

Moduli spaces from the Kuznetsov component of cubic fivefolds

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Notes for Tuesday's contributed talk at BMC2026, geometry section.

1 Hyper-Kähler & Lagrangian

One of the main reasons behind studying these moduli spaces is their connection with hyper-Kähler geometry. Since this is a geometry seminar, everything is over \mathbb{C} .

Definition 1.1. A compact Kähler (projective) manifold X is called (projective) **hyper-Kähler** if X is simply-connected, and $H^0(X, \Omega_X^2) = \mathbb{C}\omega$, where ω is a everywhere non-degenerate 2-form.

Definition 1.2. Let X be a projective hyper-Kähler manifold. A subvariety $Y \subseteq X$ (not necessarily smooth) is called **Lagrangian** if $\dim Y = \frac{1}{2} \dim X$, and $\omega|_Y = 0$.

Example 1.3 (Hyper-Kähler manifolds). 1. The only hyper-Kähler surfaces are K3 surfaces. Let S be a K3 surface. Any curve $C \subseteq S$ is Lagrangian.

2. For $n \in \mathbb{Z}_{>0}$, the Hilbert scheme of n points on a K3 surface $S^{[n]}$ is a hyper-Kähler manifold of dimension $2n$. Note that $\text{Sym}^n C \subseteq S^{[n]}$ is a Lagrangian submanifold.
3. The deformation type of $S^{[n]}$ is known as the $\text{K3}^{[n]}$ -type hyper-Kähler manifolds. In the same family, we have the moduli spaces $M_H(S, v)$ of μ_H -stable sheaves on S with (primitive) Mukai vector v .
4. Other known deformation families include Kum_n -type, OG6, and OG10.
5. Recently, new families of singular hyper-Kähler varieties have been constructed. For instance, [LLX25] constructs a 42-dimensional example by compactifying the Jacobian fibration associated with cubic 5-folds.

2 Kuznetsov components

Definition 2.1. Let $X \subseteq \mathbb{P}^{n+1}$ be a cubic hypersurface. The bounded derived category of coherent sheaves $D^b(X)$ admits a semi-orthogonal decomposition:

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \dots, \mathcal{O}_X(n-2) \rangle,$$

where $\mathcal{K}u(X) := \{E \in D^b(X) \mid \text{Ext}^\bullet(\mathcal{O}_X(k), E) = 0 \text{ for } k = 0, \dots, n-2\}$ is called the **Kuznetsov component** of X . It is expected to encode the interesting geometric information of X . For example, in a lot of cases $\mathcal{K}u(X)$ is able to determine the variety X up to isomorphism.

Example 2.2. For $n = 4$, $\mathcal{K}u(X)$, the Kuznetsov component of a cubic 4-fold, is a **K3 category** in the sense that its Serre functor is given by $S_{\mathcal{K}u(Y)} \cong [2]$ and its Hochschild homology $\text{HH}_\bullet(\mathcal{K}u(Y)) \cong \text{HH}_\bullet(S)$ for a K3 surface S .

To give a “non-commutative version” of the example $M_H(S, v)$, we need the notion of stability conditions on derived/triangulated categories. For this talk we skip the formal definition.

Slogan:

$$\begin{array}{ccc} \text{Gieseker stability of coherent sheaves} & \xrightarrow{\text{generalise}} & \text{Bridgeland stability of complexes} \\ \text{on varieties} & & \text{on triangulated categories} \\ E \in \text{Coh}(X) & & E \in \text{D}^b(X) \text{ or } \mathcal{K}u(X) \end{array}$$

For a long time, it had been a difficult problem to construct stability conditions, but...

Theorem 2.3 ([Li26]).

Let X be a smooth projective variety. Then $\text{D}^b(X)$ admits a Bridgeland stability condition.

For Kuznetsov components, the general existence is still open, but we have the following results for cubic hypersurfaces:

Theorem 2.4. *Let X be a smooth cubic n -fold. Then $\mathcal{K}u(X)$ admits a Bridgeland stability condition for the following cases:*

1. $n = 3$: Barnardara–Macrì–Mehrotra–Stellari [BMMS12];
2. $n = 4$: Bayer–Lahoz–Macrì–Stellari [BLMS23];
3. $n = 5$: L. [Liu25].

With a stability condition σ on $\mathcal{K}u(X)$, we can count the σ -semistable objects, and form a moduli space $M_\sigma(\mathcal{K}u(X), v)$ of σ -semistable objects in $\mathcal{K}u(X)$ with a fixed numerical class $v \in K_{\text{num}}(\mathcal{K}u(X))$. For X a smooth cubic 4-fold, this provides hyper-Kähler manifolds:

Theorem 2.5 ([BLMNPS21]).

Let X be a smooth cubic 4-fold. For a numerical character $v \in K_{\text{num}}(\mathcal{K}u(X))$ and a “generic” stability condition $\sigma \in \text{Stab}(\mathcal{K}u(X))$, the moduli space $M_\sigma(\mathcal{K}u(X), v)$ is a smooth projective hyper-Kähler manifold of $\text{K3}^{[n]}$ -type with dimension equal to $2 - \chi(v, v)$.

For low-dimensional moduli spaces, we can give them geometric descriptions.

- Example 2.6.**
1. The 4-dimensional moduli space $M_\sigma(\mathcal{K}u(X), \lambda_1)$ is the Fano variety $\mathcal{F}_1(X)$ of lines on X .
 2. The 8-dimensional moduli space $M_\sigma(\mathcal{K}u(X), \lambda_1 + \lambda_2)$ is called the LLSvS 8-fold, discovered by [LLSvS17], which parametrises twisted cubic curves on X .

Note that the symplectic 2-form ω on $M_\sigma(\mathcal{K}u(X), v)$ is pointwise given by the Yoneda pairing:

$$\begin{array}{ccc} \omega : \text{Ext}_X^1(E, E) \times \text{Ext}_X^1(E, E) & \longrightarrow & \text{Ext}_X^2(E, E) \xrightarrow{\sim} \mathbb{C} \\ (E \xrightarrow{\alpha} E[1], E \xrightarrow{\beta} E[1]) & \longmapsto & (E \xrightarrow{\beta[1] \circ \alpha} E[2]) \end{array}$$

3 Cubic 5-folds

Let Y be a smooth cubic 5-fold. The Kuznetsov component $\mathcal{K}u(Y)$ is a fractional Calabi–Yau 7/3-category. It is not deformation equivalent to a geometric category. Nonetheless, we will see that $\mathcal{K}u(Y)$ plays the rôle of the curve C in the pair $C \hookrightarrow S$, so in a very vague sense we can think of $\mathcal{K}u(Y)$ as a “non-commutative curve”.

We describe the moduli spaces of stable objects in $\mathcal{K}u(Y)$.

Theorem 3.1 ([Liu25]).

Let Y be a smooth cubic 5-fold. For a non-zero numerical character $v \in K_{\text{num}}(\mathcal{K}u(Y))$, the moduli space $M_{\sigma_Y}(\mathcal{K}u(Y), v)$ is non-empty; if v is primitive, then $M_{\sigma_Y}(\mathcal{K}u(Y), v)$ contains a non-empty smooth locus $\mathcal{U}_Y(v)$ of dimension $1 - \chi(v, v)$.

Example 3.2. The 2-dimensional moduli space $M_{\sigma_Y}(\mathcal{K}u(Y), \kappa_1)$ has a connected component isomorphic to the Fano surface $\mathcal{F}_2(Y)$ of 2-planes on Y . Let $i : X \hookrightarrow Y$ be an inclusion of a very general cubic 4-fold as a hyperplane section. [IM08] observes a map from $\mathcal{F}_2(Y)$ to $\mathcal{F}_1(X)$:

$$\begin{aligned} r : \mathcal{F}_2(Y) &\longrightarrow \mathcal{F}_1(X) \\ [\Pi] &\longmapsto [X \cap \Pi] \end{aligned}$$

where $\Pi \subseteq Y$ is a 2-plane and $\ell := X \cap \Pi$ is a line in X . The map r is generically injective and unramified; its image is a (singular) Lagrangian surface in the hyper-Kähler 4-fold $\mathcal{F}_1(X)$.

In the language of Bridgeland moduli spaces, we have a map $M_{\sigma_Y}(\mathcal{K}u(Y), \kappa_1) \rightarrow M_{\sigma_X}(\mathcal{K}u(X), \lambda_1)$ given by the derived pull-back $E \mapsto i^*E$. It is easy to check that the image is Lagrangian in this language, because the pull-back $i^*\omega$ of the symplectic 2-form vanishes everywhere on $M_{\sigma_Y}(\mathcal{K}u(Y), \kappa_1)$ since we have that $\text{Ext}^2(E, E) = 0$ for all $E \in M_{\sigma_Y}(\mathcal{K}u(Y), \kappa_1)$.

In higher dimensions, we have the following result:

Theorem 3.3 ([Liu25]).

For primitive $v \in K_{\text{num}}(\mathcal{K}u(Y))$, the moduli space $M_{\sigma_Y}(\mathcal{K}u(Y), v)$ contains a smooth open subset $\mathcal{U}_Y(v)$ such that there exists a rational map $r_v : \overline{\mathcal{U}_Y(v)} \dashrightarrow M_{\sigma_X}(\mathcal{K}u(X), i^*v)$ induced by the pull-back $i^* : \mathcal{K}u(Y) \rightarrow \mathcal{K}u(X)$, whose image is a Lagrangian subvariety of the hyper-Kähler variety $M_{\sigma_X}(\mathcal{K}u(X), i^*v)$. Moreover, $r_v|_{\mathcal{U}_Y(v)}$ is unramified and hence r_v is generically finite onto its image.

This extends the result of [FGLZ25] for Gushel–Mukai 3-folds and 4-folds, and that of [LLPZ26] for cubic 3-folds and 4-folds. Note that unlike these cases, we do not expect the Lagrangian subvarieties in our examples to be smooth.

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