Condensed Maths Talk (Week 3)

Before we start: Fix κ a strong limit cardinal.

Aim

- Main aim is to give a proof of

$$\mathrm{Cond}_\kappa(\mathrm{An}) \cong P_\Sigma(\mathrm{ed}\mathbb{CH}_\kappa)$$

- On the way, we will see for $\mathbb{T} \subset \mathrm{Top}_\kappa$ that if suitably nice there is a canonical topology $\tau_{\kappa\text{-cond}}$ s.t.

$$\mathrm{HypShv}_{\kappa\text{-cond}}(\mathbb{T}) \cong \mathrm{Cond}_\kappa(\mathrm{An})$$

- I'll try to explain the role of Hyper in making this work.

Recap / Learn : Hypersheaves

- last time saw hypersheaves via hypercovers

Q Suppose I give you $\mathrm{Shv}_\tau(\mathcal{C})$ [just as a category]
Can you give me $\mathrm{HypShv}_\tau(\mathcal{C})$

A Yes!

Truncation Functors

Let $\tau_{\leq n}\mathcal{A}n \hookrightarrow \mathcal{A}n$ be the full subcategory of Anima on X s.t. $\pi_k(X) = 0$ for $k > n$.

e.g. ($\tau_{\leq -1}\mathcal{A}n = \{*, \emptyset\}$)

$\tau_{\leq 0}\mathcal{A}n$ = Sets

$\tau_{\leq 1}\mathcal{A}n$ = ((2,1)-Category of ordinary) groupoids

$\vdots$

Def^ $\tau_{\leq n}\mathcal{C}$ is the full subcategory of $\mathcal{C}$ on objects X s.t. $\forall Y \in \mathcal{C}$

$$\mathrm{map}_{\mathcal{C}}(Y, X) \in \tau_{\leq n}\mathcal{A}n$$

Such objects called n-truncated.

(If $n = -2$ same as being final)

Note For $\mathcal{C}$ presentable we can recover this via Lurie tensor product

$$\tau_{\leq n}\mathcal{C} \cong \mathcal{C} \otimes \tau_{\leq n}\mathcal{A}n$$

Also the inclusion $\tau_{\leq n}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint called truncation (which we also denote $\tau_{\leq n}$)

For Topoi:

Prop $\tau_{\leq n} : \mathcal{X} \to \mathcal{X}$ preserves finite products.

proof: Bootstrap from Anima

Works for A_n $\Rightarrow$ Works for Presheafs

$\Rightarrow$ Works for Left exact localisations thereof. □

We can define homotopy groups!

Choosing a basepoint $s : * \to S^n$ define for $X \in \mathcal{X}$

$$s^* : X^{S^n} \to X \in \mathcal{X}_{/X}$$

(recall $\mathcal{X}$ presentable and so cotensored over A_n)

define $\pi_n(X) := \tau_{\leq 0}\left(X^{S^n} \to X\right) \in \mathcal{X}_{/X}$

The constant map $S^n \to *$ determines

$$\begin{array}{ccc} X & \longrightarrow & X^{S^n} \\ & \searrow^{\|} \quad \swarrow & \\ & X & \end{array}$$

$\rightsquigarrow$ basepoint of $\pi_n(X)$

For morphisms $f : X \to Y$, consider f as an object of $\mathcal{X}_{/Y}$ and take homotopy groups there.

Def^ $f : X \to Y$ in $\mathcal{X}$ is n-connective if it is an effective epimorphism and

$$\pi_k(f) = *$$ for all $0 \leq k < n$

Now for Hypercompletion:

Def^ $\mathcal{X}^\wedge$ is the full subcategory of $\mathcal{X}$ spanned by "hypercomplete objects" i.e. those local wrt ∞-connective morphisms.

i.e. Z is hypercomplete if $\forall\, f : X \to Y$ ∞-connective

$$\mathrm{map}(Y, Z) \xrightarrow{\simeq} \mathrm{map}(X, Z)$$

Fact $\mathcal{X} \to \mathcal{X}^\wedge$ is a localisation and when $\mathcal{X} = \mathrm{Shv}_\tau(C)$

$\mathcal{X}^\wedge = \mathrm{HypShv}_\tau(C)$!!

Coverages, Quasi Sites....

i) A cover of an object c is a set of maps $\{c_i \to c\}$

ii) A coverage is a set of covers $\tau(c)$ for every object $c \in \mathcal{C}$ s.t.

For every cover $\{c_i \to c\}$ and morphism $b \xrightarrow{f} c$

$\exists$ cover $\{b_j \to b\}$ s.t. $\forall j$ $b_j \to b \to c$

factors thru $c_{i(j)}$.

- Small Category $\mathcal{C}$ + Coverage = quasi-site.
- For any cover $\{c_i \to c\}$ we can make a Sieve $S(\{c_i \to c\})$ as the full subcategory of $\mathcal{C}_{/c}$ on $x \to c$ s.t. $\exists i$

s.t. $x \to c$ factors through c_i

- Thus can define Sheaf wrt quasi-site.

(Can generate a Grothendieck Topology: notions of Sheaf will agree, have no fear!)

~~Def~~ Fix $(\mathcal{C}, \tau_C)$, (D, τ_D) quasi sites and $f: \mathcal{C} \to D$.

Def^n f is cts if $\forall$ covers $\{c_i \to c\}$, $\{f(c_i) \to f(c)\}$ is a cover

~~■~~

Def^ f has covering lifting property if $\forall \{P_i : d_i \to d\} \leq \tau_D$

and $f(c) \to d$

$\exists \{q_j : c_j \to c\}$ s.t. $\forall j \quad \exists i(j)$ s.t.

$$f(c_j) \longrightarrow f(c) \longrightarrow d$$
$$\hookrightarrow d_{i(j)} \nearrow$$

Lemma If f has covering lifting property then right Kan extension

$$f_* : P(C) \longrightarrow P(D)$$

restricts to functors

$$Shv_{\tau_C}(C) \longrightarrow Shv_{\tau_D}(D)$$

$$HypShv_{\tau_C}(C) \longrightarrow HypShv_{\tau_D}(D)$$

proof (sketch)

Pick cover $\chi = \{d_j \to d\} \leq \tau_D$

Idea is to show $L_{\tau_C}(f^* S(\chi) \to f^* h_d)$ is an $\simeq$ as then for all τ_C-Sheafs F:

$$\begin{aligned} map(h_d, f_* F) &\cong map(f^* h_d, F) \\ &\cong map(L_{\tau_C} f^* h_d, F) \\ &\cong map(L_{\tau_C} f^* S(\chi), F) \\ &\cong map(f^* S(\chi), F) \\ &\cong map(S(\chi), f_* F) \end{aligned}$$

so $f_* F$ is a τ_D sheaf

Colimits universal + density means it suffices to show $\forall c$

$$f^* S(X) \underset{f^* h_d}{\times} h_c \longrightarrow h_c$$

is a equivalence after applying L_{τ_c}.

Idea Covering lifting property gives $Y = \{c_i \to c\} \subseteq \tau_c$ s.t.

$S(Y) \longrightarrow h_c$ factors thru $f^* S(X) \underset{f^* h_d}{\times} h_c$

Then observe the maps are mono so

$$S(Y) \subseteq f^* S(X) \underset{f^* h_d}{\times} h_c \subseteq h_c$$

and as $Y \in \tau_c$,

$$L_{\tau(c)} S(Y) \xrightarrow{\simeq} L_{\tau(c)} h_c$$

Thus our map is an L_{τ_c} equivalence as claimed.

Cor f is a cts functor of quasi-Sites which has covering lifting property and f is fully faithful then

$$Shv_{\tau_c}(C) \longrightarrow Shv_{\tau_D}(D)$$

is fully faithful

if further $i^*: Shv_{\tau_D}(D) \longrightarrow Shv_{\tau_c}(C)$ is conservative

$$Shv_{\tau_c}(C) \cong Shv_{\tau_D}(D)$$

(Hypersheaf statements similar)

Main Hammer we will Apply:

Cor If f is cts, has covering lifting property and is fully faithful and further $\forall d \in D$ $\exists$ cover $\{f(C_i) \longrightarrow d\}$

Then right Kan extension restricts to an $\simeq$

$$HypShv_{\tau_c}(C) \cong Hyp\,Shv_{\tau_D}(D)$$

$\longrightarrow$ Need the Hyper for the proof!

proof (Idea)

Suffices to show

$$\mathrm{Hyp}\mathrm{Shv}_{\tau_D}(D) \hookrightarrow \mathrm{Shv}_{\tau_D}(D) \xrightarrow{f^*} \mathrm{Shv}_{\tau_C}(C) \longrightarrow \mathrm{Hyp}\mathrm{Shv}_{\tau_C}(D)$$

is conservative.

i.e. if $\phi : A \to B \in \mathrm{Shv}_{\tau_C}(C)$ is s.t.

$A \circ f \to B \circ f$ is ∞-connective

then ϕ was ∞-connective

- As f^* is a left exact left adjoint f^* commutes w/ homotopy groups
- Therefore reduced to showing f^* is conservative when restricted to $\tau_{\leq 0}\mathrm{Shv}_{\tau_D}(D) \to \tau_{\leq 0}\mathrm{Shv}_{\tau_C}(C)$

- Now can use 1-categorical tricks

  e.g. replacing a limit over Δ with a limit over $\Delta_{\leq 1}$

  ~~i.e.~~ ~> simple equalisers.

Back to Condensed Maths:

For def^n of promised $\tau_{k\text{-cond}}$:

Set $\mathbb{T} \subset \mathrm{Top}$ a small full subcategory. Denote by $(-)_k : \mathrm{Top} \to \mathrm{Cond}_k(\mathrm{An})$ the map in previous talk!

A collection of maps $\{T^i \to T\} \subseteq \mathbb{T}$ is a k-condensed cover if

$$\coprod_i \underline{T^i}_k \longrightarrow \underline{T}_k$$

is an epimorphism in $\mathrm{Cond}_k(\mathrm{Set})$

Fact $\underline{X}_k$ for $X \in \mathrm{edCH}_k$ generate $\mathrm{Cond}_k(\mathrm{Set})$ under colimits

proof Take $Z \in \mathrm{Pro}(\mathrm{fin})_k$. Suffices to show $\underline{Z}$ can be written as a colimit in $\underline{X_i}_k$, $X_i \in \mathrm{edCH}_k$

$\exists\, Y \in \mathrm{edCH}_k$ with surjection $Y \twoheadrightarrow Z$

Then by def^n of topology on $\mathrm{Cond}_k(\mathrm{Set})$

$$\mathrm{Coeq}\left(\underline{Y \underset{Z}{\times} Y} \rightrightarrows \underline{Y} \right) \cong \underline{Z}$$

But (often) $Y \underset{Z}{\times} Y$ need not be in edCH_k.

choose $X \twoheadrightarrow Y \underset{Z}{\times} Y$, X edCH_k.

$$\mathrm{Coeq}\left(\underline{X} \rightrightarrows \underline{Y} \right) \cong \mathrm{Coeq}\left(\underline{Y \underset{Z}{\times} Y} \rightrightarrows \underline{Y} \right) \cong \underline{Z}$$

□

Main Point-Set Input:

Prop A family $\{X^i \to X\}$ is a k-condensed cover if and only if $\forall K \to X$ in $\overline{\text{Top}}$ with $K \in \text{edCH}_k$ $\exists$ finite family $\{K^j \to K\}$ s.t. $\coprod K^j \to K$ is surjective and $\forall j$ $\exists i(j)$ s.t.

$$K^j \to K \to X$$

factors as $K^j \to X^{i(j)} \to X$

proof:

First show: Suppose $K \in \text{edCH}_k$. A map $\coprod X^i \to \underline{K}_k$ in $\text{Cond}_k(\text{Set})$ is an epimorphism if and only if $\exists$ finite family $\{K^j \to K\} \subseteq \text{edCH}_k$ s.t. $\coprod K^j \to K$ is surjective and $\forall j \in J$ $\exists i(j) \in I$ s.t. $\underline{K^j}_k \to \underline{K}$ factors thru $X^{i(j)}$

if is obvs.

Conversely if $\coprod X^i \to \underline{K}_k$ is an epimorphism then

$$\coprod_i \coprod_{Y \in \text{edCH}_k/X^i} \underline{Y}_k \to \coprod_i X^i \to \underline{K}_k$$

is an epimorphism. As condensed topology on Profin_k is finitary $\exists$ finite $F \subseteq I$ and $P_f \subseteq \text{edCH}_k/X^i$

$\underline{K^i}_k \longrightarrow \underline{K}$ factors thru $\underline{X^{i(i)} \underset{X}{\times} K}$ ✓

$K^i \longrightarrow K$ factors thru $X^{i(i)} \underset{X}{\times} K$ proving the result.

Lemma Suppose $\mathbb{T} \subset \mathrm{Top}$ contains edCH_k.
Then the k-condensed covers form a coverage.

proof

Take $\{T^i \to T\}$ a cover and $S \to T$ a cts map.

Then

$$\coprod \underline{T^i \underset{T}{\times} S}_k = \coprod \underline{T^i}_k \underset{\underline{T}_k}{\times} \underline{S}_k \longrightarrow \underline{S}_k$$

is an epimorphism.

If $T^i \underset{T}{\times} S \in \mathrm{edCH}_k$ we are done as

$$T^i \underset{T}{\times} S \longrightarrow S \longrightarrow T \qquad \text{~~factors thru } T^i\text{~~}$$
$$\searrow T^i \nearrow \qquad \text{commutes}$$

But $T^i \underset{T}{\times} S$ might not be edCH_k

So cover via

$$\coprod_{X \in \mathrm{edCH}_k / T^i \underset{T}{\times} S} \underline{X}_k \longrightarrow \underline{T^i \underset{T}{\times} S}_k$$

□

$$\coprod_f \coprod_{Y \in P_f} \underline{Y}_k \longrightarrow \underline{K}_k \quad \text{is an epimorphism}$$

Then take $\coprod_f \coprod_{Y \in P_f} Y \longrightarrow K$ and we are done. □

Now for main proof:

in $\mathrm{Cond}_k(\mathrm{Set})$: - Colimits universal
- epimorphisms effective

+ edCH_k generates $\mathrm{Cond}_k(\mathrm{Set})$ under colimits

Thus $A \longrightarrow B \in \mathrm{Cond}_k(\mathrm{Set})$ is an epimorphism if and only if $\forall\, \underline{K}_k \longrightarrow B$ for $K \in \mathrm{edCH}_k$

$$A \underset{B}{\times} \underline{K}_k \longrightarrow \underline{K}_k$$

is an epimorphism

Given family $\{X^i \longrightarrow X\} \subseteq \mathrm{Top}$ is k-condensed cover if and only if $\forall K \in \mathrm{edCH}_k$ and $\underline{K}_k \longrightarrow \underline{X}_k$

$$\coprod_i X^i \underset{X}{\times} K \underset{k}{} = \coprod_i \underline{X}^i_k \underset{\underline{X}_k}{\times} \underline{K}_k \longrightarrow \underline{K}_k$$

is an epimorphism

Then apply what we just showed: this is epi if and only if $\exists$ finite family $\{K^j \longrightarrow K\} \in \mathrm{edCH}_k$ s.t.

$\coprod_j K^j \longrightarrow K$ surjective and $\forall j \; \exists\, i(j)$ s.t. $\underline{K}^j \longrightarrow \underline{K}$ factors thru $\underline{X}^{i(j)} \underset{X}{\times} \underline{K}$

Proof of Main Aim:

Step 1 # If $\mathbb{T}$ contcins $Profin_k$

$$HypShv_{k\text{-}cond}(\mathbb{T}) \cong Cond_k(A_n)$$

we have map of quasi-sites:

$$(Profin_k, \tau_{cond}) \longleftrightarrow (\overline{\mathbb{T}}, \tau_{k\text{-}cond})$$

τ_{cond}: original topology we asked for

Idea: hit with Ou Hennes.

- Lemma 2.3.4 is covering lifting property
- $\{X \to T\}_{X \to T \in Profin_{/T}} \leq \overline{\mathbb{T}}$

is a k-condensed cover for ay $T \in \overline{\mathbb{T}}$

Step 2 If $\mathbb{T}$ contcins $edCH_k$ then

$$HypShv_{k\text{-}cond}(edCH_k) \cong HypShv_{k\text{-}cond}(\mathbb{T})$$

Aagain

$$(edCH_k, \tau_{k\text{-}cond}) \longleftrightarrow (\overline{\mathbb{T}}, \tau_{k\text{-}cond})$$

and hit with hennes using

- Lemma 2.3.4 for covering lifting property
- $\{X \to T\}_{X \to T \in edCH_{/T}} \leq \overline{\mathbb{T}}$

Ou Cone

Step 3

$$\mathrm{Hyp}\,\mathrm{Shv}_{k\text{-cond}}(\mathrm{edCH}_k) \cong P_\Sigma(\mathrm{edCH}_k)$$

Consider the covering $\{X^i \to X^0 \amalg X^1\}_{i=0,1}$

looking at Čech Nerve sheaf condition

$$F(X^0 \amalg X^1) \cong \lim\Big(\prod_i FX^i \rightrightarrows \prod_i FX^i \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \Big)$$

$$\cong \prod FX^i \cong FX^0 \times FX^1$$

so $$\mathrm{Shv}_{k\text{-cond}}(\mathrm{edCH}_k) \subseteq P_\Sigma(\mathrm{edCH}_k)$$

Hard part is converse

Pick any cover $\{X^i \to X\}$ then $\coprod \underline{X}^i \to \underline{X}$ is an epimorphism.

$\exists$ finite subset $\hat{\coprod}_{i=1} \underline{X}^i \to \underline{X}$ is an epimorphism

Now $\hat{\coprod}_{i=1} X^i \to X$ is thus surjective so admits a section.

This implies the sheaf condition collapses (HTT Lemma 6.1.3.16)

so

$$P_\Sigma(\mathrm{edCH}_k) \cong \mathrm{Shv}_{k\text{-cond}}(\mathrm{edCH}_k)$$

Then as $\tau_{\leq n}$ preserve finite products

$$P_\Sigma(\mathrm{edCH}_k) \to \tau_{\leq n} P_\Sigma(\mathrm{edCH}_k)$$

has explicit description as

$$\left(\mathrm{edCH}_k \xrightarrow{F} A_n\right) \rightsquigarrow \left(\mathrm{edCH}_k \xrightarrow{F} A_n \xrightarrow{\tau_{\le n}} \tau_{\le n} A_n\right)$$

Thus $P_\Sigma(\mathrm{edCH}_k)$ is hypercomplete as A_n is.

Thus $P_\Sigma(\mathrm{edCH}_k) \simeq \mathrm{HypShv}_{k\text{-Cond}}(\mathrm{edCH}_k)$

$\simeq \mathrm{Cond}_k(A_n)$ (by steps 1 and 2)

□