The Hunt for Totally Real Number Fields

Vandita Patel
University of Warwick

YRM 2015

August 17, 2015

The Problem ...

Question

Given integers n and Δ, can we find all number fields of degree n and discriminant Δ?

Restrictions:

1. Primitive number fields (K primitive if there does not exist L such that $\mathbb{Q} \subsetneq L \subsetneq K$).
2. Totally Real Number Fields.

The Problem ...

Question

Given integers n and Δ, can we find all number fields of degree n and discriminant Δ?

Restrictions:

1. Primitive number fields (K primitive if there does not exist L such that $\mathbb{Q} \subsetneq L \subsetneq K$).
2. Totally Real Number Fields.

The Problem ...

Question

Given integers n and Δ, can we find all number fields of degree n and discriminant Δ?

Restrictions:

1. Primitive number fields (K primitive if there does not exist L such that $\mathbb{Q} \subsetneq L \subsetneq K$).
2. Totally Real Number Fields.

Current Method ...

Theorem (Hunter's Theorem)

Let K be a primitive totally real number field of degree n and discriminant Δ. Then there exists $\alpha \in \mathcal{O}_K \setminus \mathbb{Z}$ such that:

1. $$0 \leq Tr(\alpha) \leq \frac{n}{2},$$

2. $$0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}.$$

Current Method ...

Theorem (Hunter's Theorem)

Let K be a primitive totally real number field of degree n and discriminant Δ. Then there exists $\alpha \in \mathcal{O}_K \setminus \mathbb{Z}$ such that:

1. $$0 \leq Tr(\alpha) \leq \frac{n}{2},$$

2. $$0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}.$$

Current Method ...

Theorem (Hunter's Theorem)

Let K be a primitive totally real number field of degree n and discriminant Δ. Then there exists $\alpha \in \mathcal{O}_K \setminus \mathbb{Z}$ such that:

1. $$0 \leq Tr(\alpha) \leq \frac{n}{2},$$

2. $$0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}.$$

An Example ...

Example

Let $n = 2$ and $\Delta \leq 5$. Find all totally real number fields of degree 2 and discriminant less than or equal to 5.
Hunter's Theorem gives us some bounds.

$$0 \leq Tr(\alpha) \leq \frac{n}{2} = 1,$$

$$0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

$$\leq \frac{(Tr(\alpha))^2}{2} + \gamma_1 \left(\frac{5}{2} \right)$$

An Example ...

Example

Let $n = 2$ and $\Delta \leq 5$. Find all totally real number fields of degree 2 and discriminant less than or equal to 5.
Hunter's Theorem gives us some bounds.

$$0 \leq Tr(\alpha) \leq \frac{n}{2} = 1,$$

$$0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

$$\leq \frac{(Tr(\alpha))^2}{2} + \gamma_1 \left(\frac{5}{2} \right)$$

An Example ...

Example

Let $n = 2$ and $\Delta \leq 5$. Find all totally real number fields of degree 2 and discriminant less than or equal to 5.
Hunter's Theorem gives us some bounds.

$$0 \leq Tr(\alpha) \leq \frac{n}{2} = 1,$$

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} \sigma_i(\alpha^2) &\leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1}\left(\frac{|\Delta|}{n}\right)^{\frac{1}{n-1}} \\ &\leq \frac{(Tr(\alpha))^2}{2} + \gamma_1\left(\frac{5}{2}\right) \end{aligned}$$

An Example ...

Example

$$\text{if } Tr(\alpha) = 0 \longrightarrow Tr(\alpha^2) \leq \frac{5}{2}$$
$$\text{if } Tr(\alpha) = 1 \longrightarrow Tr(\alpha^2) \leq \frac{1}{2} + \frac{5}{2} = 3.$$

An Example ...

Example

We are looking for a polynomial $f(x) = x^2 + ax + b$. We let α and β be the roots of f. Then
$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.

1. $Tr(\alpha) = \alpha + \beta$.
2. $\alpha\beta = \frac{(\alpha+\beta)^2 - \alpha^2 - \beta^2}{2} = \frac{1}{2}\left(Tr(\alpha)^2 - Tr(\alpha^2)\right)$.
3. If $Tr(\alpha) = 0$, then $Tr(\alpha^2) \leq 2$ and $\alpha\beta \geq \frac{0-2}{2}$ and so $-1 \leq \alpha\beta \leq 0$.
4. If $Tr(\alpha) = 1$, then $Tr(\alpha^2) \leq 3$ and $\alpha\beta \geq \frac{1-3}{2}$ and so $-1 \leq \alpha\beta \leq 0$.

An Example ...

Example

We are looking for a polynomial $f(x) = x^2 + ax + b$. We let α and β be the roots of f. Then
$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.

1. $Tr(\alpha) = \alpha + \beta$.
2. $\alpha\beta = \frac{(\alpha+\beta)^2 - \alpha^2 - \beta^2}{2} = \frac{1}{2}\left(Tr(\alpha)^2 - Tr(\alpha^2)\right)$.
3. If $Tr(\alpha) = 0$, then $Tr(\alpha^2) \leq 2$ and $\alpha\beta \geq \frac{0-2}{2}$ and so $-1 \leq \alpha\beta \leq 0$.
4. If $Tr(\alpha) = 1$, then $Tr(\alpha^2) \leq 3$ and $\alpha\beta \geq \frac{1-3}{2}$ and so $-1 \leq \alpha\beta \leq 0$.

An Example ...

Example

We are looking for a polynomial $f(x) = x^2 + ax + b$. We let α and β be the roots of f. Then
$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.

1. $Tr(\alpha) = \alpha + \beta$.
2. $\alpha\beta = \frac{(\alpha+\beta)^2 - \alpha^2 - \beta^2}{2} = \frac{1}{2}\left(Tr(\alpha)^2 - Tr(\alpha^2)\right)$.
3. If $Tr(\alpha) = 0$, then $Tr(\alpha^2) \leq 2$ and $\alpha\beta \geq \frac{0-2}{2}$ and so $-1 \leq \alpha\beta \leq 0$.
4. If $Tr(\alpha) = 1$, then $Tr(\alpha^2) \leq 3$ and $\alpha\beta \geq \frac{1-3}{2}$ and so $-1 \leq \alpha\beta \leq 0$.

An Example ...

Example

We are looking for a polynomial $f(x) = x^2 + ax + b$. We let α and β be the roots of f. Then
$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.

1. $Tr(\alpha) = \alpha + \beta$.
2. $\alpha\beta = \frac{(\alpha+\beta)^2-\alpha^2-\beta^2}{2} = \frac{1}{2}\left(Tr(\alpha)^2 - Tr(\alpha^2)\right)$.
3. If $Tr(\alpha) = 0$, then $Tr(\alpha^2) \leq 2$ and $\alpha\beta \geq \frac{0-2}{2}$ and so $-1 \leq \alpha\beta \leq 0$.
4. If $Tr(\alpha) = 1$, then $Tr(\alpha^2) \leq 3$ and $\alpha\beta \geq \frac{1-3}{2}$ and so $-1 \leq \alpha\beta \leq 0$.

An Example ...

Example

We are looking for a polynomial $f(x) = x^2 + ax + b$. We let α and β be the roots of f. Then
$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$.

1. $Tr(\alpha) = \alpha + \beta$.
2. $\alpha\beta = \frac{(\alpha+\beta)^2 - \alpha^2 - \beta^2}{2} = \frac{1}{2}\left(Tr(\alpha)^2 - Tr(\alpha^2)\right)$.
3. If $Tr(\alpha) = 0$, then $Tr(\alpha^2) \leq 2$ and $\alpha\beta \geq \frac{0-2}{2}$ and so $-1 \leq \alpha\beta \leq 0$.
4. If $Tr(\alpha) = 1$, then $Tr(\alpha^2) \leq 3$ and $\alpha\beta \geq \frac{1-3}{2}$ and so $-1 \leq \alpha\beta \leq 0$.

An Example ...

Example

Possible polynomials:

1. $x^2 - 0x + 0 \longrightarrow$ reducible
2. $x^2 - 0x - 1 \longrightarrow$ reducible
3. $x^2 - x \longrightarrow$ reducible
4. $x^2 - x - 1 \checkmark$

$$Tr(\alpha^2) = \sum_{i=1}^{d} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

An Example ...

Example

Possible polynomials:

1. $x^2 - 0x + 0 \longrightarrow$ reducible
2. $x^2 - 0x - 1 \longrightarrow$ reducible
3. $x^2 - x \longrightarrow$ reducible
4. $x^2 - x - 1 \checkmark$

$$Tr(\alpha^2) = \sum_{i=1}^{d} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

An Example ...

Example

Possible polynomials:

1. $x^2 - 0x + 0 \longrightarrow$ reducible
2. $x^2 - 0x - 1 \longrightarrow$ reducible
3. $x^2 - x \longrightarrow$ reducible
4. $x^2 - x - 1 \checkmark$

$$Tr(\alpha^2) = \sum_{i=1}^{d} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

An Example ...

Example

Possible polynomials:

1. $x^2 - 0x + 0 \longrightarrow$ reducible
2. $x^2 - 0x - 1 \longrightarrow$ reducible
3. $x^2 - x \longrightarrow$ reducible
4. $x^2 - x - 1 \checkmark$

$$Tr(\alpha^2) = \sum_{i=1}^{d} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n}\right)^{\frac{1}{n-1}}$$

An Example ...

Example

Possible polynomials:

1. $x^2 - 0x + 0 \longrightarrow$ reducible
2. $x^2 - 0x - 1 \longrightarrow$ reducible
3. $x^2 - x \longrightarrow$ reducible
4. $x^2 - x - 1 \checkmark$

$$Tr(\alpha^2) = \sum_{i=1}^{d} \sigma_i(\alpha^2) \leq \frac{(Tr(\alpha))^2}{n} + \gamma_{n-1} \left(\frac{|\Delta|}{n} \right)^{\frac{1}{n-1}}$$

Motivation ...

Primitive degree 8 Number Fields.

1. $x^8 - 28x^6 - 28x^5 + 196x^4 + 308x^3 - 224x^2 - 528x - 182$
2. $x^8 - 42x^6 - 56x^5 + 420x^4 + 1092x^3 + 406x^2 - 852x - 609$
3. $x^8 - 4x^7 - 28x^6 + 60x^5 + 102x^4 - 20x^3 - 36x^2 - 4x + 1$
4. $x^8 - 2x^7 - 34x^6 + 28x^5 + 280x^4 - 210x^3 - 686x^2 + 588x + 49$

	Δ	HB	$\min(\mathrm{Tr}(\alpha^2))$	$\mathrm{Dim}(M_4(N))$	$\mathrm{Dim}(M_{7/2}(N))$
1.	$2^{18}7^{10}$	130	56	184	152
2.	$2^{8}3^{10}7^{8}$	133	84	536	864
3.	$2^{16}7^{4}11^{6}$	157	72	150	124
4.	$2^{12}7^{6}11^{6}$	184	72	150	124

An Approach via Modular Forms ...

Theorem (Hecke,Schoenberg)

1. *Let K be a primitive, totally real number field with degree n and discriminant Δ. Let $\mathcal{O}_K$ be its ring of integers, with basis $\{b_1, \ldots, b_n\}$.*
2. *Pick any $\alpha \in \mathcal{O}_K$ i.e. $\alpha = x_1 b_1 + \cdots + x_n b_n$.*
3. *Let $Q(\underline{x}) = \sum_{i=1}^{n} \sigma_i(\alpha^2)$.*
4. *$R_Q(m) = \#\{\underline{x} \in \mathbb{Z}^n | Q(\underline{x}) = m\}$.*

Then

$$f = \sum_{m=0}^{\infty} R_Q(m) q^m \in M_{n/2}(N, \chi).$$

An Approach via Modular Forms ...

Theorem (Hecke, Schoenberg)

1. *Let K be a primitive, totally real number field with degree n and discriminant Δ. Let $\mathcal{O}_K$ be its ring of integers, with basis $\{b_1, \ldots, b_n\}$.*
2. *Pick any $\alpha \in \mathcal{O}_K$ i.e. $\alpha = x_1 b_1 + \cdots + x_n b_n$.*
3. *Let $Q(\underline{x}) = \sum_{i=1}^{n} \sigma_i(\alpha^2)$.*
4. *$R_Q(m) = \#\{\underline{x} \in \mathbb{Z}^n | Q(\underline{x}) = m\}$.*

Then

$$f = \sum_{m=0}^{\infty} R_Q(m) q^m \in M_{n/2}(N, \chi).$$

An Approach via Modular Forms ...

Theorem (Hecke, Schoenberg)

1. *Let K be a primitive, totally real number field with degree n and discriminant Δ. Let $\mathcal{O}_K$ be its ring of integers, with basis $\{b_1, \ldots, b_n\}$.*
2. *Pick any $\alpha \in \mathcal{O}_K$ i.e. $\alpha = x_1 b_1 + \cdots + x_n b_n$.*
3. *Let $Q(\underline{x}) = \sum_{i=1}^{n} \sigma_i(\alpha^2)$.*
4. *$R_Q(m) = \#\{\underline{x} \in \mathbb{Z}^n | Q(\underline{x}) = m\}$.*

Then

$$f = \sum_{m=0}^{\infty} R_Q(m) q^m \in M_{n/2}(N, \chi).$$

An Approach via Modular Forms ...

Theorem (Hecke, Schoenberg)

1. *Let K be a primitive, totally real number field with degree n and discriminant Δ. Let $\mathcal{O}_K$ be its ring of integers, with basis $\{b_1, \ldots, b_n\}$.*
2. *Pick any $\alpha \in \mathcal{O}_K$ i.e. $\alpha = x_1 b_1 + \cdots + x_n b_n$.*
3. *Let $Q(\underline{x}) = \sum_{i=1}^{n} \sigma_i(\alpha^2)$.*
4. $R_Q(m) = \#\{\underline{x} \in \mathbb{Z}^n | Q(\underline{x}) = m\}$.

Then

$$f = \sum_{m=0}^{\infty} R_Q(m) q^m \in M_{n/2}(N, \chi).$$

An Approach via Modular Forms ...

Theorem (Hecke,Schoenberg)

1. *Let K be a primitive, totally real number field with degree n and discriminant Δ. Let $\mathcal{O}_K$ be its ring of integers, with basis $\{b_1, \ldots, b_n\}$.*
2. *Pick any $\alpha \in \mathcal{O}_K$ i.e. $\alpha = x_1 b_1 + \cdots + x_n b_n$.*
3. *Let $Q(\underline{x}) = \sum_{i=1}^{n} \sigma_i(\alpha^2)$.*
4. $R_Q(m) = \#\{\underline{x} \in \mathbb{Z}^n | Q(\underline{x}) = m\}$.

Then

$$f = \sum_{m=0}^{\infty} R_Q(m) q^m \in M_{n/2}(N, \chi).$$

An Example ...

Let

$$K = \mathbb{Q}(\sqrt{5}).$$

$$\alpha = a + b\left(\frac{1+\sqrt{5}}{2}\right) \in \mathcal{O}_K, a, b, \in \mathbb{Z}.$$

$$Q = \sigma_1(\alpha^2) + \sigma_2(\alpha^2) = 2a^2 + 2ab + 3b^2.$$

Then

$$f = 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots \in M_1(\Gamma_1(20), \chi)$$

where

$$\chi(11) = \chi(17) = -1, \text{ character modulo 20, conductor 20.}$$

An Example ...

Let

$$K = \mathbb{Q}(\sqrt{5}).$$

$$\alpha = a + b\left(\frac{1+\sqrt{5}}{2}\right) \in \mathcal{O}_K, a, b, \in \mathbb{Z}.$$

$$Q = \sigma_1(\alpha^2) + \sigma_2(\alpha^2) = 2a^2 + 2ab + 3b^2.$$

Then

$$f = 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots \in M_1(\Gamma_1(20), \chi)$$

where

$$\chi(11) = \chi(17) = -1, \text{ character modulo 20, conductor 20.}$$

An Example ...

Let

$$K = \mathbb{Q}(\sqrt{5}).$$

$$\alpha = a + b\left(\frac{1+\sqrt{5}}{2}\right) \in \mathcal{O}_K, a, b, \in \mathbb{Z}.$$

$$Q = \sigma_1(\alpha^2) + \sigma_2(\alpha^2) = 2a^2 + 2ab + 3b^2.$$

Then

$$f = 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots \in M_1(\Gamma_1(20), \chi)$$

where

$$\chi(11) = \chi(17) = -1, \text{ character modulo 20, conductor 20.}$$

An Example ...

Let

$$K = \mathbb{Q}(\sqrt{5}).$$

$$\alpha = a + b\left(\frac{1+\sqrt{5}}{2}\right) \in \mathcal{O}_K, a, b, \in \mathbb{Z}.$$

$$Q = \sigma_1(\alpha^2) + \sigma_2(\alpha^2) = 2a^2 + 2ab + 3b^2.$$

Then

$$f = 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots \in M_1(\Gamma_1(20), \chi)$$

where

$$\chi(11) = \chi(17) = -1, \text{ character modulo 20, conductor 20.}$$

An Example ...

Let

$$K = \mathbb{Q}(\sqrt{5}).$$

$$\alpha = a + b\left(\frac{1+\sqrt{5}}{2}\right) \in \mathcal{O}_K, a, b, \in \mathbb{Z}.$$

$$Q = \sigma_1(\alpha^2) + \sigma_2(\alpha^2) = 2a^2 + 2ab + 3b^2.$$

Then

$$f = 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots \in M_1(\Gamma_1(20), \chi)$$

where

$$\chi(11) = \chi(17) = -1, \text{ character modulo 20, conductor 20.}$$

Backwards Compatibility ...

Question

Given $M_k(N,\chi)$*, can we find all number fields of degree* $2k$ *and discriminant* Δ *where* $N \mid \Delta \mid N^{2k}$*?*

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.
Equate coefficients:
$q : x_2 \geq 0$
$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$
and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.
Equate coefficients:
$q : x_2 \geq 0$
$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$
and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1.$
Equate coefficients:
$q : x_2 \geq 0$
$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$
and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0.$

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.

Equate coefficients:

$q : x_2 \geq 0$

$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$

and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.
Equate coefficients:
$q : x_2 \geq 0$
$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$
and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + \cdots$$
$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.
Equate coefficients:
$q : x_2 \geq 0$
$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$
and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

An Example ...

$M_1(20, \chi)$ has dimension 2, where $\chi(11) = \chi(17) = -1$, character modulo 20, conductor 20.

Question

Is $f := x_1 f_1 + x_2 f_2$ a modular form coming from a number field?

$$f_1 := 1 + 2q^2 + \boxed{4q^3} + 4q^7 + 2q^8 + \cdots$$

$$f_2 := q - q^2 - 2q^3 + q^4 + q^5 + 2q^6 - 2q^7 - q^8 + \cdots$$

$x_1 = 1$.

Equate coefficients:

$q : x_2 \geq 0$

$q^2 : 2 - x_2 \geq 0 \longrightarrow x_2 \leq 2$

and $2 - x_2 \geq 2 \longrightarrow x_2 \leq 0$.

Further Work ...

1. This method gives us a very high dimensional linear programming problem.
2. As the dimension increases, our solution space looks like an unbounded cone.
3. A bound from above!

Further Work ...

1. This method gives us a very high dimensional linear programming problem.
2. As the dimension increases, our solution space looks like an unbounded cone.
3. A bound from above!

Further Work ...

1. This method gives us a very high dimensional linear programming problem.
2. As the dimension increases, our solution space looks like an unbounded cone.
3. A bound from above!

Bound From Above ...

1. Let $n \geq 2$ be a natural number.
2. Let $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form with symmetric matrix.
3. Let $A(X)$ be the number of lattice points in the region $Q(\underline{x}) \leq X$
4. Then, $A(X) = V(X) + P(X)$.
5. Application: The sum of the coefficients of the modular form must be less than or equal to $A(X)$.

Bound From Above ...

1. Let $n \geq 2$ be a natural number.
2. Let $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form with symmetric matrix.
3. Let $A(X)$ be the number of lattice points in the region $Q(\underline{x}) \leq X$
4. Then, $A(X) = V(X) + P(X)$.
5. Application: The sum of the coefficients of the modular form must be less than or equal to $A(X)$.

Bound From Above ...

1. Let $n \geq 2$ be a natural number.
2. Let $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form with symmetric matrix.
3. Let $A(X)$ be the number of lattice points in the region $Q(\underline{x}) \leq X$
4. Then, $A(X) = V(X) + P(X)$.
5. Application: The sum of the coefficients of the modular form must be less than or equal to $A(X)$.

Bound From Above ...

1. Let $n \geq 2$ be a natural number.
2. Let $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form with symmetric matrix.
3. Let $A(X)$ be the number of lattice points in the region $Q(\underline{x}) \leq X$
4. Then, $A(X) = V(X) + P(X)$.
5. Application: The sum of the coefficients of the modular form must be less than or equal to $A(X)$.

Bound From Above ...

1. Let $n \geq 2$ be a natural number.
2. Let $Q(\underline{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a positive definite quadratic form with symmetric matrix.
3. Let $A(X)$ be the number of lattice points in the region $Q(\underline{x}) \leq X$
4. Then, $A(X) = V(X) + P(X)$.
5. Application: The sum of the coefficients of the modular form must be less than or equal to $A(X)$.

Thank you for Listening ...