On the Gauss-Bonnet-Chern Theorem

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Abstract

The Gauss-Bonnet Theorem is a simple relationship between the topological and geometric information of a Riemannian surface. We will explore this relationship further in higher dimensions and for general vector bundles. Only the basic definitions from geometry and topology will be assumed.

0 Introduction

Theorem 0.1 (Gauss-Bonnet 1848)**.** Let (M^2, g) be a closed (oriented) Riemannian surface.

$$\overbrace{\int_{M^2} \frac{K}{2\pi}\, dA}^{\text{Geometric}} = \overbrace{\chi(M^2)}^{\text{Topological}} \tag{1}$$

where K denotes Gaussian curvature, dA the Area form and $\chi(M)$ the Euler characteristic.

Example 0.2. For Σ_g, the orientable surface of genus g, we have that $\chi(\Sigma_g) = 2 - 2g$. So the total curvature on the torus is 0. Thus, the only non-positive or non-negative curvatures on the torus are identically zero.

$$K \geq 0 \text{ or } \leq 0 \implies K \equiv 0 \quad \text{on } \Sigma_1$$

Moreover, if on some orientable surface, $K > 0$, then $g = 0$ and we are on a sphere, whereas if $K < 0$, then we know $g \geq 2$.

There are multiple ways to generalise this Theorem:

- Include a boundary:

$$\int_M K\, dA + \int_{\partial M} \kappa\, ds = 2\pi\chi(M^2)$$

  where κ denotes the geodesic curvature of the boundary.

- Drop compactness (Cohn-Vossen's inequality): If M is complete and has finite total curvature, then

$$\int_M K\, dA \leq 2\pi\chi(M)$$

- Generalise to higher dimensions/general vector bundles (Gauss-Bonnet-Chern): Let $E \xrightarrow{\pi} M$ be an oriented vector bundle over a closed and oriented manifold M...

We are going to do the last. The plan is as follows.

- Do some Geometry to generalise the LHS.
- Do some Topology to generalise the RHS.
- Find a relationship between these new ideas.

1 Geometry

1.1 Invariant Polynomials

Let G be a Lie Group and $\mathfrak{g}$ its Lie Algebra. Recall, $Sym^r(\mathfrak{g}^*)$ consists of all symmetric r-multilinear maps $\phi : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$. We note that, by the polarisation law, any such ϕ is uniquely determind by its associated homogeneous degree r polynomial P_ϕ, where

$$P_\phi(v) := \phi(v, \dots, v) \quad \forall v \in \mathfrak{g}$$

The Adjoint action $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ induces an action $G \curvearrowright \text{Sym}^r(\mathfrak{g}^*)$. Let $I_r(G)$ denote those polynomials invariant under this action.

Example 1.1. $G = \text{GL}(k, \mathbb{R})$, $\mathfrak{g} = \text{End}(k, \mathbb{R})$. Then we have that $\text{Ad}_X(Y) = XYX^{-1}$. So, $I_r(G)$ consists of those polynomials invariant under a change of basis. e.g, $\text{tr} \in I_1(G)$ and $\det \in I_k(G)$.

Suppose our bundle E has structure group G, and is equipped with a connection matrix ω. The associated curvature form is then given by a matrix $F \in \Omega^2(\mathfrak{g})$. Thus, for $P \in I_r(G)$, we can apply it to our curvature form F to get $P(F) \in \Omega^{2r}(M)$.

Lemma 1.2.

1. $P(F)$ is closed
2. If ω_0, ω_1 are two connections with curvature forms F_0, F_1, then $P(F_0)$ and $P(F_1)$ differ by an exact form.

Proof. If $P \in I_r$, then for B invertible

$$P(AB) = P(B(AB)B^{-1}) = P(BA)$$

By the continuity of P, the above holds for any $A, B \in \mathfrak{g}$. In particular, $P((I + tE_{ij})F) = P(F(I + tE_{ij}))$ where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Taking the derivative at $t = 0$ of the left hand side we have

$$dP(F) \cdot E_{ij}F = dP(F)_{kl} \cdot (E_{ij}F)_{kl} = dP(F)_{kl}\delta_{ik}F_{jl} = dP(F)_{il}F_{jl} = (FdP(F)^T)_{ji}$$

and similarily for the right hand side

$$dP(F) \cdot FE_{ij} = dP(F)_{kj} \cdot F_{ki} = (dP(F)^T F)_{ji}$$

Thus $dP(F)^T$ commutes with F. Using the second Bianchi identity and the fact that F is a 2-form, we get

$$d(P(F)) = Tr(dP(F)^T \wedge dF) = Tr(dP(F)^T \wedge (\omega \wedge F - F \wedge \omega)) = Tr(F \wedge dP(F)^T \wedge \omega - dP(F)^T \wedge F \wedge \omega) = 0$$

For the second part, consider the following diagram with $f(x, t) := x$ and $g_i(x) := (x, i)$.

$$\begin{array}{ccccc}
(E,\omega_i) & \longrightarrow & (f^*(E),\tilde{\omega}) & \longrightarrow & (E,\omega_i) \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{g_i} & M \times [0,1] & \xrightarrow{f} & M
\end{array}$$

We have the two induced connections $f^*(\omega_i)$ on $f^*(E)$ for $i = 0, 1$. Note that $\tilde{\omega} := tf^*(\omega_0) + (1-t)f^*(\omega_1)$ is also a connection on $f^*(E)$, with curvature $\tilde{F}$. In particular, $(g_i)^*(\tilde{\omega}) = \omega_i$ and therefore $(g_i)^*[P(\tilde{F})] = [P(F_i)]$, for $i = 0, 1$. Since cohomology is invariant under homotopy, we are done. □

This lemma shows that $[P(F)]$ is an element of $H^{2k}(M)$, **independent** of the connection.

Definition 1.3. The homomorphism $I_\bullet(G) \to H^{2\bullet}(M)$, $P \mapsto [P(F)]$ is known as the *Chern-Weil* homomorphism.

1.2 Geometric Euler Class

Let us now fix a metric g on M, and assume E has even rank $k = 2m$. In this situation, our structure group can be reduced to $G = SO(2m)$ with $\mathfrak{g} = \text{Skew}(2m, \mathbb{R})$. We will construct a special invariant polynomial in $I_m(G)$.

Let $A \in \mathfrak{g}$. Define Π to be the partitions of $\{1, \ldots, 2m\}$ into unordered pairs. Then, any element $\alpha \in \Pi$ has a canonical representation

$$\alpha = \{(i_1, j_1), \ldots, (i_m, j_m)\} \text{ such that } i_1 < \ldots < i_m \text{ and } i_s < j_s$$

Let σ_α be the permutation

$$\begin{bmatrix} 1 & 2 & \ldots & 2m \\ i_1 & j_1 & \ldots & j_m \end{bmatrix}$$

and $A_\alpha := A_{i_1 j_1} \cdots A_{i_m j_m}$.

Definition 1.4 (Pfaffian)**.** The *Pfaffian* $Pf \in \text{Sym}^m(\mathfrak{g}^*)$ is given by

$$Pf(A) := \sum_{\alpha \in \Pi} sgn(\sigma_\alpha) A_\alpha$$

Claim. $Pf \in I_m$, and so $[Pf(F)] \in H^k(M)$ is a canonically defined cohomology class for the bundle E.

To prove the basis invariance of the Pfaffian, we show a basis invariant equality.

Proof. Let $\eta \in \Lambda^2(\mathbb{R}^{2m})$ and e_i be an oriented, orthonormal basis of $\mathbb{R}^{2m}$. Then $\exists A \in \mathfrak{g}$ such that $\eta = A_{ij}\ e_i \wedge e_j$. Wedging η with itself m times we have,

$$\begin{aligned}
\eta^m &= \sum_{i_1 < j_1} \cdots \sum_{i_m < j_m} A_{i_1 j_1} \ldots A_{i_m j_m} (e_{i_1} \wedge e_{j_1}) \wedge \ldots \wedge (e_{i_m} \wedge e_{j_m}) \\
&= m! \sum_{\alpha \in \Pi} A_\alpha (e_{i_1} \wedge e_{j_1}) \wedge \ldots \wedge (e_{i_m} \wedge e_{j_m}) \\
&= m! \sum_{\alpha \in \Pi} sgn(\sigma_\alpha)\ A_\alpha\ e_1 \wedge \ldots \wedge e_k \\
&= m!\ Pf(A)\ dV \quad \square
\end{aligned}$$

We can now finally introduce the geometric Euler class.

Definition 1.5 (Geometric Euler Class)**.** The *Geometric Euler Class* of $E \xrightarrow{\pi} M$ is

$$e_{\mathrm{G}}(E) := \begin{cases} \frac{Pf(F)}{(-2\pi)^{\frac{k}{2}}} & \text{if } 2 \mid k \\ 0 & \text{if } 2 \nmid k \end{cases} \in H^k(M)$$

Example 1.6. $TM \to (M^2, g)$ and let ω be the Levi-Civita connection. Then,

$$F = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}, \quad Pf(F) = F_{12}\ \theta_1 \wedge \theta_2 = -K\ dA, \quad e_{\mathrm{G}}(TM) = \frac{K}{2\pi}\ dA$$

Therefore, we can rewrite the LHS of (1) as

$$\int_M \frac{K}{2\pi}\ dA = \int_M e_{\mathrm{G}}(TM) \tag{2}$$

2 Topology

2.1 Poincare Duality

For a manifold N^{n+d} without boundary, consider the following pairing on forms

$$(\cdot\ ,\cdot) : \Omega^p(N) \times \Omega_c^{n+d-p}(N) \to \mathbb{R}, \quad (\alpha, \beta) := \int_N \alpha \wedge \beta$$

By Stokes Theorem, for β closed we have

$$(d\alpha, \beta) = \int_N d(\alpha \wedge \beta) + (-1)^p \int_N \alpha \wedge d\beta = \int_{\partial N = \emptyset} \alpha \wedge \beta$$

Thus, the pairing descends to the cohomology $(\cdot\ ,\cdot) : H^p(N) \times H_c^{n+d-p}(N) \to \mathbb{R}$. Using Hodge Theory, we have that this pairing is perfect, and hence have the following Isomorphism.

Definition 2.1 (Poincare Duality)**.**

$$H_c^{n+d-p}(N) \cong (H^p(N))^*$$

Now suppose we have a smooth cycle $\sigma : M^n \to N^{n+d}$. This gives a linear map

$$\Omega^n(N) \to \mathbb{R}, \quad \alpha \mapsto \int_M \sigma^*(\alpha)$$

Again, using Stokes Theorem, we have that

$$\int_M \sigma^*(d\alpha) = \int_M d\sigma^*(\alpha) = 0$$

and so it descends to an element of $(H^n(N))^*$. Let $\delta_\sigma \in H_c^d(N)$ denote a representation of its Poincare dual, so that

$$\int_N \alpha \wedge \delta_\sigma = \int_M \sigma^*(\alpha) \quad \forall\ \alpha \in \Omega^n(N) \text{ closed.}$$

Example 2.2. Set $N := M \times M$, $(d = n)$, and take $\sigma := \Delta : M \to M \times M$, the diagonal map. Then $\exists\ \delta_\Delta \in H^n_c(M \times M)$ such that

$$\int_{M\times M} \alpha \wedge \delta_\Delta = \int_M \Delta^*(\alpha) \quad \forall\ \alpha \in \Omega^n(M \times M) \text{ closed.}$$

Claim. Let e_i be a homogeneous basis of $H^\bullet(M)$, and f_i another basis which is dual to e_i under our pairing. Then,

$$\delta_\Delta = (-1)^{|e_i|} e_i \times f_i \in H^n_c(M \times M)$$

Proof. For any homogeneous $\alpha, \beta \in \Omega^\bullet(M)$ with $|\alpha| + |\beta| = n$, we have that

$$\int_{M\times M} (\alpha \times \beta) \wedge (-1)^{|e_i|}(e_i \times f_i) = (-1)^{|e_i|}(-1)^{|e_i||\beta|} \int_{M\times M} (\alpha \wedge e_i) \times (\beta \wedge f_i)$$

These terms are non-zero iff $|\beta| = |e_i|$ and $|\alpha| = |f_i|$. Hence

$$\int_{M\times M} (\alpha \times \beta) \wedge (-1)^{|e_i|}(e_i \times f_i) = \left(\int_M \alpha \wedge e_i\right)\left(\int_M \beta \wedge f_i\right) = (\alpha, e_i)(\beta, f_i)$$

Finally, we have

$$\int_{M\times M} (\alpha \times \beta) \wedge \delta_\Delta = \int_M \Delta^*(\alpha \times \beta) = \int_M \alpha \wedge \beta = (\alpha, e_i)(\beta, f_j) \int_M e_i \wedge f_j = (\alpha, e_i)(\beta, f_i) \quad \square$$

Example 2.3. Set $N := E$, $(d = k)$, and take $\sigma : M \to E$ to be any section. Then $\exists\ \delta_\sigma \in H^k_c(E)$ such that

$$\int_E \alpha \wedge \delta_\sigma = \int_M \sigma^*(\alpha) \quad \forall\ \alpha \in \Omega^n(E) \text{ closed.}$$

2.2 Topological Euler Class

Poincare Duality as in Example 2.3 provides the topological invariant for our bundle $E \xrightarrow{\pi} M$.

Definition 2.4 (Thom Class)**.** Let $s_0 : M \to E$ be the zero section. Then the *Thom class* of E is defined by

$$\tau_E := \delta_{s_0} \in H^k_c(E)$$

Definition 2.5 (Topological Euler Class)**.** The *Topological Euler Class* of E is the pullback of the Thom class of E via the zero section. That is,

$$e_{\mathrm{T}}(E) := s_0^*(\tau_E) \in H^k(M)$$

The Topological Euler Class has the following properties:

- Invariance: If $E \cong F$, then $\tau_E = \tau_F$
- Localisation: Let U be an open neighbourhood of M inside E. Then, there exists a representative $\tau^U_E \in \Omega^k_c(E)$ for the cohomology class τ_E with support in U.

Example 2.6. Fixing a metric on M, we have the orthogonal decomposition

$$TM \oplus TM|_\Delta \cong T(M \times M)|_\Delta \cong N\Delta \oplus T\Delta$$

where $N\Delta$ denotes the normal bundle to the diagonal. Since $TM \cong T\Delta$, we have that $TM \cong N\Delta$ and hence $\tau_{TM} = \tau_{N\Delta}$.

We can use Example 2.2 to give an alternative description of the Euler characteristic.

Theorem 2.7.

$$\chi(M) = \int_M e_T(TM) \tag{3}$$

Therefore we can rewrite the RHS of (1) as (3).

Proof. Since $(e_i, f_j) = (-1)^{|e_i||f_j|}(f_j, e_i)$, the dual basis to f_j is $(-1)^{|e_j||f_j|}e_j$ and hence, by the previous characterisation of δ_Δ, we have

$$\begin{aligned}\int_{M\times M} \delta_\Delta \wedge \delta_\Delta &= (-1)^{|e_i|+|e_j||f_j|+|f_j|} \int_{M\times M} (e_i \times f_i) \wedge (f_j \times e_j) \\ &= (-1)^{|e_i|}(-1)^{|e_j||f_j|+|f_j|+|f_i||f_j|}(e_i, f_j)(f_i, e_j) \\ &= (-1)^{|e_i|}(-1)^{|f_j|+|f_i||f_j|}\delta_{ij}^2 = (-1)^{|e_i|} = (-1)^k b_k = \chi(M)\end{aligned}$$

However, if U is a neighbourhood of Δ in $M \times M$ such that $\exp : U \to M \times M$ is an embedding, we have

$$\int_{M\times M} \delta_\Delta \wedge \delta_\Delta = \int_U \delta_\Delta^U \wedge \delta_\Delta^U = \int_\Delta \Delta^*(\tau_{N\Delta}) = \int_M s_0^*(\tau_{TM}) = \int_M e_{\mathrm{T}}(TM)$$

□

3 Gauss-Bonnet-Chern

Our original equation (1) has now become:

$$\int_M e_{\mathrm{G}}(TM) = \int_M \frac{K}{2\pi}\, dA = \chi(M) = \int_M e_{\mathrm{T}}(TM) \tag{4}$$

Theorem 3.1 (Gauss-Bonnet-Chern)**.** Let $E \xrightarrow{\pi} M$ be an oriented vector bundle over a closed and oriented manifold M. Then

$$e_{\mathrm{G}}(E) = e_{\mathrm{T}}(E) \tag{5}$$

Sketch Proof. We will need the following Theorem:

Theorem 3.2 (Thom Isomorphism Theorem)**.** The Thom map

$$H^\bullet(M) \to H_c^{\bullet+k}(E), \quad \omega \to \tau_E \wedge \pi^*(\omega)$$

is an isomorphism, with inverse

$$(-1)^{kn} \int_{E\setminus M} : H_c^{\bullet+k}(E) \to H^\bullet(M)$$

Suppose E has odd rank. Consider the isomorphism $\rho : E \to E$, $v \to -v$. This reverses orientation, so

$$\int_{E\setminus M} \rho^*(\tau_E) = -\int_{E\setminus M} \tau_E$$

By the Thom Isomorphism Theorem, $\rho^*(\tau_E) = -\tau_E$. But, $\rho \circ s_0 = s_0$, and so

$$e_{\mathrm{T}}(E) = (\rho \circ s_0)^*(\tau_E) = s_0^*(-\tau_E) = -e_{\mathrm{T}}(E)$$

Now suppose E has rank $2k$. It suffices to prove that there is $\omega \in \Omega_c^{2k}(E)$ such that

(i) $d\omega = 0$

(ii) $s_0^*(\omega) = e_{\mathrm{G}}(E)$

(iii) $\int_{E\setminus M} \omega = 1 \in \Omega^0(M)$

Since, by the Thom isomorphism theorem,

$$\tau_E = \tau_E \wedge \pi^*(\int_{E\setminus M} \omega) = \omega$$

and hence $e_{\mathrm{G}}(E) = s_0^*(\omega) = e_{\mathrm{T}}(E)$, as required.

Let S be the unit sphere bundle of E and $S \xrightarrow{\pi_S} M$. Pulling back, we have the bundle $\pi_S^*(E) \to S$, which has a natural $SO(2k)$-structure, and a canonical non-vanishing section $S \ni e \mapsto e \in E_{\pi(e)} = (\pi_S^*(E))_e$.

<u>Fact:</u> $\quad e_{\mathrm{G}}(\pi_S^*(E)) = 0 \in H^{2k}(S)$

So, there exists $\psi \in \Omega^{2k-1}(S)$ such that $d\psi = e_{\mathrm{G}}(\pi_S^*(E))$. In fact, we can choose ψ to be the *global angular form*. That is,

$$\int_{S\setminus M} \psi = -1 \in \Omega^0(M)$$

Under the identification $E \setminus M \cong (0,\infty) \times S$, $e \mapsto (|e|, \frac{e}{|e|})$, we define our form

$$\omega := -g'(r)dr \wedge \psi - g\pi^*(e_{\mathrm{G}}(E)) \tag{6}$$

where g is any cut-off function on $(0,\infty)$ with $g = -1$ below $\frac{1}{4}$ and $g = 0$ above $\frac{3}{4}$.

Remark. ω is well defined since $g' = 0$ near the zero section.

As $\pi = \pi_S$ on supp g', we have

$$d\omega = g'(r)dr \wedge d\psi - g'(r) \wedge \pi^*(e_{\mathrm{G}}(E)) = g'(r)dr \wedge [\pi_S^*(e_{\mathrm{G}}(E)) - \pi^*(e_{\mathrm{G}}(E))] = 0$$

So we have shown (i). (ii) follows since,

$$s_0^*(\omega) = -g(0)\ s_0^*(\pi^*(e_{\mathrm{G}}(E))) = e_{\mathrm{G}}(E)$$

Finally, (iii) follows since

$$\begin{aligned}\int_{E\setminus M}\omega &= -\int_{E\setminus M} g'(r)dr\wedge\psi - \int_{E\setminus M} g\pi^*(e_{\mathrm{G}}(E))\\ &= -\int_0^\infty g'(r)\int_{S\setminus M}\psi\\ &= (g(0)-g(1))\int_{S\setminus M}\psi = 1\end{aligned}$$

□