Week 4 Notes

Warning: These are unofficial notes loosely related to the support class or the module in general. The contents are not necessarily part of the lectures and may not be examinable. Please use them at your own discretion.

In this course, all rings are assumed to be commutative rings with multiplicative identity 1.

3.1 Localisations and local rings

Lemma 3.1

Let R be a ring and $I \subseteq R$. Then I is the unique maximal ideal of R if and only if I is the set of all non-units of R.

Proof. " $\Longleftarrow$ ": Note that an ideal $I \triangleleft R$ of containing a unit $u \in R$ is necessarily equal to R by definition. Hence a proper ideal of R contains only non-units. If the set of all non-units of R forms an ideal, then it is the unique maximal ideal of R.

" $\Longrightarrow$ ": Suppose that R has a unique maximal ideal $\mathfrak{m}$. We claim that it is exactly the set of all non-units of R. If not, suppose that $a \notin \mathfrak{m}$ is a non-unit. Then $\langle a \rangle \neq R$. By the standard argument using Zorn's lemma, R is contained in some maximal ideal. But $\mathfrak{m}$ is the unique maximal ideal of R. Contradiction. □

Remark. I think I have made a false claim in the class. It is not true in general that the set of all non-units form a ideal of any ring. In fact, if the set of all non-units form a ideal I of R, then R is local and I is the unique maximal ideal of R.

Recall that for $\mathfrak{p} \in \operatorname{Spec} R$, the localisation of R at $\mathfrak{p}$ is the ring

$$R_{\mathfrak{p}} := R[(R \setminus \mathfrak{p})^{-1}] = \{a/b \mid a \in R,\ b \in R \setminus \mathfrak{p}\} /\sim,$$

where $a/b \sim a'/b'$ if and only if there exists $u \in R \setminus \mathfrak{p}$ such that $u(ab' - a'b) = 0$. The ideal extension of $\mathfrak{p}$ in $R_{\mathfrak{p}}$ is given by

$$\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}} = \{a/b \mid a \in \mathfrak{p},\ b \in R \setminus \mathfrak{p}\} /\sim .$$

Proposition 3.2

Let R be a ring and $\mathfrak{p} \in \operatorname{Spec} R$. Then $R_{\mathfrak{p}}$ is a local ring with the unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.

We proved this directly in the lecture / class. Another way of showing this is via the ideal correspondence:

Proposition 3.3

Let R be a ring and $\mathfrak{p} \in \operatorname{Spec} R$. There is a order-preserving bijective correspondence:

$$\begin{aligned} \{I \triangleleft R \mid I \subseteq \mathfrak{p}\} &\longleftrightarrow \{I \triangleleft R_{\mathfrak{p}}\} \\ I &\longmapsto IR_{\mathfrak{p}} \\ \varphi^{-1}(J) &\longleftarrow\!\shortmid J \end{aligned}$$

Compare this with the ideal correspondence for quotient rings:

$$
\begin{array}{ccc}
\{I \lhd R \mid \mathfrak{p} \subseteq I\} & \longleftrightarrow & \{I \lhd R/\mathfrak{p}\} \\
I & \longmapsto & I/\mathfrak{p} \\
\pi^{-1}(J) & \longleftarrow\!\shortmid & J
\end{array}
$$

3.2 More remarks on localisations

- For the more categorical-minded students, I find it useful to think of any algebraic constructions in terms of universal properties. For the localisation, it is the **universal** way to construct a ring $R[U^{-1}]$ by adding the inverses of $u \in U$ in the ring R. The universal property of $R[U^{-1}]$ can be stated as follows:

  For any ring S and any ring homomorphism $f : R \to S$ such that $f(u)$ is a unit in S for all $u \in U$, there exists a unique ring homomorphism $\widetilde{f} : R[U^{-1}] \to S$ such that $f = \widetilde{f} \circ \varphi$:

  $$
  \begin{array}{ccc}
  R & \xrightarrow{\ \forall f\ } & S \\
  {\scriptstyle\varphi}\big\downarrow & \nearrow_{\exists ! \widetilde{f}} & \\
  R[U^{-1}] & &
  \end{array}
  $$

  We can also represent $R[U^{-1}]$ as a quotient ring of a polynomial ring over R with lots of indeterminates (one for each $u \in U$). That is:

  $$R[U^{-1}] \cong \frac{R[x_u \mid u \in U]}{\langle u x_u - 1 \mid u \in U \rangle}.$$

- If R is a domain, $\varphi : R \to R[U^{-1}]$ is injective, so $R[U^{-1}]$ has "more" elements than R in general. But when R and U contain zero-divisors, things get more complicated.

Example 3.4

Let $R = \mathbb{Z}/6\mathbb{Z}$.

1. $U = \{1, 3, 5\}$. $R[U^{-1}] = (\mathbb{Z}/6\mathbb{Z})_{\langle 2 \rangle} \cong \mathbb{Z}/2\mathbb{Z}$. $\ker \varphi = \{0, 2, 4\}$.
2. $U = \{1, 2, 4\}$. $R[U^{-1}] = (\mathbb{Z}/6\mathbb{Z})[2^{-1}] \cong \mathbb{Z}/3\mathbb{Z}$. $\ker \varphi = \{0, 3\}$.
3. $U = \{1, 5\}$, $R[U^{-1}] \cong R$ and φ is an isomorphism because U consists of units of R.