_2 = 2^{-2}$, $r=2$, and so we find

$$\begin{aligned} & 1+\log F+\sum_{q|F}\frac{\log q}{q-1} < 2r\log p+\frac{2}{(p-1)}\log p \\ \Rightarrow\quad & 1+\log 4+\sum_{q|4}\frac{\log q}{q-1} < 4\log 2+2\log 2 \\ \Rightarrow\quad & 1+\log 4+\log 2 < 6\log 2 \\ \Rightarrow\quad & 1 < 5\log 2-\log 4 \approx 2.07944 \end{aligned}$$

and again we see that the assumptions stated in Theorem 8.1 are all satisfied, therefore $L_2(2,\chi_8)$ is irrational.

□

As a final remark, we make a few similar calculations to those above for $L_3(2,\chi_{12}) = -(1/36)\Theta_3(1/6)$ which results in

$$1 < (5/2)\log 3-\log 6-\log 2 \approx 0.261624.$$

The inequality needed to apply Theorem 8.1 doesn't hold here, therefore we cannot apply the Theorem to deduce any results on the irrationality of $L_2(2,\chi_8)$.

## Appendix A. Binomial Coefficients

**Definition A.1.** *We know that the Binomial Coefficients can be defined as follows. For positive integers, $n$ and $k$, we have*

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-(k-1))}{k!} = \frac{n!}{(n-k)!k!}$$

*where we can use the factorial notation when $n \geq k$.*

*When $k < 0$, the Binomial coefficients are identically zero.*

For negative integers $n$ and positive integers $k$, we can derive a formula for the binomial coefficients.

$$\binom{-n}{k} = \frac{-n \cdot (-n-1) \cdots (-n-k+1)}{k!}$$
$$= (-1)^k \cdot \frac{(n+k-1) \cdots (n+1) \cdot n}{k!} = (-1)^k \binom{n+k-1}{k}.$$

Substituting $n = -1$ and $n = -2$, we can simplify the above.

$$\binom{-1}{k} = \frac{(-1)(-2) \cdots (-1-k+1)}{k!} = (-1)^k.$$

and

$$\binom{-2}{k} = \frac{(-2)(-3) \cdots (-2-k+1)}{k(k-1) \cdots (2)(1)}$$
$$= (-1)^k \frac{\cancel{(2)(3) \cdots (k)}(k+1)}{\cancel{k(k-1) \cdots (2)(1)}} = (-1)^k (k+1).$$

**Definition A.2.** *The Binomial series can be defined for any $n \in \mathbb{C}$.*

$$(1+x)^n = \sum_{k \geq 0} \binom{n}{k} x^k.$$

We simplify the above definition to derive a formula for any integer $n \geq 0$.

$$(1+x)^{-n-1} = \sum_{k \geq 0} \binom{-n-1}{k} x^k \tag{A.1}$$
$$= \sum_{k \geq 0} \binom{n+k}{k} (-x)^k = \sum_{k \geq 0} \binom{n+k}{n} (-x)^k.$$

## Appendix B. Differential Operator of Order 3

The operator $L_2(y)$ as defined in chapter 3 has a symmetric square $L_3(y)$ which we define as:

$$L_3(y) := z^2 P(z) y''' + Q(z) y'' + R(z) y' + S(z) y$$

with $P, Q, R, S \in R[z]$, $P(0) = 1$. The symmetric square is characterised by the property that the solution space of $L_3(y) = 0$ is spanned by the squares of solutions of $L_2(y) = 0$. The equation $L_3(y) = 0$ has a unique power series solution $y_0^2(z)$ with $y_0^2(0) = 1$.

**Lemma B.1.** *$y_0(z) y_1(z)$ is a solution to $L_3(y) = 0$.*

*Proof.* We know that $y_0(z)$ and $y_1(z)$ are both solutions to $L_2(y) = 0$. Therefore, by linearity of $L_2$, $(y_0(z) + y_1(z))/2$ and $(y_0(z) - y_1(z))/2$ are also solutions to $L_2(y) = 0$. Therefore $(y_0(z) + y_1(z))^2/4$ and $(y_0(z) - y_1(z))^2/4$ are solutions to $L_3(y) = 0$. Using the linearity of $L_3$ we conclude

$$\frac{(y_0(z) + y_1(z))^2}{4} - \frac{(y_0(z) - y_1(z))^2}{4} = y_0(z)y_1(z)$$

is a solution to $L_3(y) = 0$. □

Before stating the next Lemma, we note that $y_0(t), y_1(t), P(t)$ and $W_0(t)$ are all functions. However, to avoid a mass of notation in the following equations, we shall denote these as $y_0, y_1, P$ and $W_0$. The variables $w$, $x$, $y$, and $t$ shall be used as dummy variables and differentiation is always with respect to $z$, unless specified.

**Lemma B.2.** *Define:*

$$h(z) := y_0^2 \int_0^z \frac{y_1^2}{PW_0^2} dt - 2y_0y_1 \int_0^z \frac{y_0y_1}{PW_0^2} dt + y_1^2 \int_0^z \frac{y_0^2}{PW_0^2} dt. \tag{B.1}$$

*Then $h(z)$ is a solution to $L_3(y) = 2$.*

*Proof.* First we calculate the first, second and third derivatives of $h(z)$.

$$\begin{aligned} h'(z) &= (y_0^2)' \int_0^z \frac{y_1^2}{PW_0^2} dt + y_0^2 \cancel{\left[\frac{y_1^2}{PW_0^2}\right]} - 2(y_0y_1)' \int_0^z \frac{y_0y_1}{PW_0^2} dt \\ &\qquad - 2y_0y_1 \cancel{\left[\frac{y_0y_1}{PW_0^2}\right]} + (y_1^2)' \int_0^z \frac{y_0^2}{PW_0^2} dt + y_1^2 \cancel{\left[\frac{y_0^2}{PW_0^2}\right]} \\ &= (y_0^2)' \int_0^z \frac{y_1^2}{PW_0^2} dt - 2(y_0y_1)' \int_0^z \frac{y_0y_1}{PW_0^2} dt + (y_1^2)' \int_0^z \frac{y_0^2}{PW_0^2} dt. \end{aligned}$$

$$\begin{aligned} h''(z) &= (y_0^2)'' \int_0^z \frac{y_1^2}{PW_0^2} dt - 2(y_0y_1)'' \int_0^z \frac{y_0y_1}{PW_0^2} dt + (y_1^2)'' \int_0^z \frac{y_0^2}{PW_0^2} dt \\ &\qquad \cancel{+ 2y_0'y_0 \frac{y_1^2}{PW_0^2} - (2y_0'y_1 + 2y_0y_1') \frac{y_0y_1}{PW_0^2} + 2y_1y_1' \frac{y_0^2}{PW_0^2}}. \end{aligned}$$

$$\begin{aligned} h'''(z) &= (y_0^2)''' \int_0^z \frac{y_1^2}{PW_0^2} dt - 2(y_0y_1)''' \int_0^z \frac{y_0y_1}{PW_0^2} dt + (y_1^2)''' \int_0^z \frac{y_0^2}{PW_0^2} dt \\ &\qquad + 2((y_0')^2 + \cancel{y_0y_0''}) \cdot \left(\frac{y_1^2}{PW_0^2}\right) - 2(\cancel{y_0''y_1} + 2y_0'y_1' + \cancel{y_0y_1''}) \cdot \left(\frac{y_0y_1}{PW_0^2}\right) \\ &\qquad + 2((y_1')^2 + \cancel{y_1y_1''}) \cdot \left(\frac{y_0^2}{PW_0^2}\right) \\ &= (y_0^2)''' \int_0^z \frac{y_1^2}{PW_0^2} dt - 2(y_0y_1)''' \int_0^z \frac{y_0y_1}{PW_0^2} dt + (y_1^2)''' \int_0^z \frac{y_0^2}{PW_0^2} dt \\ &\qquad + 2(y_0')^2 \cdot \left(\frac{y_1^2}{PW_0^2}\right) - 4y_0'y_1' \cdot \left(\frac{y_0y_1}{PW_0^2}\right) + 2(y_1')^2 \cdot \left(\frac{y_0^2}{PW_0^2}\right). \end{aligned}$$

Substituting $h(z)$ and its derivatives into $L_3$ gives,

$$\begin{aligned}L_3(h(z)) &= z^2 P\left(2(y_0')^2\cdot\left(\frac{y_1^2}{PW_0^2}\right) - 4y_0'y_1'\cdot\left(\frac{y_0y_1}{PW_0^2}\right) + 2(y_1')^2\cdot\left(\frac{y_0^2}{PW_0^2}\right)\right)\\ &= \frac{2z^2}{W_0^2}\cdot\left((y_0')^2(y_1^2) - 2y_0'y_1'(y_0y_1) + (y_1')^2(y_0^2)\right)\\ &= \frac{2z^2}{W_0^2}\cdot(y_1'y_0 - y_1y_0')^2 = \frac{2z^2}{W_0^2}\cdot\left(\frac{W_0^2}{z^2}\right) = 2\end{aligned}$$

where we use the fact that $y_0^2$, $y_1^2$ and $y_0y_1$ are solutions to $L_3(y) = 0$ and so we can cancel all of the other terms. □

**Proposition B.1.** *$L_3(y) = 1$ has a unique solution $h_1(z) \in Q(R)[[z]]$ beginning with $z + O(z^2)$. Moreover, the $n$-th coefficient of $h_1(z)$ has denominator dividing $lcm(1, 2, \ldots, n)^3$.*

*Proof.* We instantly see that $h_1(z) = (1/2)h(z)$ since $L_3(y)$ is linear. Therefore we can make calculations for $h(z)$ and factor by $1/2$ to deduce results for $h_1(z)$. Using $y_1 = y_0\log(z) + \tilde{y}_0$ and the identities from Lemma 3.1 we see that

$$\begin{aligned}y_0^2\int_0^z \frac{y_1^2}{PW_0^2}dt &= y_0^2\int_0^z \frac{y_0^2(\log t)^2 + 2y_0\tilde{y}_0\log t + \tilde{y}_0^2}{PW_0^2}dt\\ &= y_0^2(\log z)^2\int_0^z\frac{y_0^2}{PW_0^2}dt - 2y_0^2(\log z)\int_0^z\frac{1}{x}\int_0^x\frac{y_0^2}{PW_0^2}dtdx\\ &\quad + 2y_0^2\int_0^z\frac{1}{w}\int_0^w\frac{1}{x}\int_0^x\frac{y_0^2}{PW_0^2}dtdxdw + y_0^2(\log z)\int_0^z\frac{2y_0\tilde{y}_0}{PW_0^2}dt\\ &\quad - y_0^2\int_0^z\frac{1}{x}\int_0^x\frac{2y_0(t)\tilde{y}_0(t)}{P(t)W_0^2(t)}dtdx + y_0^2\int_0^z\frac{\tilde{y}_0^2(t)}{P(t)W_0^2(t)}dt\end{aligned}$$

and

$$\begin{aligned}-2y_0y_1\int_0^z\frac{y_0y_1}{PW_0^2}dt &= (-2y_0^2\log z - 2y_0\tilde{y}_0)\int_0^z\frac{y_0^2\log z + y_0\tilde{y}_0}{PW_0^2}dt\\ &= (-2y_0^2\log z - 2y_0\tilde{y}_0)\left(\log z\int_0^z\frac{y_0^2}{PW_0^2}dt\right)\\ &\quad + (2y_0^2\log z + 2y_0\tilde{y}_0)\left(\int_0^z\frac{1}{x}\int_0^x\frac{y_0^2}{PW_0^2}dtdx\right)\\ &\quad - (2y_0^2\log z + 2y_0\tilde{y}_0)\left(\int_0^z\frac{y_0\tilde{y}_0}{PW_0^2}dt\right)\end{aligned}$$

and

$$y_1^2\int_0^z\frac{y_0^2}{PW_0^2}dt = (y_0^2(\log z)^2 + 2y_0\tilde{y}_0\log z + \tilde{y}_0^2)\int_0^z\frac{y_0^2}{PW_0^2}dt.$$

Substituting all of the above into equation (B.1) gives,

$$
\begin{aligned}
h(z) &= \cancel{y_0^2(\log z)^2 \int_0^z \frac{y_0^2}{PW_0^2}dt} - \cancel{2y_0^2 \log z \int_0^z \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdx} \\
&+ 2y_0^2 \int_0^z \frac{1}{w}\int_0^w \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdxdw + \cancel{y_0^2 \log z \int_0^z \frac{2y_0\tilde{y}_0}{PW_0^2}dt} \\
&- y_0^2 \int_0^z \frac{1}{x}\int_0^x \frac{2y_0\tilde{y}_0}{PW_0^2}dtdx + y_0^2 \int_0^z \frac{\tilde{y}_0^2}{PW_0^2}dt - \cancel{2y_0^2(\log z)^2 \int_0^z \frac{y_0^2}{PW_0^2}dt} \\
&+ \cancel{2y_0^2 \log z \int_0^z \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdx} - \cancel{2y_0^2 \log z \int_0^z \frac{y_0\tilde{y}_0}{PW_0^2}dt} \\
&- \cancel{2y_0\tilde{y}_0 \log z \int_0^z \frac{y_0^2}{PW_0^2}dt} + 2y_0\tilde{y}_0 \int_0^z \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdx - 2y_0\tilde{y}_0 \int_0^z \frac{y_0\tilde{y}_0}{PW_0^2}dt \\
&+ \cancel{y_0^2(\log z)^2 \int_0^z \frac{y_0^2}{PW_0^2}dt} + \cancel{2y_0\tilde{y}_0 \log z \int_0^z \frac{y_0^2}{PW_0^2}dt} + \tilde{y}_0^2 \int_0^z \frac{y_0^2}{PW_0^2}dt \\
&= 2y_0^2 \left( \int_0^z \frac{1}{w}\int_0^w \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdxdw - \int_0^z \frac{1}{x}\int_0^x \frac{y_0\tilde{y}_0}{PW_0^2}dtdx + \frac{1}{2}\int_0^z \frac{\tilde{y}_0^2}{PW_0^2}dt \right) \\
&\qquad + 2y_0\tilde{y}_0 \int_0^z \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdx - 2y_0\tilde{y}_0 \int_0^z \frac{y_0\tilde{y}_0}{PW_0^2}dt + \tilde{y}_0^2 \int_0^z \frac{y_0^2}{PW_0^2}dt.
\end{aligned}
$$

We can write out the solution of $L_3(y) = 1$ as

$$
\begin{aligned}
h_1(z) = y_0^2 \int_0^z \frac{1}{w}\int_0^w \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdxdw - y_0^2 \int_0^z \frac{1}{x}\int_0^x \frac{y_0\tilde{y}_0}{PW_0^2}dtdx + \frac{1}{2}y_0^2 \int_0^z \frac{\tilde{y}_0^2}{PW_0^2}dt \\
+ y_0\tilde{y}_0 \int_0^z \frac{1}{x}\int_0^x \frac{y_0^2}{PW_0^2}dtdx - y_0\tilde{y}_0 \int_0^z \frac{y_0\tilde{y}_0}{PW_0^2}dt + \frac{1}{2}\tilde{y}_0^2 \int_0^z \frac{y_0^2}{PW_0^2}dt.
\end{aligned}
$$

where we have an additional term in our solution compared to the solution that Beukers' derives in [4].

We expand the functions below into their power series to check that the solution $h_1(z)$ of the differential operator $L_3 = 1$ begins with $z + O(z^2)$, and the $n$-th coefficient of $h_1(z)$ has denominator dividing $\mathrm{lcm}(1, 2, \ldots, n)^3$.

$$y_0(z) = 1 + b_1 z + b_2 z^2 + \cdots$$

$$y_0^2(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

$$\tilde{y}_0(z) = a_1 z + a_2 z^2 + \cdots$$

$$P(z) = 1 + P_1 z + P_2 z^2 + \cdots$$

$$W_0(z) = 1 + d_1 z + d_2 z^2 + \cdots$$

We look at each term of $h_1(z)$ separately, just as we did in the proof of Lemma 3.3. This is especially important, since we have a new extra term to deal with.

Some simple calculations shows that

$$\begin{aligned}
&y_0^2(z)\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x \frac{y_0^2(w)}{P(w)W_0^2(w)}dwdxdy \\
&\quad= y_0^2(z)\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\int_0^x (1+e_1w+e_2w^2+\cdots)dwdxdy \qquad \text{with } e_i \in Q(R) \\
&= y_0^2(z)\int_0^z \frac{1}{y}\int_0^y \frac{1}{x}\left(x+\frac{e_1x^2}{2}+\frac{e_3x^3}{3}+\cdots\right)dxdy \\
&\quad= y_0^2(z)\int_0^z \frac{1}{y}\left(y+\frac{e_1y^2}{2\cdot 2}+\frac{e_3y^3}{3\cdot 3}+\cdots\right)dy \\
&\quad= y_0^2(z)\left(z+\frac{e_1z^2}{2\cdot 2\cdot 2}+\frac{e_3z^3}{3\cdot 3\cdot 3}+\cdots\right) \\
&\quad= (1+c_1z+c_2z^2+\cdots)\left(z+\frac{e_1z^2}{2\cdot 2\cdot 2}+\frac{e_3z^3}{3\cdot 3\cdot 3}+\cdots\right) \\
&\quad= z+O(z^2)
\end{aligned}$$

where it is clear that the coefficient of $z^n$ in the above has denominator which divides $\mathrm{lcm}(1,2,\ldots,n)^3$. We continue to check the remaining terms. Below we have used the letter $e_i$ to represent elements in the quotient ring. These are not necessarily the same elements as above. I shall continue to use this (awful) notation, however please note that it does not affect the conclusions we draw about the denominators in any form.

$$\begin{aligned}
&y_0^2(z)\int_0^z \frac{1}{y}\int_0^y \frac{y_0(x)\tilde{y}_0(x)}{P(x)W_0^2(x)}dxdy \\
&\quad= y_0^2(z)\int_0^z \frac{1}{y}\int_0^y \left(e_1x+e_2x^2+\cdots\right)dxdy \\
&\quad= y_0^2(z)\int_0^z \frac{1}{y}\left(\frac{e_1y^2}{2}+\frac{e_2y^3}{3}+\cdots\right)dy \\
&\quad= y_0^2(z)\left(\frac{e_1z^2}{2\cdot 2}+\frac{e_2z^3}{3\cdot 3}+\cdots\right) \\
&\quad= \frac{e_1}{4}z^2+O(z^3).
\end{aligned}$$

The $n$-th coefficient of $\tilde{y}_0$ has denominator dividing $\mathrm{lcm}(1,2,\ldots,n)$, and in taking integrals twice, we have the denominator of the $n$-th coefficient of the above term dividing $\mathrm{lcm}(1,2,\ldots,n)^3$.

Next we check

$$\begin{aligned}
&\frac{1}{2}y_0^2(z)\int_0^z \frac{\tilde{y}_0^2(x)}{P(x)W_0^2(x)}dx \\
&= \frac{1}{2}y_0^2(z)\int_0^z \left(e_1x+e_2x^2+e_3x^3+\cdots\right)dx \\
&= \frac{1}{2}y_0^2(z)\left(\frac{e_1z^2}{2}+\frac{e_2z^3}{3}+\cdots\right) \\
&= \frac{1}{2}\left(\frac{e_1z^2}{2}+O(z^3)\right)
\end{aligned}$$

Similar observations tells us that the $n$-th coefficient of $\tilde{y}_0^2(z)$ already has denominator which divides $\mathrm{lcm}(1,2,\ldots,n)^2$ and taking an integral gives us the desired results.

We shall analyse the last three terms together, once we have stated their power series expansions.

$$\begin{aligned}
&y_0(z)\tilde{y}_0(z)\int_0^z \frac{1}{x}\int_0^x \frac{y_0^2(w)}{P(w)W_0^2(w)}dwdx\\
&= y_0(z)\tilde{y}_0(z)\left(z+\frac{e_1z^2}{2\cdot 2}+\frac{e_3z^3}{3\cdot 3}+\cdots\right)\\
&= a_1z^2+O(z^3)
\end{aligned}$$

and

$$\begin{aligned}
&y_0(z)\tilde{y}_0(z)\int_0^z \frac{y_0(x)\tilde{y}_0(x)}{P(x)W_0^2(x)}dx\\
&= y_0(z)\tilde{y}_0(z)\left(\frac{e_1z^2}{2}+\frac{e_2z^3}{3}+\cdots\right)\\
&= \frac{a_1e_1}{2}z^3+O(z^4)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\tilde{y}_0^2(z)\int_0^z \frac{y_0^2(w)}{P(w)W_0^2(w)}dw\\
&= \frac{1}{2}\tilde{y}_0^2(z)\left(z+\frac{e_1z^2}{2}+\frac{e_3z^3}{3}+\cdots\right)\\
&= \frac{1}{2}\left(a_1^2z^3+O(z^4)\right).
\end{aligned}$$

We notice that the above three terms have some combination of integrals and $\tilde{y}_0(z)$. From each integral and every appearance of $\tilde{y}_0$, we get the denominator of the $n$-th coefficient dividing $\mathrm{lcm}(1,2,\ldots,n)$. In the above three terms, we easily see that there are three instances of this in every term, hence we arrive at the results we were hoping for. We conclude that the solution $h_1(z)$ to $L_3(y)=1$ begins with $z+O(z^2)$, and the $n$-th coefficient has denominator which divides $\mathrm{lcm}(1,2,\ldots,n)^3$. □

## Appendix C. Identities for $R(x)$ and $T(x)$

**Proposition C.1.** *Recall the Laurent series $R(x)$ and $T(x)$ defined in Definition 4.2. Then we have the following identities.*

$$R(x) = -\sum_{n=0}^{\infty}\frac{1}{n+1}\left[\begin{array}{c} n \\ x \end{array}\right]$$

*and*

$$T(x) = -\sum_{n=0}^{\infty}\frac{1}{n+1}\left[\begin{array}{c} n \\ x \end{array}\right]\left[\begin{array}{c} n \\ 1-x \end{array}\right].$$

*Proof.* To prove the identity for $R(x)$, we let

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left[ \begin{array}{c} n \\ x \end{array} \right].$$

Some simple calculations shows

$$\begin{aligned}
\left[ \begin{array}{c} n \\ x+1 \end{array} \right] - \left[ \begin{array}{c} n \\ x \end{array} \right] &= \frac{n!}{(x+1)\cdots(x+n+1)} - \frac{n!}{x(x+1)\cdots(x+n)} \\
&= \frac{x \cdot n! - (x+n+1)n!}{x(x+1)\cdots(x+n+1)} \\
&= -\frac{(n+1)n!}{x(x+1)\cdots(x+n+1)} \\
&= -\frac{n+1}{x} \left[ \begin{array}{c} n \\ x+1 \end{array} \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
S(x+1) - S(x) &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \left[ \begin{array}{c} n \\ x+1 \end{array} \right] - \left[ \begin{array}{c} n \\ x \end{array} \right] \right) \\
&= \sum_{n=0}^{\infty} -\frac{(n+1)}{(n+1)x} \left[ \begin{array}{c} n \\ x+1 \end{array} \right] \\
&= -\frac{1}{x} \sum_{n=0}^{\infty} \left[ \begin{array}{c} n \\ x+1 \end{array} \right].
\end{aligned}$$

Notice that,

$$\begin{aligned}
&-\frac{1}{x}\Delta_n \left( (1+n+x) \left[ \begin{array}{c} n \\ x+1 \end{array} \right] \right) \\
&\quad = -\frac{1}{x} \left( (2+n+x) \left[ \begin{array}{c} n+1 \\ x+1 \end{array} \right] - (1+n+x) \left[ \begin{array}{c} n \\ x+1 \end{array} \right] \right) \\
&\quad = -\frac{1}{x} \left[ \begin{array}{c} n \\ x+1 \end{array} \right] \left( \frac{(2+n+x)(n+1)}{n+2+x} - (1+n+x) \right) \\
&\quad = \left[ \begin{array}{c} n \\ x+1 \end{array} \right].
\end{aligned}$$

Summing over positive $n$ gives

$$\begin{aligned}
S(x+1) - S(x) &= -\frac{1}{x} \sum_{n=0}^{\infty} \left[ \begin{array}{c} n \\ x+1 \end{array} \right] \\
&= \frac{1}{x^2} \left( \lim_{k\to\infty} \left| (1+k+x) \left[ \begin{array}{c} k \\ x+1 \end{array} \right] \right|_p - (1+x) \cdot \frac{0!}{1+x} \right) \\
&= -\frac{1}{x^2}.
\end{aligned}$$

Therefore, $S(x+1) + R(x+1) - S(x) - R(x) = 0$ and so $R(x) + S(x)$ is periodic with period 1, hence $R(x) = -S(x)$.

To prove the identity for $T(x)$, we set

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.$$

Some simple calculations shows

$$\begin{aligned}
&\begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ -x \end{bmatrix} - \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \\
&= \frac{n!}{(x+1)\cdots(x+1+n)} \cdot \frac{n!}{(-x)\cdots(n-x)} \\
&\qquad - \frac{n!}{x\cdots(x+n)} \cdot \frac{n!}{(1-x)\cdots(n+1-x)} \\
&= \frac{-(n+1-x)(n!)^2 - (x+1+n)(n!)^2}{x\cdots(x+1+n)(1-x)\cdots(n+1-x)} \\
&= \frac{-2(n+1)(n!)^2}{x(x+1)\cdots(x+1+n)(1-x)\cdots(n+1-x)} \\
&= \frac{-2(n+1)}{x} \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
S(x+1) - S(x) &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ -x \end{bmatrix} - \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \frac{-2(n+1)}{x} \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \right) \\
&= -\frac{2}{x} \sum_{n=0}^{\infty} \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.
\end{aligned}$$

We notice that

$$\begin{aligned}
&-\frac{1}{x^2} \Delta_n \left( (x^2 - (n+1)^2) \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \right) \\
&= -\frac{1}{x^2} \left( (x^2 - (n+2)^2) \begin{bmatrix} n+1 \\ x+1 \end{bmatrix} \begin{bmatrix} n+1 \\ 1-x \end{bmatrix} \right. \\
&\qquad \left. - (x^2 - (n+1)^2) \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \right) \\
&= -\frac{1}{x^2} \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \cdot \left( \frac{(x^2 - (n+2)^2)(n+1)^2}{(n+2+x)(n+2-x)} - (x^2 - (n+1)^2) \right) \\
&= -\frac{1}{x^2} \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix} \cdot \left( \frac{\cancel{(x^2 - (n+2)^2)}(n+1)^2}{\cancel{(n+2)^2 - x^2}} - (x^2 - (n+1)^2) \right) \\
&= \begin{bmatrix} n \\ x+1 \end{bmatrix} \begin{bmatrix} n \\ 1-x \end{bmatrix}.
\end{aligned}$$

Summing then gives

$$\begin{aligned} S(x+1)-S(x) &= -\frac{2}{x}\sum_{n=0}^{\infty}\left[\begin{array}{c} n \\ x+1 \end{array}\right]\left[\begin{array}{c} n \\ 1-x \end{array}\right] \\ &= \frac{2}{x^3}\left(\lim_{k\to\infty}\left|(x^2-(k+1)^2)\left[\begin{array}{c} k \\ x+1 \end{array}\right]\left[\begin{array}{c} k \\ 1-x \end{array}\right]\right|_p\right) \\ &- \frac{2}{x^3}\left((x^2-1)\frac{(0!)^2}{(x+1)(1-x)}\right) \\ &= \frac{2}{x^3} \end{aligned}$$

Therefore, $S(x+1)-S(x)=\frac{2}{x^3}$. Since we have proven $T(x+1)-T(x)=-\frac{2}{x^3}$ it follows that $S(x+1)+T(x+1)-S(x)-T(x)=0$ and so $S(x)+T(x)$ is periodic with period 1. Hence $S(x)=T(x)$. □

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