Week 6 Notes

Warning: These are unofficial notes loosely related to the support class or the module in general. The contents are not necessarily part of the lectures and may not be examinable. Please use them at your own discretion.

In this course, all rings are assumed to be commutative rings with multiplicative identity 1.

5.1 Identifying free modules

Proposition 5.1

Let R be a ring and M an R-module. A **torsion element** of M is $m \in M$ such that there exists a non-zero-divisor $r \in R \setminus \{0\}$ with $rm = 0$. If M has a non-zero torsion element, then it is not a free R-module.

Proof. Suppose that M is free with a basis B. Let $m \in M$ be a non-zero trosion element. Then there exists unique $r_1, ..., r_n \in R \setminus \{0\}$ and $m_1, ..., m_n \in B$ such that $m = \sum_i r_i m_i$. By assumption we have some non-zero divisor $r \in R$ such that $rm = 0$. Therefore $\sum_i r r_i m_i = 0$. By linear independence of $m_1, ..., m_n$, we must have $rr_i = 0$. This is a contradiction. Hence M is not free. □

For example, the $\mathbb{Z}$-modules $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ are not free due to the presence of torsion.

Remark. An R-module is called **torsion** if every element of M is torsion. It is called **torsion-free** if no non-zero element of M is torsion. Therefore we have

$$\text{free} \implies \text{torsion-free}.$$

The converse is not true. it is proven in the lectures that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module, yet it is still torsion-free.

Remark. For a general module M, the set of torsion elements of M is a submodule of M and is called the torsion submodule of M. In fact, when R is a PID and M finitely generated R-module, a preliminary version of the **structure theorem** says

$$M \cong \underbrace{R^{\mathrm{rk}(M)}}_{\text{free module}} \oplus \underbrace{R/\langle r_1 \rangle \oplus \cdots \oplus R/\langle r_s \rangle}_{\text{torsion module}}.$$

5.2 An alternative proof of Cayley–Hamilton theorem

Theorem 5.2. Cayley–Hamilton Theorem

Let R be a ring and M a finitely generated R-module. For any R-module homomorphism $f : M \to M$, there exists a monic polynomial $\chi \in R[x]$ such that $\chi(f) = 0$.

A proof by using the adjoint (adjugate) matrix is presented in the lectures. There is an alternative proof without using matrices. The idea is to reduce the problem to what is known in linear algebra.

Proof. We shall prove that the characteristic polynomial $\chi(x) := \det(x\,\mathrm{id} - f)$ annihilates $f \in \mathrm{End}(M)$. The strategy is a sequence of relaxations on the assumption of R:

$$R \text{ splitting field} \Longrightarrow R \text{ field} \Longrightarrow R \text{ integral domain} \Longrightarrow R \text{ commutative ring}.$$

- *Step 1: R is a field such that χ splits as linear factors and M is an n-dimensional vector space over R.*

Let $\chi(x) = \det(x \operatorname{id} - f) \in R[x]$ and suppose that χ splits over the field R. Let $\lambda_1, \ldots, \lambda_n$ be the roots of χ, which are the eigenvalues of f. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors such that the matrix of M is upper triangular with respect to this basis of M. That is, $f(V_i) \subseteq V_i$ for all $1 \leqslant i \leqslant n$ where $V_i := \operatorname{span}\{v_1, \ldots, v_i\}$. For $v = \sum_{i=1}^n a_n v_n \in V$,

$$(f - \lambda_n \operatorname{id})v = \sum_{i=1}^{n-1} a_i (f(v_i) - \lambda_n v_i) \in V_{n-1}.$$

By induction, we have

$$\chi(f)v = (f - \lambda_1 \operatorname{id}) \cdots (f - \lambda_n \operatorname{id})v = 0.$$

Therefore $\chi(x)$ annihilates f.

- *Step 2: R is a field and M is an n-dimensional vector space over R.*

  Let K be the splitting field of $\chi \in R[x]$. By passing to $K[x]$ we have by Step 1 that $\chi(x)$ annihilates f.

- *Step 3: R is an integral domain and M is generated by $v_1, \ldots, v_n \in M$.*

  Suppose that $f(v_i) = \sum_{j=1}^n a_{ij} v_j$. So we can represent $\chi(x) = \det(x \operatorname{id} - f) \in R[x]$ by the determinant of the matrix $(\delta_{ij} x - a_{ij})_{i,j=1}^n$. Let $K = \operatorname{Frac} R$ be the field of fractions of R. By passing to $K[x]$ we have by Step 2 that $\chi(x)$ annihilates f.

- *Step 4: R is a commutative ring and M is generated by $v_1, \ldots, v_n \in M$.*

  Suppose that $f(v_i) = \sum_{j=1}^n a_{ij} v_j$. Let $S := \mathbb{Z}[x_{11}, \ldots, x_{nn}]$ and consider the ring homomorphism $\pi \colon S \longrightarrow R$ which sends x_{ij} to a_{ij}. Let $\widetilde{\pi} \colon S[x] \longrightarrow R[x]$ be induced by π. Note that M is naturally an S-module M_S, where $x_{ij} \cdot v := a_{ij} v$, and f is naturally an S-module homomorphism $f_S : M_S \longrightarrow M_S$. Since S is an integral domain, $\chi(x) = \det\big(\delta_{ij} x - x_{ij}\big)_{i,j=1}^n \in S[x]$ annihilates f_S. Therefore $\widetilde{\pi}(\chi)(x) = \det\big(\delta_{ij} x - a_{ij}\big)_{i,j=1}^n \in R[x]$ annihilates f. □