MA4J7 Cohomology & Poincaré Duality
Sheet 3 Solutions

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Exercise 3.1

Let X be a topological space and R a commutative ring with unit. Let $\epsilon \in C^0(X;R)$ be the *augmentation homomorphism*: for all singular zero-simplices σ^0 we have $\epsilon\left(\sigma^0\right) = 1_R$. Prove that the cup product at the level of cochains is R-linear in both variables, associative, and has $\epsilon \in C^0(X;R)$ as its identity element.

Recall that the cup product is defined on singular cochains by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}), \qquad \varphi \in C^k(X;R),\ \psi \in C^\ell(X;R),\ \sigma \in C_{k+\ell}(X).$$

***R*-linearity:** For $a_i, b_i \in R$, $\varphi_i \in C^k(X;R)$, $\psi_i \in C^\ell(X;R)$

$$\begin{aligned}
\left(\sum_{i=1}^n a_i\varphi_i \smile \psi\right)(\sigma) &= \left(\sum_{i=1}^n a_i\varphi_i\right)(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\
&= \sum_{i=1}^n a_i\varphi_i(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\
&= \sum_{i=1}^n a_i(\varphi_i \smile \psi)(\sigma); \\
\left(\varphi \smile \sum_{i=1}^n b_i\psi_i\right)(\sigma) &= \varphi(\sigma|_{[v_0,\dots,v_k]})\left(\sum_{i=1}^n b_i\psi_i\right)(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\
&= \sum_{i=1}^n b_i\varphi(\sigma|_{[v_0,\dots,v_k]})\psi_i(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\
&= \sum_{i=1}^n b_i(\varphi \smile \psi_i)(\sigma).
\end{aligned}$$

Associativity: For $\varphi \in C^i(X;R)$, $\psi \in C^j(X;R)$, $\chi \in C^k(X;R)$, $\sigma \in C_{i+j+k}(X)$,

$$\begin{aligned}
(\varphi \smile (\psi \smile \chi))(\sigma) &= \varphi(\sigma|_{[v_0,\dots,v_i]})(\psi \smile \chi)(\sigma|_{[v_i,\dots,v_{i+j+k}]}) \\
&= \varphi(\sigma|_{[v_0,\dots,v_i]})\psi((\sigma|_{[v_i,\dots,v_{i+j+k}]})|_{[v_0,\dots,v_j]})\chi((\sigma|_{[v_i,\dots,v_{i+j+k}]})|_{[v_j,\dots,v_{j+k}]}) \\
&= \varphi(\sigma|_{[v_0,\dots,v_i]})\psi(\sigma|_{[v_i,\dots,v_{i+j}]})\chi(\sigma|_{[v_{i+j},\dots,v_{i+j+k}]}) \\
&= \varphi((\sigma|_{[v_0,\dots,v_{i+j}]})|_{[v_0,\dots,v_i]})\psi((\sigma|_{[v_0,\dots,v_{i+j}]})|_{[v_i,\dots,v_{i+j}]})\chi(\sigma|_{[v_{i+j},\dots,v_{i+j+k}]}) \\
&= (\varphi \smile \psi)(\sigma|_{[v_0,\dots,v_{i+j}]})\chi(\sigma|_{[v_{i+j},\dots,v_{i+j+k}]}) \\
&= ((\varphi \smile \psi) \smile \chi)(\sigma).
\end{aligned}$$

Identity: For $\varphi \in C^k(X;R)$,

$$\begin{aligned}
(\varphi \smile \epsilon)(\sigma) &= \varphi(\sigma|_{[v_0,\dots,v_k]})\epsilon(\sigma|_{[v_k]}) = \varphi(\sigma) \cdot 1_R = \varphi(\sigma); \\
(\epsilon \smile \varphi)(\sigma) &= \epsilon(\sigma|_{[v_0]})\varphi(\sigma|_{[v_0,\dots,v_k]}) = 1_R \cdot \varphi(\sigma) = \varphi(\sigma).
\end{aligned}$$

Exercise 3.2

(i) Give a Δ-complex structure on the torus $T^2 = S^1 \times S^1$.

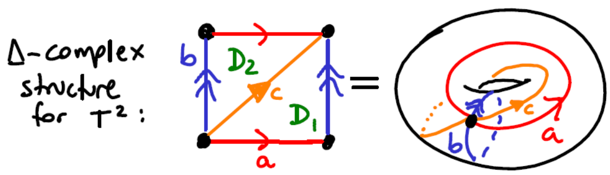
(ii) Using (i), describe the simplicial chain complex associated to the Δ-complex structure of item (i) (with $R = \mathbb{Z}$).

(iii) Using (ii), compute the homology groups of T^2.

(iv) Using (i) and (ii), describe the simplicial cochain complex associated to the Δ-complex structure of item (i) (with $R = Q = \mathbb{Z}$).

(v) Using (iv), compute the cohomology groups of T^2.

(vi) Compute the cohomology ring $H^*(T^2; \mathbb{Z})$.



(ii)
- 0-simplex: v.
- 1-simplices: a, b, c: $\partial a = \partial b = \partial c = v - v = 0$.
- 2-simplices: D_1, D_2: $\partial D_1 = \partial D_2 = a + b - c$.

Therefore the simplicial chain complex is given by:

$$0 \longrightarrow \underset{2}{\mathbb{Z}D_1 \oplus \mathbb{Z}D_2} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}} \underset{1}{\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c} \xrightarrow{0} \underset{0}{\mathbb{Z}v} \longrightarrow 0$$

(iii) Denote by $\varphi : \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \longrightarrow \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$ the linear map associated to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$. Then

- $\mathrm{H}_0(T^2) = \mathbb{Z}v \cong \mathbb{Z}$;
- $\mathrm{H}_1(T^2) = \operatorname{coker} \varphi = \dfrac{\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c}{\langle a + b - c \rangle} \cong \mathbb{Z}^2$, with a, b being the generators.
- $\mathrm{H}_2(T^2) = \ker \varphi = \mathbb{Z}(D_1 - D_2) \cong \mathbb{Z}$.
- $\mathrm{H}_i(T^2) = 0$ for $i \neq 0, 1, 2$.

(iv) Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the simplicial chain complex:

$$0 \longleftarrow \underset{2}{\mathbb{Z}D_1^* \oplus \mathbb{Z}D_2^*} \xleftarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \underset{1}{\mathbb{Z}a^* \oplus \mathbb{Z}b^* \oplus \mathbb{Z}c^*} \xleftarrow{0} \underset{0}{\mathbb{Z}v^*} \longleftarrow 0$$

(v)
- $\mathrm{H}^0(T^2; \mathbb{Z}) = \mathbb{Z}v^* \cong \mathbb{Z}$;
- $\mathrm{H}^1(T^2; \mathbb{Z}) = \ker \varphi^* = \mathbb{Z}(a^* + c^*) \oplus \mathbb{Z}(b^* + c^*) \cong \mathbb{Z}^2$;
- $\mathrm{H}^2(T^2; \mathbb{Z}) = \operatorname{coker} \varphi^* = \dfrac{\mathbb{Z}D_1^* \oplus \mathbb{Z}D_2}{\langle D_1 + D_2 \rangle} \cong \mathbb{Z}$, with $D_1^* = -D_2^*$ being the generator.
- $\mathrm{H}^i(T^2; \mathbb{Z}) = 0$ for $i \neq 0, 1, 2$.

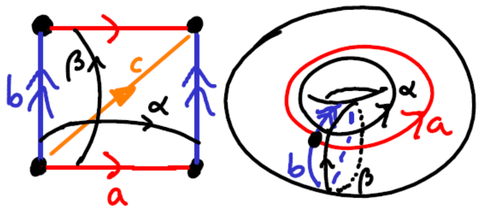
(vi) Let $\alpha^* := a^* + c^*$ and $\beta^* := b^* + c^*$ be the generators of $\mathrm{H}^1(T^2; \mathbb{Z})$. We need to compute the cup product of them. Note that $2(\alpha^* \smile \alpha^*) = 2(\beta^* \smile \beta^*) = 0 \in \mathrm{H}^2(T^2; \mathbb{Z})$ by graded commutativity. Since $\mathrm{H}^2(T^2; \mathbb{Z})$ has no torsion, we must have $\alpha^* \smile \alpha^* = \beta^* \smile \beta^* = 0$. It remains to compute $\alpha^* \smile \beta^*$. We use the definition:

$$(\alpha^* \smile \beta^*)(D_1) = \alpha^*(D_1|_{[v_0,v_1]})\beta^*(D_1\_[v_1,v_2]) = \alpha^*(a)\beta^*(b) = 1;$$
$$(\alpha^* \smile \beta^*)(D_2) = \alpha^*(D_2|_{[v_0,v_1]})\beta^*(D_2\_[v_1,v_2]) = \alpha^*(b)\beta^*(a) = 0.$$

Hence $\alpha^* \smile \beta^* = D_1^*$ on the cochain level. We lift this to cohomology and deduce that $\mathrm{H}^2(T^2;\mathbb{Z}) = \mathbb{Z}D_1^*$ is generated by $\alpha^* \smile \beta^*$. In conclusion, we have the isomorphism as graded rings:

$$\mathrm{H}^\bullet(T^2;\mathbb{Z}) \cong \frac{\mathbb{Z}\langle x,y\rangle}{\langle x^2, y^2, xy+yx\rangle} \cong \frac{\mathbb{Z}[x]}{\langle x^2\rangle} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[y]}{\langle y^2\rangle} \cong \bigwedge\nolimits^\bullet(\mathbb{Z}x \oplus \mathbb{Z}y), \qquad |x| = |y| = 1.$$

Another way to visualise this cup product computation is through ***intersection theory***[1]: let $\alpha, \beta \in \mathrm{H}_1(T^2)$ be the Poincaré duals of α^*, β^* respectively. They are represented by the 1-cycles α, β as shown in the diagram:



We identify $\mathrm{H}^2(T^2;\mathbb{Z})$ with $\mathbb{Z}$ by $D_1^* \mapsto 1$. Then the cup product $\alpha^* \smile \beta^*$ is the same as the intersection product $\alpha \cdot \beta$, which is the number of intersection points of the cycles α and β, counted with signs and multiplicities.

From the diagram, we observe that $\alpha \cdot \beta = 1$, as they intersect transversely at one point. $\alpha \cdot \alpha = 0$ because we can deform α to α' within the same homology class such that α and α' are disjoint cycles. Similarly $\beta \cdot \beta = 0$. This visualisation is verified by our previous computation:

$$\alpha^* \smile \alpha^* = \beta^* \smile \beta^* = 0, \qquad \alpha^* \smile \beta^* = D_1^*.$$

[1]We do not make this rigourous, so it is only for intuition and visualisation purpose.