Condensed objects without cardinals

All cardinals assumed small.

So far: κ uncountable cardinal $\rightsquigarrow$ $\mathrm{Pro}(\mathrm{Fin})_\kappa \subseteq \mathrm{Pro}(\mathrm{Fin})$

($\mathrm{Pro}(\mathrm{Fin}) \longrightarrow \mathrm{Top}$ is fully faithful with essential image the totally disconnected compact Hausdorff spaces)

κ-light profinite sets, i.e. those which admit a basis of open subsets of cardinality $< \kappa$ when considered as a space $\Leftrightarrow$ κ-cocompact objects of $\mathrm{Pro}(\mathrm{Fin})$

τ_{cond} Grothendieck topology on $\mathrm{Pro}(\mathrm{Fin})_\kappa$ generated by jointly surjective covering families

$$\mathrm{Cond}_\kappa(\mathrm{An}) := \mathcal{Sh}^\wedge_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\kappa; \mathrm{An}) \qquad \kappa\text{-condensed anima}$$

and for $\mathcal{C}$ presentable:

$$\mathrm{Cond}_\kappa(\mathcal{C}) := \mathrm{Cond}_\kappa(\mathrm{An}) \otimes_{\mathcal{P}\mathrm{L}} \mathcal{C} \qquad \kappa\text{-condensed objects in } \mathcal{C}$$

$$\simeq \mathcal{Sh}^\wedge_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\kappa; \mathcal{C}) \simeq \mathrm{Fun}^{\lim}(\mathrm{Cond}_\kappa(\mathrm{An})^{\mathrm{op}}, \mathcal{C})$$

Now remove dependency on κ:

τ_{cond} Grothendieck topology on $\mathrm{Pro}\mathrm{Fin}$ generated by jointly surjective covering families.

$$\mathrm{Cond}(\mathcal{C}) := \mathcal{Sh}^{\wedge,\mathrm{acc}}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C}) \qquad \text{condensed objects in } \mathcal{C}$$

$$:= \mathcal{Sh}^\wedge_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C}) \cap \mathrm{Fun}^{\mathrm{acc}}(\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}, \mathcal{C})$$

accessible functors, i.e. functors which preserve κ-filtered colimits for some regular cardinal κ

Will use that $(\mathrm{Pro}(\mathrm{Fin}), \text{surjections})$ is an explicit covering site which is κ-hyperaccessible for every uncountable regular cardinal κ:

$\mathrm{Pro}(\mathrm{Fin})$ is κ-coaccessible, and hypercoverings are κ-cofiltered limits of hypercoverings in κ-cocompact objects.

$\rightsquigarrow$ $F: \mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}} \longrightarrow \mathcal{C}$ is a condensed object in $\mathcal{C}$ iff

- F is accessible,
- F preserves finite products,
- for every condensed hypercover $U_\bullet: \Delta^{\mathrm{op}}_{s,+} \longrightarrow \mathrm{Pro}(\mathrm{Fin})$, $F(U_{-1}) \simeq \lim_{\Delta_s} F(U_\bullet)$.

for $\mathcal{C} = \mathrm{Set}$, get an equalizer diagram

$\forall$ regular κ $\exists$ regular $\lambda \geq \kappa$ st. $(\mathcal{C}, S)$ λ-hyperaccessible

$\downarrow$

For $(\mathcal{C}, S)$ a hyperaccessible covering site with S subcanonical:

$\mathcal{Sh}^{\wedge,\mathrm{acc}}_S(\mathcal{C})$ is accessible $\Leftrightarrow$ $\mathcal{C}$ essentially small.

$\mathrm{Pro}(\mathrm{Fin})$ is not essentially small, so $\mathrm{Cond}(\mathrm{An})$ is not accessible and in particular not an ∞-topos. But all of Giraud's axioms except accessibility are satisfied:

Lem $\mathrm{Cond}(\mathrm{An})$ is locally small and satisfies:
(i) Small colimits and small limits exist.
(ii) Small colimits are universal.
(iii) Small coproducts are disjoint.
(iv) Every groupoid object of $\mathrm{Cond}(\mathrm{An})$ is effective.

Spectrum objects in $\mathrm{Cond}(\mathrm{An})$ still behave like the stabilisation of a presentable category:

- $\mathrm{Sp}(\mathrm{Cond}(\mathrm{An})) \simeq \mathrm{Cond}(\mathrm{Sp})$ is stable

reduced excisive functors $\mathrm{An}^{\mathrm{fin}}_* \longrightarrow \mathrm{Cond}(\mathrm{An})$

- $\Omega^\infty: \mathrm{Cond}(\mathrm{Sp}) \longrightarrow \mathrm{Cond}(\mathrm{An})$ has a left adjoint Σ^∞_+ which factors through $\mathrm{CMon}^{\mathrm{gp}}(\mathrm{Cond}(\mathrm{An}))$
- The ess. image of $\mathrm{CMon}^{\mathrm{gp}}(\mathrm{Cond}(\mathrm{An})) \longrightarrow \mathrm{Cond}(\mathrm{Sp})$ is the connective part of a t-structure with heart $\mathrm{Cond}(\mathrm{Ab})$
- Σ^∞_+ enhances to a sym. mon. functor
- The sym. mon. structure on $\mathrm{Cond}(\mathrm{Sp})$ is compatible with the t-structure

Condensed and κ-condensed categories

For $\mathcal{C}$ presentable, want to exhaust $\mathrm{Cond}(\mathcal{C})$ by $\mathrm{Cond}_\kappa(\mathcal{C})$ for κ ranging over a family of uncountable cardinals.

For such κ, $\mathrm{Pro}(\mathrm{Fin})_\kappa \hookrightarrow \mathrm{Pro}(\mathrm{Fin})$ induces

$$r^\kappa : \mathrm{Cond}(\mathcal{C}) = \mathrm{Sh}^{\wedge,\mathrm{acc}}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C}) \longrightarrow \mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\kappa; \mathcal{C}) \simeq \mathrm{Cond}_\kappa(\mathcal{C}).$$

Lem This functor has a left adjoint.

Proof idea: Choose a regular cardinal μ s.t. $\mathcal{C}$ μ-accessible, choose reg. card. λ s.t. $\kappa \le \lambda$, $\mu \ll \lambda$ (i.e. $\forall$ cardinals $\mu_0 < \mu$ and $\lambda_0 < \lambda$, $\lambda_0^{\mu_0} < \lambda$).
Then r^κ factors as

$$\mathrm{Cond}(\mathcal{C}) \xrightarrow{r^\lambda} \mathrm{Cond}_\lambda(\mathcal{C}) \xrightarrow{r^\lambda_\kappa} \mathrm{Cond}_\kappa(\mathcal{C}).$$

r^λ: since $\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}$ is compactly generated, λ-bypresentable and Δ-indexed limits commute with λ-filtered colimits in $\mathcal{C}$, $\mathrm{Fun}^{\mathrm{acc}}(\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}_\lambda, \mathcal{C})$ has a fully faithful left adjoint and the adjunction descends to hypersheaves.

r^λ_κ: has a left adjoint given by left Kan extension followed by sheafification.

□

Λ poset of uncountable cardinals, $\Lambda^\triangleright := \Lambda \cup \{\infty\}$. Restrictions give a functor $(\Lambda^\triangleright)^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$, $\lambda \mapsto \begin{cases} \mathrm{Cond}_\lambda(\mathcal{C}), & \lambda \neq \infty \\ \mathrm{Cond}(\mathcal{C}), & \lambda = \infty \end{cases}$,
and it factors through $\widehat{\mathrm{Cat}}^R_\infty$.

Thm The opposite diagram $\mathrm{Cond}_*(\mathcal{C}) : \Lambda^\triangleright \longrightarrow \widehat{\mathrm{Cat}}^L_\infty \longrightarrow \widehat{\mathrm{Cat}}_\infty$ is a colimit diagram.

Proof idea: Choose reg. cardinal μ st. $\mathcal{C}$ μ-accessible. Then the uncountable regular cardinals $\lambda \ge \mu$ form a cofinal subcategory of Λ.

$$\begin{aligned} &\operatorname{colim}_{\lambda \in \Lambda} \mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\lambda; \mathcal{C}) \\ \simeq\ &\operatorname{colim}_{\lambda \in \Lambda_{\ge\mu}} \mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\lambda; \mathcal{C}) \xrightarrow{(*)} \mathrm{Sh}^{\wedge,\mathrm{acc}}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C}) \end{aligned}$$

$(*)$: filtered colimit of the maps

$$(i_\lambda, r^\lambda) : \mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\lambda; \mathcal{C}) \rightleftarrows \mathrm{Sh}^{\wedge,\mathrm{acc}}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C})$$

for which i_λ factors through the equivalence

$$\mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin})_\lambda; \mathcal{C}) \simeq \mathrm{Fun}^{\lambda\text{-colim}}(\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}}, \mathcal{C}) \cap \mathrm{Sh}^{\wedge}_{\mathrm{cond}}(\mathrm{Pro}(\mathrm{Fin}); \mathcal{C}),$$

in particular i_λ and $(*)$ are fully faithful.

The same identification shows essential surjectivity. □

Can also replace Λ by regular or strong limit cardinals.
Λ^r Λ^s

In particular, $\mathrm{Cond}_\kappa(\mathcal{C}) \longrightarrow \mathrm{Cond}(\mathcal{C})$ is fully faithful with essential image the condensed hypersheaves $\mathrm{Pro}(\mathrm{Fin})^{\mathrm{op}} \longrightarrow \mathcal{C}$ preserving κ-filtered colimits. This full subcategory is closed under μ-small limits and small colimits.

⇝ "exhaustion by topoi" of $\mathrm{Cond}(\mathcal{C})$ when restricting to $\Lambda^{r/s}$ (fully faithful left exact left adjoints between topoi)

Condensed sets and topological spaces

T_1 : for each pair of points $x \neq y$, both have a neighbourhood not containing the other

T_2 = Hausdorff : distinct points have disjoint neighbourhoods

Prop Suppose X is a $T1$ space. If $\kappa > |X|$ is an uncountable regular cardinal, then $\mathcal{C}(-, X) : \mathrm{Ro}(\mathrm{Fin})^{op} \longrightarrow \mathrm{Set}$ is κ-accessible.
In particular, $\underline{X} := \mathcal{C}(-, X) \in \mathrm{Cond}(\mathrm{Set})$.

(It already is a condensed hypersheaf for every space X since continuous surjections between totally disconnected compact Hausdorff spaces are quotient maps.)

Cor $T1\,\mathrm{Top}_k$ category of $T1$ spaces with k-continuous maps.
Then $(\underline{-}) : T1\,\mathrm{Top}_k \longrightarrow \mathrm{Cond}(\mathrm{Set})$ is fully faithful.

($X \longrightarrow Y$ is k-cont. if for all compact Hausdorff spaces K and all continuous maps $K \longrightarrow X$, the composition $K \longrightarrow X \longrightarrow Y$ is continuous)

Proof: Choose an uncountable regular cardinal $\kappa > |X| + |Y|$. Then $\underline{X}, \underline{Y}$ are κ-accessible and

$$\mathrm{Map}_{\mathrm{Cond}(\mathrm{Set})}(\underline{X}, \underline{Y}) \simeq \mathrm{Map}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(\underline{X}, \underline{Y}),$$

and $\mathrm{Map}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(\underline{X}, \underline{Y}) \simeq \mathcal{C}_\kappa(X, Y)$.

($(\underline{-})_\kappa : \mathrm{Top}^{\kappa\text{-c}} \longrightarrow \mathrm{Cond}_\kappa(\mathrm{Set})$, $X \longmapsto \mathcal{C}(-, X)$ is fully faithful.)

($\mathrm{Top}^{\kappa\text{-c}}$: top. spaces and κ-continuous maps, i.e. maps $X \to Y$ for which for all $K \to X$ continuous and K κ-light, $K \to X \to Y$ is continuous)

Same holds for all regular cardinals $\lambda \geq \kappa$, so $\mathcal{C}_k(X, Y) = \mathcal{C}_\kappa(X, Y)$. □

Rem Can define "quasi-compact" and "T1" internal to $\mathrm{Cond}(\mathrm{Set})$.
Then for K compact Hausdorff, $\underline{K}$ is quasi-compact, and
$T1\,\mathrm{Top}_k \hookrightarrow \mathrm{Cond}(\mathrm{Set})$ factors over $T1\,\mathrm{Cond}(\mathrm{Set}) \subseteq \mathrm{Cond}(\mathrm{Set})$,
the induced map $T1\,\mathrm{Top}_k \longrightarrow T1\,\mathrm{Cond}(\mathrm{Set})$ admits a left adjoint which lands in compactly generated $T1$-spaces.