MA4J7 Cohomology & Poincaré Duality
Sheet 2 Solutions

Peize Liu

5 Feb 2024

Exercise 2.1

Consider the continuous map $\exp\colon \mathbb{R} \to S^1, t \mapsto \exp(2\pi i t)$. Given a singular simplex $\sigma : \Delta^1 \to S^1$, let $w(\sigma) := (\widetilde{\sigma}(1) - \widetilde{\sigma}(0)) \in \mathbb{R}$, where $\widetilde{\sigma}$ is any continuous map $\widetilde{\sigma} : \Delta^1 \to \mathbb{R}$ such that $\sigma = \widetilde{\sigma} \circ \exp$.

(i) Show that $w(\sigma)$ is well-defined in the sense that it does not depends on the choice of $\widetilde{\sigma}$.
(ii) Extend the assignment $\sigma \mapsto w(\sigma)$ by $\mathbb{R}$-linearity and obtain a 1-cochain $w : C_1^{\mathrm{sing}}(S;\mathbb{R}) \to \mathbb{R}$.
(iii) Show that $w \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$.
(iv) Show that $w \notin B^1_{\mathrm{sing}}(S^1;\mathbb{R})$.
(v) Show that $\mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R}) \simeq \mathbb{R}$ is generated by w.

(i) Suppose that $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are two liftings of σ. Then by definition we have

$$\sigma(t) = \exp(2\pi \mathrm{i}\widetilde{\sigma}(t)) = \exp(2\pi \mathrm{i}\widetilde{\sigma}'(t))$$

for all $t \in \Delta^1 \cong I = [0,1]$. Hence $1 = \exp(2\pi \mathrm{i}(\widetilde{\sigma}(t) - \widetilde{\sigma}'(t)))$. It follows that $\widetilde{\sigma}(t) - \widetilde{\sigma}'(t) \in \mathbb{Z}$ for all $t \in \Delta^1$. Since Δ^1 is connected, we have $\widetilde{\sigma} - \widetilde{\sigma}'$ is constant. In particular,

$$\widetilde{\sigma}(1) - \widetilde{\sigma}'(1) = \widetilde{\sigma}(0) - \widetilde{\sigma}'(0) \implies \widetilde{\sigma}(1) - \widetilde{\sigma}(0) = \widetilde{\sigma}'(1) - \widetilde{\sigma}'(0).$$

Hence $w(\sigma)$ is well-defined.

(ii) There is nothing to prove here.

(iii) For any singular 2-simplex $\tau\colon \Delta^2 \to S^1$,

$$\begin{aligned}(\delta w)(\tau) = w(\partial\tau) &= \sum_{i=0}^{2}(-1)^i w(\tau|_{\Delta^1_i}) \\ &= w(\tau|_{[v_0,v_1]}) + w(\tau|_{[v_1,v_2]}) - w(\tau|_{[v_0,v_2]}) \\ &= \widetilde{\tau_{01}}(v_1) - \widetilde{\tau_{01}}(v_0) + \widetilde{\tau_{12}}(v_2) - \widetilde{\tau_{12}}(v_1) - \widetilde{\tau_{02}}(v_2) + \widetilde{\tau_{02}}(v_0)\end{aligned}$$

Since $\mathbb{R}$ is a covering space of S^1, $\tau\colon \Delta^2 \to S^1$ lifts to a continuous map $\widetilde{\tau}\colon \Delta^2 \to \mathbb{R}$, so that $\widetilde{\tau}_{ij} = \widetilde{\tau}|_{[v_{i-1},v_{i+1}]}\colon \Delta^1 \to \mathbb{R}$ is a lift of $\tau|_{[v_i,v_j]}\colon \Delta^1 \to S^1$. In particular, we have $\widetilde{\tau_{01}}(v_1) = \widetilde{\tau}(v_1) = \widetilde{\tau_{12}}(v_1)$, and similar for v_0 and v_2. Hence $\delta w = 0$. $w \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$.

(iv) Suppose that $w = \delta u$ for some $u \in C^0_{\mathrm{sing}}(S^1;\mathbb{R})$. Then, for any $\sigma\colon \Delta^1 \to S^1$, we have

$$w(\sigma) = (\delta u)(\sigma) = u(\partial\sigma) = u(\sigma(1)) - u(\sigma(0)).$$

In particular if $\sigma_1\colon \Delta^1 \cong [0,1] \to S^1$ is given by $\sigma_1(t) = \exp(2\pi i t)$, then $\sigma_1(0) = \sigma_1(1)$ and hence $w(\sigma_1) = 0$. On the other hand, $\widetilde{\sigma}_1\colon [0,1] \to \mathbb{R}$, $\widetilde{\sigma}_1(t) = t$ is a lift of σ_1, which implies that $w(\sigma_1) = \widetilde{\sigma}_1(1) - \widetilde{\sigma}_1(0) = 1$. This is a contradiction. Hence $w \notin B^1_{\mathrm{sing}}(S^1;\mathbb{R})$.

(v) By (iii) and (iv), w define a non-zero class $[w] \in \mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R})$. It suffices to show that $\mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R})$ is one-dimensional. There are a lot of ways: cohomological Mayer–Vietoris, universal coefficient theorem, or direct computation by definition.

- **Cohomological Mayer–Vietoris sequence**: Let $a, b \in S^1$ be two distinct points. Put $A := S^1 \setminus \{a\}$ and $B := S^1 \setminus \{b\}$. Then $A, B \cong \mathbb{R}$ are contractible, and $A \cap B = \{a, b\}$ consists of two points. Consider the cohomological reduced Mayer–Vietoris long exact sequence:

$$\cdots \longrightarrow \widetilde{\mathrm{H}}^0(A;\mathbb{R}) \oplus \widetilde{\mathrm{H}}^0(B;\mathbb{R}) \longrightarrow \widetilde{\mathrm{H}}^0(A\cap B;\mathbb{R}) \longrightarrow \widetilde{\mathrm{H}}^1(S^1;\mathbb{R}) \longrightarrow \widetilde{\mathrm{H}}^1(A;\mathbb{R}) \oplus \widetilde{\mathrm{H}}^1(B;\mathbb{R}) \longrightarrow \cdots$$

Since $\widetilde{\mathrm{H}}^\bullet(A;\mathbb{R}) = \widetilde{\mathrm{H}}^\bullet(B;\mathbb{R}) = 0$, we have $\widetilde{\mathrm{H}}^1(S^1;\mathbb{R}) \cong \widetilde{\mathrm{H}}^0(A\cap B;\mathbb{R}) = \mathbb{R}$. Hence $\mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R}) \cong \widetilde{\mathrm{H}}^1(S^1;\mathbb{R}) \cong \mathbb{R}$ and it is generated by $[w]$.

- **Universal coefficient theorem**: we have an isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(S^1),\mathbb{R}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_0(S^1),\mathbb{R}).$$

Note that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$ is free and $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. We deduce that $\mathrm{H}^1_{\mathrm{sing}}(S^1;\mathbb{R}) \cong \mathbb{R}$.

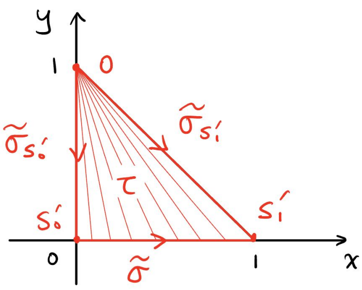
- **Proof by definition**: We claim that $Z^1_{\mathrm{sing}}(S^1;\mathbb{R}) = \mathbb{R}w \oplus B^1_{\mathrm{sing}}(S^1;\mathbb{R})$. The only non-trivial part is to show $Z^1_{\mathrm{sing}}(S^1;\mathbb{R}) \subseteq \mathbb{R}w + B^1_{\mathrm{sing}}(S^1;\mathbb{R})$. Let $\alpha \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$. Consider the singular simplex

$$\sigma_s :\ \Delta^1 \longrightarrow S^1; \qquad \sigma_s(t) = \mathrm{e}^{2\pi \mathrm{i} st}.$$

Let $\beta := \alpha - \alpha(\sigma_1)w \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$. Note that $\beta(\sigma_1) = \alpha(\sigma_1) - \alpha(\sigma_1)w(\sigma_1) = 0$. We claim that $\beta \in B^1_{\mathrm{sing}}(S^1;\mathbb{R})$. Let $u \in C^0_{\mathrm{sing}}(S^1;\mathbb{R})$ be such that $u(\mathrm{e}^{2\pi \mathrm{i}s}) = \beta(\sigma_s)$ for $s \in [0,1]$. Since $\beta \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$, then $\beta(\sigma_s) = \beta(\sigma_{s'}) + \beta(\sigma_{s-s'})$; and since $\beta(\sigma_1) = 0$, we have $\beta(\sigma_s) = \beta(\sigma_{s'})$ if $s \equiv s' \bmod \mathbb{Z}$. It suffices to show that $\beta(\sigma) = u(s_1) - u(s_0)$ for any $\sigma :\ \Delta^1 \longrightarrow S^1$, where $s_0 := \sigma(0)$ and $s_1 := \sigma(1)$. Choose a lift $\widetilde{\sigma} :\ \Delta^1 \longrightarrow \mathbb{R}$ and put $s'_i := \widetilde{\sigma}(i)$ for $i = 0, 1$. Then $s_i \equiv s'_i \bmod \mathbb{Z}$. Consider the singular 2-simplex $\tau :\ \Delta^2 \longrightarrow S^1$ defined by its lift $\widetilde{\tau} :\ \Delta^2 \longrightarrow \mathbb{R}$, given by

$$\widetilde{\tau}(x,y) = (1-y)\widetilde{\sigma}\left(\frac{x}{1-y}\right),$$

where the 2-simplex is modelled as follows:



Since $\beta \in Z^1_{\mathrm{sing}}(S^1;\mathbb{R})$, we have

$$0 = \delta\beta(\tau) = \beta(\partial\tau) = \beta(\sigma_{s'_0}) + \beta(\sigma) - \beta(\sigma_{s'_1}) = \beta(\sigma_{s_0}) + \beta(\sigma) - \beta(\sigma_{s_1}) = \beta(\sigma) - u(s_1) + u(s_0).$$

Hence $\beta(\sigma) = u(s_1) - u(s_0) = \delta u(\sigma)$. It follows that $\beta \in B^1_{\mathrm{sing}}(S^1;\mathbb{R})$. This finishes the proof. (*I think the argument could be simplified.*)

Exercise 2.2

Let $f : S^n \longrightarrow S^n$ be a continuous map. Review the definition of the degree of f from Section 2.2 (page 134) of Hatcher. Now, prove that that the degrees of the following two homomorphisms are equal:

$$f_n : \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n) \qquad f^n : \mathrm{H}^n(S^n;\mathbb{Z}) \longrightarrow \mathrm{H}^n(S^n;\mathbb{Z}).$$

Recall that the sphere S^n has the homology groups

$$\mathrm{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

The continuous map $f : S^n \to S^n$ induces the group homomorphism $f_* : \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$, which is uniquely determined by the image $f_*(\alpha)$, where $\alpha \in \mathrm{H}_n(S^n) \cong \mathbb{Z}$ is any chosen generator. The integer $f_*(\alpha) \in \mathrm{H}_n(S^n) \cong \mathbb{Z}$ is called the degree of f, denoted by $\deg f$.

To show that f_n and f^n are "equal", we must first identify $\mathrm{H}^n(S^n)$ and $\mathrm{H}_n(S^n)$ with $\mathbb{Z}$ by choosing generators. Consider the cellular chain complex $C_\bullet^{\mathrm{CW}}(S^n)$: S^n has a unique n-cell α and no other k-cells for $k > 0$. The cellular chain complex is given by

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathbb{Z}\alpha & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ n+1 & & n & & n-1 & & & & 1 & & 0 & & \end{array}$$

Hence $\mathrm{H}_n(S^n) = \mathbb{Z}\alpha$. $f : S^n \to S^n$ induces f_n by $f_n(\alpha) = \deg f \cdot \alpha \in \mathrm{H}_n(S^n)$. By dualisation, the cellular cochain complex is given by

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathbb{Z}\alpha^* & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ n+1 & & n & & n-1 & & & & 1 & & 0 & & \end{array}$$

where $\alpha^* \in \mathrm{Hom}(C_n^{\mathrm{CW}}(S^n), \mathbb{Z})$ is the cocycle such that $\alpha^*(\alpha) = 1$. The cellular cohomology $\mathrm{H}^n(S^n) = \mathbb{Z}\alpha^*$. Now f^n acts by

$$f^n(\alpha^*)(k\alpha) = \alpha^* \circ f_n(k\alpha) = \alpha^*(k \deg f \cdot \alpha) = k \deg f = \deg f \cdot \alpha^*(k\alpha).$$

Hence $f^n(\alpha^*) = \deg f \cdot \alpha^*$. So both f_n and f^n acts as multiplication by $\deg f \in \mathbb{Z}$.

In fact, the result can be summarised as a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^n(S^n) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(S^n), \mathbb{Z}) \\ \downarrow f^n & & \downarrow (f_n)^* \\ \mathrm{H}^n(S^n) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(S^n), \mathbb{Z}) \end{array}$$

So f^n should be regarded as the dual map (i.e. transpose) of f_n. This also holds for general space X with $\mathrm{H}_{n-1}(X)$ free (by universal coefficient theorem).