Introduction to Comparison Theorems in Geometry

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21/10/19

Abstract

Given a Riemannian manifold, we may compare its geometric quantites with those of suitably nice model spaces (e.g hyperbolic space). If these quantities are reasonably similar, a typical comparison theorem would show that the manifold retains geometric properties of the model space. In this talk we will motivate and introduce one such comparison theorem, the Bishop-Gromov inequality, and its consequence to the manifolds underlying topology.

0 Introduction

What is a Comparison Theorem (In Riemannian geometry)?

Comparison Theorems involve comparing quantities in a Riemannian manifold (M^n, g) (which we will always assume is complete), to quantities in a model space, e.g $\mathbb{R}^n$.

Definition 0.1 (Model Spaces)**.** For $\sigma \in \mathbb{R}$, let M^n_σ be the simply connected, n-dimensional manifold with constant sectional curvature σ. In particular,

$$M^n_\sigma = \begin{cases} \mathbb{R}^n & : \sigma = 0 \\ \mathbb{S}^n & : \sigma > 0 \\ \mathbb{H}^n & : \sigma < 0 \end{cases}$$

Examples of Comparison Theorems:

- Bonnet's Theorem:
  If sec $\geq \sigma > 0$, then $\text{diam}(M) \leq \text{diam}(M^n_\sigma) = \frac{\pi}{\sqrt{\sigma}}$.
- Myer's Theorem:
  If Ric $\geq (n-1)\sigma > 0$, then $\text{diam}(M) \leq \text{diam}(M^n_\sigma) = \frac{\pi}{\sqrt{\sigma}}$.
- Toponogov's Theorem:
  If sec $\geq \sigma$, let pqr be a geodesic triangle in M and $p'q'r'$ a geodesic triangle in M^n_σ such that: pq is minimal, $pr < \frac{\pi}{\sqrt{\sigma}}$ if $\sigma > 0$ and $pq = p'q'$, $pr = p'r'$, $\angle qpr = \angle q'p'r'$. Then, $qr \leq q'r'$.

  Remark. Can use this to define lower curvature bounds for a general metric space.

- Sphere Theorem:
  If $\pi_1(M) = 0$ and $\sec \in (\frac{1}{4}, 1]$, then $M \simeq S^n$.
  Remark. If $\sec \in [\frac{1}{4}, 1]$, then $(\mathbb{CP}^n, g_{FS})$ is a counterexample. Moreover, due the existence of exotic spheres, the original proof of this theorem does not show a diffeomorphism.

- Differentiable Sphere Theorem:
  If $\pi_1(M) = 0$ and $\sec \in (\frac{1}{4}, 1]$, then $M \cong S^n$. The proof is by Schoen and Brendle in 07' and uses Ricci flow.

In this talk, we are going to be interested in comparing volumes. We will now try to motivate why Ricci curvature is the correct quantity to control.

1 Motivation

For a metric g in normal coordinates, it has the following expansion

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj} x^k x^l + O(|x|^3) \tag{1}$$

where R_{iklj} are the coefficients of the Riemann Curvature. (higher order terms also depend on R, so R is the direct obstruction to flatness.)

Remark. This expansion comes from Jacobi fields: these are vector fields J along geodesics γ satisfying the 2nd order ODE

$$J'' + R(J, \gamma')\gamma' = 0$$

Recall, the volume form for a Riemannian manifold (M, g) is given by

$$d\mu_g := \sqrt{\det(g_{ij})}\ dx^1 \wedge \cdots \wedge dx^n$$

We now plug (1) into this formula to get an expansion for $d\mu_g$.

Recall, $\det \circ \exp = \exp \circ \operatorname{tr}$. Also, $\det(g_{ij}) = \det(\delta_{ij} - \frac{1}{3} R_{iklj} x^k x^l) + O(|x|^3)$. Thus,

$$\begin{aligned}
\det(g_{ij}) &= \exp(\operatorname{tr}(\log(I - \frac{1}{3} R_{iklj} x^k x^l)) + O(|x|^3) \\
&= \exp(\operatorname{tr}(-\frac{1}{3} R_{iklj} x^k x^l)) + O(|x|^3) \\
&= \exp(-\frac{1}{3} Ric_{kl} x^k x^l) + O(|x|^3) \\
&= 1 - \frac{1}{3} Ric_{kl} x^k x^l + O(|x|^3)
\end{aligned}$$

Therefore,

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} \mathrm{Ric}_{kl} x^k x^l + O(|x|^3)$$

and

$$d\mu_g = (1 - \frac{1}{6} Ric_{kl} x^k x^l + O(|x|^3))\ dx^1 \wedge \cdots \wedge dx^n$$

Thus, Ricci curvature measures the difference of the volume form from flat space. Also, a larger Ricci curvature implies a smaller volume form.

2 Volume Estimates

Theorem 2.1 (Bishop-Gromov 1964-80)**.** Suppose (M^n, g) is complete and $\text{Ric} \geq (n-1)\sigma$ for some $\sigma \in \mathbb{R}$. Then, fixing $p \in M$, the function

$$r \mapsto \frac{\text{Vol}(B_p(r))}{\text{Vol}(B_\sigma^n(r))}$$

is decreasing and converges to 1 as $r \to 0$.

Remark. A simple corollary to this Theorem is that

$$\text{Vol}(B_p(r)) \leq \text{Vol}(B_\sigma^n(r)) \quad \forall r > 0 \tag{2}$$

as expected from our motivation. However, we can also extend this inequality to Annuli.

$$\frac{\text{Vol}(B_p(r_2))}{\text{Vol}(B_\sigma^n(r_2))} \leq \frac{\text{Vol}(B_p(r_1))}{\text{Vol}(B_\sigma^n(r_1))} \leq 1 \quad \forall r_2 > r_1 > 0$$

Implies

$$\begin{aligned}
\text{Vol}(A_p(r_2, r_1)) &= \text{Vol}(B_p(r_2)) - \text{Vol}(B_p(r_1)) \\
&= \frac{\text{Vol}(B_p(r_2))}{\text{Vol}(B_\sigma^n(r_2))} \cdot \text{Vol}(B_\sigma^n(r_2)) - \frac{\text{Vol}(B_p(r_1))}{\text{Vol}(B_\sigma^n(r_1))} \cdot \text{Vol}(B_\sigma^n(r_1)) \\
&\leq \frac{\text{Vol}(B_p(r_1))}{\text{Vol}(B_\sigma^n(r_1))} [\text{Vol}(A_\sigma^n(r_2, r_1))] \\
&\leq \text{Vol}(A_\sigma^n(r_2, r_1))
\end{aligned}$$

Proof Idea. Using the exponential map gives radial coords (r, θ) about p. We have the Bochner formula

$$\Delta(\frac{1}{2}|\nabla f|^2) = \langle \nabla \Delta f, \nabla f \rangle + \text{Ric}(\nabla f, \nabla f) + |\nabla^2 f|^2 \quad \forall f \in C^\infty(M)$$

Setting f to be the distance function $f(x) := d(p, x)$, then $\nabla f = \partial_r$, $|\nabla f| = 1$ and $\nabla^2 f$ has eigenvalues $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$, so that

$$|\nabla^2 f|^2 = \lambda_2^2 + \cdots + \lambda_n^2 \geq \frac{(\lambda_2 + \cdots + \lambda_n)^2}{n-1} = \frac{(\Delta f)^2}{n-1}$$

Substituting in the Bochner formula gives

$$\partial_r(\Delta f) + \frac{(\Delta f)^2}{n-1} \leq \langle \nabla \Delta f, \nabla f \rangle + |\nabla^2 f|^2 = -\text{Ric}(\partial_r, \partial_r)$$

Also, if $d\mu_g = \lambda(r, \theta) dr \wedge d\theta$, then

$$\partial_r(\lambda) = \Delta f \cdot \lambda$$

Combining the two

$$\partial_r^2(\lambda^{\frac{1}{n-1}}) \leq \frac{-\text{Ric}(\partial_r, \partial_r)}{n-1} \lambda^{\frac{1}{n-1}}$$

By ODE theory (Gromwell inequality applied to the ODE $f'' + \sigma f = 0$), we have that $\lambda \leq \lambda_\sigma$.

Note: By intergrating, this is enough to give (2).

Finally, since
$$\partial_r(\log(\frac{\lambda}{\lambda_\sigma})) = \frac{\partial_r \lambda}{\lambda} - \frac{\partial_r \lambda_\sigma}{\lambda_\sigma} = \Delta(f) - \Delta(f_\sigma) \leq 0$$
it follows that $\int_{S^{n-1}} \frac{\lambda}{\lambda_\sigma}\, d\theta$ is decreasing, and hence
$$\partial_r \left(\frac{\mathrm{Vol}(B_p(R))}{\mathrm{Vol}(B^n_\sigma(R))} \right) = \partial_r \left(\frac{\int_0^R \int_{S^{n-1}} \lambda\; dr \wedge d\theta}{\int_0^R \int_{S^{n-1}} \lambda_\sigma\; dr \wedge d\theta} \right) \leq 0 \qquad \square$$

3 An Application

Definition 3.1. Let $\mathfrak{M}(n, \sigma, \nu, D)$ denote the class of n-dimensional, compact manifolds such that:

- $\mathrm{Ric} \geq (n-1)\sigma$
- $\mathrm{vol} \geq \nu$
- $\mathrm{diam} \leq D$

Theorem 3.2 (M. Anderson 90')**.** There are only finitely many fundamental groups in $\mathfrak{M}$. i.e $|\pi_1(\mathfrak{M})| < \infty$.

Example 3.3. Recall the Lens spaces $L(p,1) := S^3/\mathbb{Z}_p$, where $S^3 \subseteq \mathbb{C}^2$ and $\mathbb{Z}_p \curvearrowright S^3$ via
$$k \cdot (z_1, z_2) := (e^{2\pi i \frac{k}{p}} z_1, e^{2\pi i \frac{k}{p}} z_2) \quad \forall\, k \in \mathbb{Z}_p$$
These have constant curvature 1, diameter $\frac{\pi}{p}$ and $\pi_1(L_p) = \mathbb{Z}_p$. However, as $p \to \infty$, their volume goes to 0, so for fixed ν, $\mathfrak{M}(3, 1, \nu, \pi)$ contains only finitely many Lens spaces.

The proof relies on the following Lemma by Gromov.

Lemma 3.4 (Gromov 80')**.** Given $M \in \mathfrak{M}(n, \sigma, \nu, D)$, and its universal cover $\tilde{x} \in \tilde{M}$, there always exists a finite set of generators $\{\gamma_1, \ldots, \gamma_m\}$ for $\pi_1(M)$ such that

(i) $d(\tilde{x}, \gamma_i(\tilde{x})) \leq 2\ \mathrm{diam}(M)\ \forall i$.

(ii) All the relatios for these generators are of the form $\gamma_i \cdot \gamma_j \cdot \gamma_k^{-1} = 1$.

Proof. For $p \in M$, $i(p) > 0$ is the largest radius such that
$$\exp_p : B(0, i(p)) \subseteq T_pM \to B(p, i(p)) \subseteq M$$
is a diffeomorphism. Since i is continuous and M is compact,
$$\mathrm{inj}(M) := \inf_{p \in M} i(p) > 0$$
Choose $\epsilon \in (0, \mathrm{inj}(M))$, and a triangulation of M where adjacent vertices are joined by curves of length $\leq \epsilon$.

— Picture —

We let the vertices $V := \{x_1, \ldots, x_k\}$ and the edges $E := \{e_{ij}\}$. Join $x = \pi(\tilde{x})$ and x_i by a segment σ_i, and if $e_{ij} \in E$, we construct the loop $\sigma_{ij} := \sigma_i \; e_{ij} \; \sigma_j^{-1}$.

— Picture —

Since $\epsilon < \operatorname{inj}(M)$,

- Any loop in $\pi_1(M)$ is homotopic to a loop in the 1 skeleton.
- Any of these loops is homotopic to a product of loops σ_{ij}

That is, our generating set of loops are $\{\sigma_{ij}\}$. Furthermore, if x_i, x_j, x_k are adjacent, then they span a 2-simplex, and hence

$$\sigma_{ij} \; \sigma_{jk} \; \sigma_{ik}^{-1} = \sigma_{ij} \; \sigma_{jk} \; \sigma_{ki} = 1 \tag{3}$$

Conversely, for any loop $\sigma = 1$ in the 1-skeleton, since σ contracts in the 2-skeleton, the relation $\sigma = 1$ can be expressed as a product of relations of the form (3).

— Picture —

Thus, we have shown (ii). Finally, for σ_{ij} we have $d(\tilde{x}, \sigma_{ij}(\tilde{x})) \leq 2D + \epsilon$. Since $\pi_1(M)$ acts discretely on $\tilde{M}$, for $\epsilon > 0$ sufficiently small,

$$\{\gamma : d(\tilde{x}, \gamma(\tilde{x})) < 2D + \epsilon\} = \{\gamma : d(\tilde{x}, \gamma(\tilde{x})) \leq 2D\}$$

Therefore, choosing our original ϵ sufficiently small, we have (i). □

Proof of Theorem. Choose generators $\{\gamma_1, \ldots \gamma_m\}$ as in the Lemma. By (ii), any relation in $\pi_1(M)$ is given by an element of the set

$$R := \{(i, j, k) : 1 \leq i, j, k \leq m\}$$

So, any possible fundamental group for M is given by an element $\pi_1(M) \in \mathcal{P}(R)$, where $|\mathcal{P}(R)| = 2^{|R|} = 2^{m^3}$. Thus, it sufficies to show m is bounded.

Let $F \subseteq \tilde{M}$ be a fundamental domain. This is a closed set such that $\pi : F \twoheadrightarrow M$ and $\operatorname{vol}(F) = \operatorname{vol}(M)$. In particular, given $\tilde{x} \in \tilde{M}$, we could choose $F := \{z \in \tilde{M} : \; d(\tilde{x}, z) \leq d(\gamma(\tilde{x}), z) \; \forall \gamma \in \pi_1(M)\}$.

Example 3.5. Given $M = T^2$, $\tilde{M} = \mathbb{R}^2$, and $\tilde{x} = 0$, we have $F = [\frac{-1}{2}, \frac{-1}{2}]^2$, and $\pi_1(T^2) = \mathbb{Z}^2$ acting via translations.

Note: $\gamma_i(F)$ are disjoint a.e, and by the Lemma, lie within $B(\tilde{x}, 4D)$. Therefore,

$$m \leq \frac{\operatorname{Vol}(B(x, 4D))}{\operatorname{Vol}(F)} \leq \frac{\operatorname{Vol}(B^n_\sigma(4D))}{\nu} < \infty$$ □