MA4J7 Cohomology & Poincaré Duality
Sheet 4 Solutions

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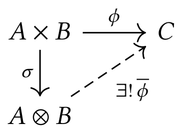
Exercise 4.1

Let R be a commutative ring. Let $\phi : A \times B \to C$ be a R-bilinear pairing. Show that the exists a unique R-linear map $\overline{\phi} : A \otimes_R B \to C$ such that $\overline{\phi}(a \otimes b) = \phi(a, b)$.

Remark. This is the universal property of the tensor product of two R-modules, which characterises the tensor product uniquely up to a unique isomorphism.



Recall that we define $A \otimes_R B$ as the R-module $R^{A\times B}/N$, where $R^{A\times B}$ is the free R-module with basis $\{(a, b) \mid a \in A,\ b \in B\}$, and N is the submodule of U generated by

$$\left\{ \begin{array}{c} (a + a', b) - (a, b) - (a', b), \\ (a, b + b') - (a, b) - (a, b'), \\ (ra, b) - r(a, b), \\ (a, rb) - r(a, b) \end{array} \;\middle|\; \begin{array}{c} a, a' \in A, \\ b, b' \in B, \\ r \in R \end{array} \right\}.$$

And we denote that image of $(a, b) \in U$ in $A \otimes_R B$ as $a \otimes b$. This gives an R-bilinear map $\sigma \colon A \times B \to A \otimes B$.

For an R-bilinear map $\phi \colon A \times B \to C$, it induces the R-linear map $\widetilde{\phi} \colon R^{A\times B} \to C$,

$$\widetilde{\phi}\left(\sum_i r_i(a_i, b_i)\right) = \sum_i r_i \phi(a_i, b_i).$$

Since ϕ is bilinear, we have that

$$\begin{aligned} &\widetilde{\phi}((a + a', b) - (a, b) - (a', b)) = \phi(a + a', b) - \phi(a, b) = \phi(a', b) = 0 \\ &\widetilde{\phi}((a, b + b') - (a, b) - (a, b')) = \phi(a, b + b') - \phi(a, b) = \phi(a, b') = 0 \\ &\widetilde{\phi}((ra, b) - r(a, b)) = \phi(ra, b) - r\phi(a, b) = 0 \\ &\widetilde{\phi}((a, rb) - r(a, b)) = \phi(a, rb) - r\phi(a, b) = 0 \end{aligned}$$

for any $a, a' \in A$, $b, b' \in B$, and $r \in R$. So $N \subseteq \ker \widetilde{\phi}$. It follows from the universal property of quotient that there is a unique map $\overline{\phi} \colon R^{A\times B}/N \to C$ such that $\widetilde{\phi} = \overline{\phi} \circ \pi$. *In plain words, this means we define* $\overline{\phi}(\sum_i r_i a_i \otimes b_i) = \widetilde{\phi}(\sum_i r_i(a_i, b_i))$ *and* $N \subseteq \ker \widetilde{\phi}$ *ensures that* $\overline{\phi}(\sum_i r_i a_i \otimes b_j) = \overline{\phi}(\sum_j r_j a_j \otimes b_j)$ *if* $\sum_i r_i a_i \otimes b_j = \sum_j r_j a_j \otimes b_j$, *and hence* $\overline{\phi}$ *is well-defined.*

Uniqueness is clear: if $\overline{\phi}(a \otimes b) = \overline{\phi}'(a \otimes b) = \phi(a, b)$, then $(\overline{\phi} - \overline{\phi}')(a \otimes b) = 0$ for all $a \in A$ and $b \in B$. But $A \otimes B$ is generated by $\{a \otimes b \mid a \in A,\ b \in B\}$. We must have $\overline{\phi} - \overline{\phi}' = 0$. Hence $\overline{\phi}$ is unique.

Exercise 4.2

Let R be a commutative ring. Let A and B be two $\mathbb{Z}$-graded commutative R-algebras. Show that $A \otimes_R B$ is also a $\mathbb{Z}$-graded commutative R-algebra with $\deg(a \otimes b) := \deg(a) + \deg(b)$ and $(a \otimes b) \cdot (a' \otimes b') := (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$.

We have the following things to check:

1. $A \otimes_R B$ is a $\mathbb{Z}$-graded R-module.

   Write $A = \bigoplus_{i \in \mathbb{Z}} A_i$ and $B = \bigoplus_{j \in \mathbb{Z}} B_j$. Then the requirement that $|a \otimes b| = |a| + |b|$ induces a natural grading on $A \otimes_R B$, given by
   $$A \otimes_R B = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i+j=n} (A_i \otimes_R B_j).$$

2. The product on $A \otimes_R B$ is well-defined.

   We need the following lemma as preparation:

   > **Lemma. (Tensor product is associative.)** Let A, B, C be R-modules. Then there exists a natural isomorphism $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$.

   Proof.
   - Let $\phi : (A \otimes_R B) \times C \to D$ be an R-bilinear map.
   - It induces an R-trilinear map $\phi' : A \times B \times C \to D$ by $\phi'(a, b, c) = \phi(a \otimes b, c)$.
   - For each $a \in A$, consider the R-bilinear map $\phi'_a : B \times C \to D$, $\phi'_a(b, c) = \phi'(a, b, c) = \phi(a \otimes b, c)$. By universal property of tensor product, there exists a unique R-linear map $\overline{\phi}'_a : B \otimes_R C \to D$ such that $\overline{\phi}'_a(b \otimes c) = \phi'(a, b, c)$.
   - Now $\phi'' : A \times (B \otimes_R C) \to D$, $\phi''(a, b \otimes c) = \overline{\phi}'_a(b \otimes c) = \phi'(a, b, c)$ is R-bilinear. So there exists a unique R-linear map $\overline{\phi}'' : A \otimes_R (B \otimes_R C) \to D$ such that $\overline{\phi}''(a \otimes (b \otimes c)) = \phi'(a, b, c)$.
   - Since $\overline{\phi}''(a \otimes (b \otimes c)) = \phi(a \otimes b, c)$, again by universal property of tensor product, we have an natural isomorphism $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$. □

   By the lemma, we can write $A \otimes B \otimes A \otimes B$ without brackets unambiguously. Consider the R-tetralinear map $\mu : A \times B \times A \times B \to A \otimes_R B$ defined by
   $$\mu(a, b, a', b') = (-1)^{|a| \cdot |b|}(aa' \otimes bb').$$

   By the lemma, μ induces the R-linear map $\overline{\mu} : (A \otimes_R B) \otimes_R (A \otimes_R B) \to A \otimes_R B$, which then induces the R-bilinear map $m : (A \otimes_R B) \times (A \otimes_R B) \to A \otimes_R B$, given by the R-bilinear extension of the following product:
   $$(a \otimes b) \cdot (a' \otimes b') = m(a \otimes b, a' \otimes b') = \mu(a, b, a', b') = (-1)^{|a| \cdot |b|}(aa' \otimes bb').$$

   So the multiplication is well-defined.

3. $A \otimes_R B$ is a $\mathbb{Z}$-graded R-algebra.

   We have checked that the product is R-bilinear. It is clear that $1_A \otimes 1_B$ is the multiplicative identity. Next we check that the product is associative.
   $$\begin{aligned}((a \otimes b) \cdot (a' \otimes b')) \cdot (a'' \otimes b'') &= -1^{|b| \cdot |a'|}(aa' \otimes bb') \cdot (a'' \otimes b'') \\ &= -1^{|b| \cdot |a'| + |bb'| \cdot |a''|} aa'a'' \otimes bb'b'' \\ &= -1^{|b| \cdot |a'a''| + |b| \cdot |a''|} aa'a'' \otimes bb'b'' \\ &= -1^{|b| \cdot |a''|}(a \otimes b) \cdot (a'a'' \otimes b'b'') \\ &= (a \otimes b) \cdot ((a' \otimes b') \cdot (a'' \otimes b'')).\end{aligned}$$

It remains to check that the product respects the grading: $(A_i \otimes_R B_j) \cdot (A_k \cdot B_\ell) \subseteq A_{i+k} \otimes_R B_{j+\ell}$. Indeed,

$$|(a_i \otimes b_j)(a_k \otimes b_\ell)| = |(-1)^{jk} a_i a_k \otimes b_j b_\ell| = i + j + k + \ell = |a_i \otimes b_j| + |a_k \otimes b_\ell|.$$

4. The product on $A \otimes_R B$ is $\mathbb{Z}$-graded commutative.

   For $c_m \in (A \otimes_R B)_m$ and $c_n \in (A \otimes_R B)_n$, we need to show that $c_m c_n = (-1)^{mn} c_n c_m$. Note that a general $c_m \in (A \otimes_R B)_m$ takes the form

$$c_m = \sum_{i+j=m} \underbrace{c_m^{(ij)}}_{\in A_i \otimes_R B_j} = \sum_{i+j=m} \sum_k r_{ij}^{(k)} \underbrace{a_i^{(k)}}_{\in A_i} \otimes \underbrace{b_j^{(k)}}_{\in B_j}.$$

   Since the product is bilinear, it suffices to verify $c_m c_n = (-1)^{mn} c_n c_m$ with the assumption that $c_m = a_i \otimes b_j$ and $c_n = a_k \otimes b_\ell$, where $i + j = m$ and $k + \ell = n$. Indeed,

$$\begin{aligned} c_m c_n &= (a_i \otimes b_j)(a_k \otimes b_\ell) \\ &= (-1)^{jk} a_i a_k \otimes b_j b_\ell \\ &= (-1)^{jk+ik+j\ell} a_k a_i \otimes b_\ell b_j \\ &= (-1)^{(i+j)(k+\ell)} (a_k \otimes b_\ell)(a_i \otimes b_j) \\ &= (-1)^{mn} c_n c_m. \end{aligned}$$

Exercise 4.3

Show that the cross product map $\mathrm{H}^\bullet(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^\bullet(Y; \mathbb{Z}) \longrightarrow \mathrm{H}^\bullet(X \times Y; \mathbb{Z})$ is not an isomorphism when X and Y are infinite discrete sets. Which assumption of the Künneth formula does not hold?

If X is an infinite discrete set, then $X = \coprod_{x \in X} \{x\}$, and hence

$$\mathrm{H}^\bullet(X; \mathbb{Z}) = \prod_{x \in X} \mathrm{H}^\bullet(\{x\}; \mathbb{Z}) = \prod_{x \in X} \mathrm{H}^0(\{x\}; \mathbb{Z}) = \prod_{x \in X} \mathbb{Z} \mathbf{1}_{\{x\}}.$$

Here $\mathbf{1}_x : X \longrightarrow \mathbb{Z}$ is the indicator function on x, i.e.

$$\mathbf{1}_{\{x\}}(x') = \delta(x, x') = \begin{cases} 1, & x = x' \\ 0, & x \neq x' \end{cases}$$

Similarly $\mathrm{H}^\bullet(Y; \mathbb{Z}) = \prod_{y \in Y} \mathbb{Z} \mathbf{1}_{\{y\}}$, and $\mathrm{H}^\bullet(X \times Y; \mathbb{Z}) = \prod_{(x,y) \in X \times Y} \mathbb{Z} \mathbf{1}_{\{(x,y)\}}$. The cross product of $\mathbf{1}_{\{x\}}$ and $\mathbf{1}_{\{y\}}$ is given by

$$\begin{aligned} \left(\pi_1^* \mathbf{1}_{\{x\}} \smile \pi_2^* \mathbf{1}_{\{y\}}\right)(x', y') &= \left(\mathbf{1}_{\{x\} \times Y} \smile \mathbf{1}_{X \times \{y\}}\right)(x' y') \\ &= \mathbf{1}_{\{x\} \times Y}(x', y') \mathbf{1}_{X \times \{y\}}(x', y') \\ &= \delta(x, x') \delta(y, y') \\ &= \mathbf{1}_{\{(x,y)\}}(x', y'). \end{aligned}$$

Hence the cross product is induced by $\mathbf{1}_{\{x\}} \otimes \mathbf{1}_{\{y\}} \longmapsto \mathbf{1}_{\{(x,y)\}}$.

We claim that $\mathrm{H}^\bullet(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^\bullet(Y; \mathbb{Z}) \longrightarrow \mathrm{H}^\bullet(X \times Y; \mathbb{Z})$ is not surjective. Since X and Y are infinite sets. we may take countable subsets $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ of them respectively, and consider

$$\varphi := \prod_{i \in \mathbb{N}} \mathbf{1}_{\{(x_i, y_i)\}} \in \mathrm{H}^0(X \times Y; \mathbb{Z}).$$

Suppose that φ is in the image of the cross product. Then $\varphi = \sum_{i=1}^k n_i \alpha_i \times \beta_i$, where $n_i \in \mathbb{Z}$, $\alpha_i \in \mathrm{H}^0(X; \mathbb{Z})$ and $\beta_i \in \mathrm{H}^0(Y; \mathbb{Z})$. Note that the support, $\operatorname{supp} \varphi = \bigcup_{i=1}^k \operatorname{supp}(\alpha_i \times \beta_i)$ is infinite. It follows that there is some

$i \in \{1, ..., k\}$ such that $\operatorname{supp}(\alpha_i \times \beta_i)$ is infinite. But $\operatorname{supp}(\alpha_i \times \beta_i) = \operatorname{supp}(\alpha)_i \times \operatorname{supp}(\beta_i) \in X \times Y$ is a rectangular set, and by our construction of φ, the only rectangular subsets of $\operatorname{supp}(\varphi)$ are the singletons $\{(x_i, y_i)\}$, which is a contradiction. This finishes the proof that the cross product is not an isomorphism.

The reason that the Künneth formula does not apply is that neither $\mathrm{H}^\bullet(X;\mathbb{Z})$ nor $\mathrm{H}^\bullet(Y;\mathbb{Z})$ is finitely generated, and $\mathrm{H}^0(Y;\mathbb{Z}) = \prod_{y \in Y} \mathbb{Z}\mathbf{1}_{\{y\}}$ is not a free $\mathbb{Z}$-module.

Here is a quick proof that $\mathrm{H}^0(Y;\mathbb{Z})$ is not free, adapted from `math.stackexchange.com/questions/500607/`.

Suppose that $\mathrm{H}^0(Y;\mathbb{Z})$ is free. Then its submodule $M := \prod_{i=1}^\infty \mathbb{Z}$ is also free. Consider the submodule of M:

$$S := \left\{ (a_i)_{i=1}^\infty \in M : \forall m \in \mathbb{N} \, \exists N \in \mathbb{N} \, \forall n \geqslant N \, (p^m \mid a_n) \right\},$$

where $p \in \mathbb{Z}$ is some prime. It follows that S is also free, and there is an injective $\mathbb{Z}$-linear map $M \hookrightarrow S$ given by multiplication by the sequence $(p^i)_{i=1}^\infty$. The cardinality of M is $\aleph_0^{\aleph_0}$, which is uncountable. Hence S is also uncountable. As a free $\mathbb{Z}$-module, any basis of S is uncountable (because $\mathbb{Z}^{\oplus \mathbb{N}}$ is countable).

On the other hand, S/pS is naturally a $\mathbb{Z}/p$-vector space. By definition of S, every sequence in S/pS has a representative $(a_i)_{i=1}^\infty$ with finitely many non-zero terms. In particular, $S/pS \cong (\mathbb{Z}/p)^{\oplus \mathbb{N}}$ is countable. But any $\mathbb{Z}$-basis of S descends to a $\mathbb{Z}/p$-basis of S/pS. So S/pS is a countable vector space with uncountable dimension, which is absurd.