MA4J7 Cohomology & Poincaré Duality
Sheet 1 Solutions

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Exercise 1.1

Let $X = S^2 \times S^4$ and $Y = \mathbb{CP}^3$.

1. Show that X and Y are compact and connected.
2. Prove that $\pi_1(X)$ and $\pi_1(Y)$ are both trivial.
3. Give a CW-complex structure on X and on Y.
4. Using the CW-complex structure, compute the homology groups of X and Y.

1. For X, we note that the Euclidean topology on the n-sphere is compact, because

$$S^n = \left\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\right\}$$

is closed and bounded in $\mathbb{R}^{n+1}$, and hence compact by Heine–Borel theorem. It is also connected because it is path-connected (for two points on S^n we can connect them with an arc of a great circle). Next, we use that the product topology on $X_1 \times X_2$ is compact and connected if both X_1, X_2 are compact and connected to conclude that $X = S^2 \times S^4$ is compact and connected.

For $Y = \mathbb{CP}^3$, we have $S^7 \cong \left\{z \in \mathbb{C}^4 \mid \|z\| = 1\right\} \subseteq \mathbb{C}^4$ a quotient map $\pi : S^7 \to \mathbb{CP}^3$ such that $\pi(z) = \pi(w)$ if and only if $z = \lambda w$ for some $\lambda \in \mathbb{C}^\times$. Since S^7 is compact and connected, so is the image $\mathbb{CP}^3 = \pi(S^7)$.

2. For X, first we note that S^n is simply-connected (i.e. $\pi_1(S^n) = \{e\}$) for $n \geqslant 2$. The standard method is to use Seifert–van Kampen Theorem. Let $a, b \in S^n$ be the north and the south pole of S^n. Let $U := S^n \setminus \{a\}$ and $V := S^n \setminus \{b\}$. Then $S^n = U \cup V$, $U \cap V$ is path-connected for $n \geqslant 2$, and $U \cong V \cong \mathbb{R}^n$. By Seifert–van Kampen Theorem, $\pi_1(S^n)$ is isomorphic to the push-out

$$\pi_1(U) \longleftarrow \pi_1(U \cap V) \longrightarrow \pi_1(V)$$

Since both $\pi_1(U)$ and $\pi_1(V)$ are trivial, the fundamental group $\pi_1(S^n)$ is also trivial. Next we use that $\pi_1(X_1 \times X_2) \cong \pi_1(X) \times \pi_1(Y)$ to conclude that $\pi_1(S^2 \times S^4)$ is trivial.

To compute $\pi_1(Y)$, we will assume for now that $Y = \mathbb{CP}^3$ have a cellular decomposition such that it has a unique 0-cell and no 1-cells (an explicit CW-structure for Y is given in the next part). For such spaces we claim that $\pi_1(Y) = \{e\}$. The strategy is that attaching an n-cell with $n \geqslant 3$ does not change the fundamental group.

Let Y' be a CW-complex obtained by attaching an n-cell e^n with $n \geqslant 3$ to a CW-complex Y'' via the gluing map $\varphi : S^{n-1} \to Y''$. Let V be the interior of e^n and $a \in V$. Then $U := Y' \setminus \{a\}$ deformation retracts to Y'', which means $\pi_1(U) = \pi_1(Y'')$. Note that $Y' = U \cup V$ and $U \cap V \simeq S^{n-1}$ is connected. By Seifert–van Kampen Theorem, $\pi_1(Y')$ is isomorphic to the push-out

$$\pi_1(Y'') \longleftarrow \pi_1(S^{n-1}) \longrightarrow \pi_1(V)$$

Since $\pi_1(V) \cong \pi_1(\mathbb{R}^n)$ and $\pi_1(S^{n-1})$ are trivial, we deduce that $\pi_1(Y') \cong \pi_1(Y'')$. This in particular shows that $\pi_1(Y) \cong \pi_1(S^0) \cong \{e\}$ if Y has only one 0-cell and no 1-cells.

3. For $X = S^2 \times S^4$, we may use the following result (Theorem A.6 in [Hatcher]): if X_1 is a CW-complex

with cells e_α and attaching maps φ_α, and X_2 is a CW-complex with cells e_β and attaching maps φ_β, then the product space $X_1 \times X_2$ has a natural CW-complex structure with cells $e_\alpha \times e_\beta$ and characteristic maps $\Phi_\alpha \times \Phi_\beta$[1]. Moreover, if both X_1, X_2 have at most countably many cells, then the product topology on $X_1 \times X_2$ coincides with the weak topology for the CW-complex.

The CW-complex structure on S^n consists of a 0-cell e^0 and an n-cell e^n with attaching map $\operatorname{im} \varphi^n = e^0$. Hence $X = S^2 \times S^4$ has the CW-complex structure consisting of a 0-cell e^0, a 2-cell $e^2 \cong e^0 \times e^2$, a 4-cell $e^4 \cong e^4 \times e^0$, and a 6-cell $e^6 \cong e^2 \times e^4$. The attaching maps are induced by the product of the corresponding characteristic maps.

For $Y = \mathbb{CP}^3$, we can in fact inductively construct the CW-complex structure on $\mathbb{CP}^n$. The base case $\mathbb{CP}^1 \cong S^2$ is clear — it has a 0-cell e^0 and a 2-cell e^2. Suppose that we have constructed the CW-complex $\mathbb{CP}^{n-1}$. Then let $\mathbb{CP}^n = \mathbb{CP}^{n-1} \cup e^{2n}$, where the attaching map $\varphi : S^{2n-1} \to \mathbb{CP}^{n-1}$ is simply the quotient map. In homogeneous coordinates, we have

$$\mathbb{CP}^{n-1} = \{[z_0 : \cdots : z_{n-1} : 0]\} \subseteq \mathbb{CP}^n$$

and $\mathbb{D}^{2n} \cong \left\{(z_0, \dots, z_{n-1}) \mid \sum_{i=0}^{n-1} |z_i|^2 \leqslant 1\right\}$. The characteristic map $\mathbb{D}^{2n} \to \mathbb{CP}^n$ of e^n is given by

$$(z_0, \dots, z_{n-1}) \longmapsto \left[z_0 : \cdots : z_{n-1} : \sqrt{1 - \textstyle\sum_{i=0}^{n-1} |z_i|^2}\right].$$

Therefore $Y = \mathbb{CP}^3$ has a 0-cell e^0, a 2-cell e^2, a 4-cell e^4, and a 6-cell e^6, and the attaching maps are given as described above.

4. For both $S^2 \times S^4$ and $\mathbb{CP}^3$, the cellular complexes have the following form:

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

This forces all cellular differentials to be zero. Taking homology, we deduce that

$$\mathrm{H}_n(S^2 \times S^4) \cong \mathrm{H}_n(\mathbb{CP}^3) \cong \begin{cases} \mathbb{Z}, & n = 0, 2, 4, 6; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 1.2

Let Pairs be the category of pairs of topological spaces and Ab the category of abelian groups. Fix n a positive integer. Consider the following functors

$$F : \mathsf{Pairs} \to \mathsf{Ab}, \quad (X, A) \mapsto H_n(X, A) \qquad G : \mathsf{Pairs} \to \mathsf{Ab}, \quad (X, A) \mapsto H_{n-1}(A).$$

Prove that the connecting homomorphism $\delta : F \to G$ is a natural transformation.

Recall the a **natural transformation** (or a morphism of functors) $\eta : F \to G$ is a collection of morphisms $\eta_X : FX \to GX$ for each object $X \in \operatorname{Obj} \mathsf{C}$, such that, for any morphism $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

Next we recall the definition of the relative homology. A pair of topological spaces (X, A) is a space X with a subspace $A \subseteq X$. Let $C_\bullet(X)$ be the (singular/cellular) chain complex of X. The inclusion map $\iota : A \hookrightarrow X$ induces

[1]Suppose that $\varphi_\alpha : S^{n-1} \cong \partial \mathbb{D}^n_\alpha \to X^{n-1}$ is the attaching map. Then the characteristic map $\Phi_\alpha : \mathbb{D}^n_\alpha \to X$ is the composition $\mathbb{D}^n_\alpha \to X^{n-1} \sqcup \coprod_\beta \mathbb{D}^n_\beta \to X^n \hookrightarrow X$, and the n-cell $e^n_\alpha := \operatorname{im} \Phi_\alpha$.

the inclusion of complexes $C_\bullet(A) \hookrightarrow C_\bullet(X)$ and hence the short exact sequence of chain complexes:

$$0 \longrightarrow C_\bullet(A) \xrightarrow{\iota_*} C_\bullet(X) \longrightarrow C_\bullet(X,A) \longrightarrow 0$$

where $C_\bullet(X,A) := \operatorname{coker} \iota$. The associated long exact sequence for relative homology takes the form:

$$\cdots \longrightarrow \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(X,A) \xrightarrow{\delta_{X,A}} \mathrm{H}_{n-1}(A) \longrightarrow \cdots$$

A morphism $f\colon (X,A) \to (Y,B)$ in Pairs is a continuous map $f\colon X \to Y$ such that $f(A) \subseteq B$. That is, we have a commutative diagram:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \hookrightarrow & Y \end{array}$$

Using the functoriality of $C_\bullet$, f induces the morphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X,A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_\bullet(B) & \longrightarrow & C_\bullet(Y) & \longrightarrow & C_\bullet(Y,B) & \longrightarrow & 0 \end{array}$$

We claim that this could be extends to a morphism between the long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X,A) & \xrightarrow{\delta_{X,A}} & \mathrm{H}_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}_n(B) & \longrightarrow & \mathrm{H}_n(Y) & \longrightarrow & \mathrm{H}_n(Y,B) & \xrightarrow{\delta_{Y,B}} & \mathrm{H}_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

Then it follows from the claim that δ is a natural transformation. More specifically, we need to check the commutativity of the square at the right-hand side.

In fact, the connecting homomorphism in the relative LES has a direct topological interpretation: it maps the class $[\alpha]$ of a relative cycle $\alpha \in Z_n(X,A)$ to the class $[\partial\alpha]$ of its boundary $\partial\alpha \in Z_{n-1}(A)$. Since the continuous map $f\colon (X,A) \to (Y,B)$ commutes with the boundary operator, then

$$f_*\delta_{X,A}([\alpha]) = f_*[\partial\alpha] = [\partial(f_*\alpha)] = \delta_{Y,B}f_*[\alpha].$$

Alternative, the functoriality of any connecting homomorphism in the LES can be verified by standard diagram chasing. Recall the construction of the connecting morphism δ by snake lemma. In general. suppose that we have a short exact sequence of chain complexes $0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$. Then the connecting morphism $\delta\colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_{n-1}(A_\bullet)$ is given by $[c] \longmapsto [f_{n-1}^{-1}(d_n^B(b))]$, where $b \in g_n^{-1}(c)$.

$$\begin{array}{ccccccccc} & & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n^A} & & \downarrow{\scriptstyle d_n^B} & & \downarrow{\scriptstyle d_n^C} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & & \end{array}$$

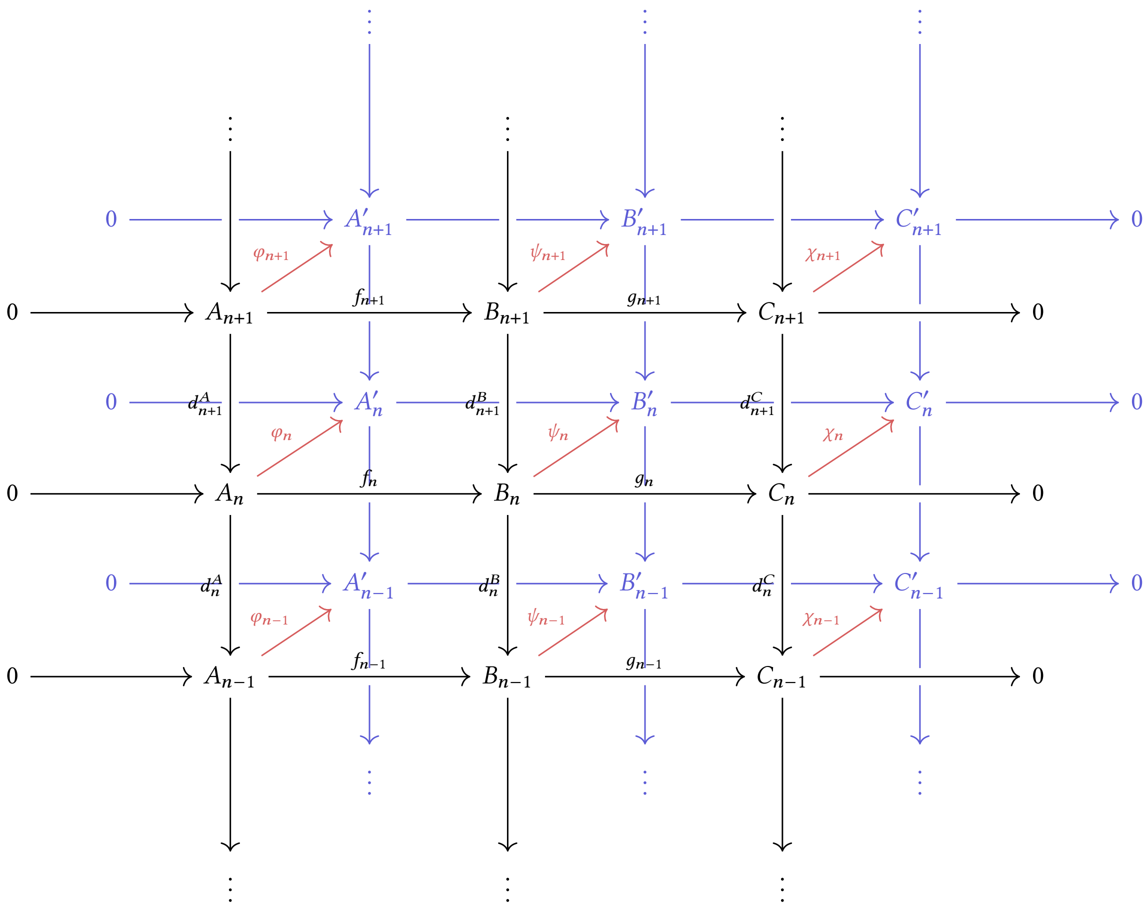
Suppose that there is a morphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{f_\bullet} & B_\bullet & \xrightarrow{g_\bullet} & C_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_\bullet} & & \downarrow{\scriptstyle \psi_\bullet} & & \downarrow{\scriptstyle \chi_\bullet} & & \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{f'_\bullet} & B'_\bullet & \xrightarrow{g'_\bullet} & C'_\bullet & \longrightarrow & 0 \end{array}$$

Then we can check:

$$\varphi_{n-1}(\delta(c)) = \varphi_{n-1}(f_{n-1}^{-1}(d_n^B(b))) = (f'_{n-1})^{-1}(\psi_{n-1}(d_n^B(b))) = (f'_{n-1})^{-1}(d_n^{B'}(\psi_n(b))).$$

Since $c = g_n(b)$, we have that $g'_n(\psi_n(b)) = \chi_n(g_n(b)) = \chi_n(c)$. Hence $\varphi_{n-1}(\delta(c)) = \delta'(\chi_n(c))$. We deduce that $\varphi_{n-1} \circ \delta = \delta' \circ \chi_n$, verifying the commutativity as desired.



Exercise 1.3

Suppose that $A \to B \to C \to 0$ is an exact sequence of abelian groups. Prove that $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is also exact.

For a homomorphism $f : A \to B$, the dual homomorphism $f^* : B^* \to A^*$ is given by $f^*(g) := g \circ f$ for any $g \in B^* = \mathrm{Hom}(B, G)$. Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. That is, g is surjective and $\ker g = \operatorname{im} f$. To show that the dualised sequence $0 \to C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$ is exact, we need to show that (1) g^* is injective; (2) $\operatorname{im} g^* = \ker f^*$.

(1) Let $\varphi \in \ker g^*$ so that $g^*(\varphi) = \varphi \circ g = 0$. For any $c \in C$, there exists $b \in B$ with $b = g(c)$ by surjectivity of g. Then $\varphi(c) = (\varphi \circ g)(b) = 0$. Hence $\varphi = 0$, proving that g^* is injective.

(2) Since $g \circ f = 0$, the dualisation is $f^* \circ g^* = 0$. Hence $\operatorname{im} g^* \subseteq \ker f^*$.

Conversely, let $\psi \in \ker f^*$. For any $a \in A$, $f^*(\psi)(a) = (\psi \circ f)(a) = 0$. Hence $\operatorname{im} f \subseteq \ker \psi$. Since $\operatorname{im} f = \ker g$, this means $\ker g \subseteq \ker \psi$. Define $\chi \in C^*$ by $\chi(c) = \psi(b)$ for some $b \in g^{-1}(c)$. This is well-defined: indeed, if $g(b) = g(b') = c$ for $b, b' \in B$, then $b - b' \in \ker g \subseteq \ker \psi$. Hence $\psi(b) = \psi(b')$. In particular, we have $\psi = \chi \circ g = g^*(\chi)$, showing that $\psi \in \operatorname{im} g^*$. We conclude that $\ker f^* = \operatorname{im} g^*$.

Exercise 1.4

Suppose that $0 \to A \to B \to C \to 0$ is a split short exact sequence of abelian groups. Prove that $0 \leftarrow A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is again a split short exact sequence.

Recall the **splitting lemma**: we say that the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split, if the following equivalent conditions are satisfied:

(1) There exists an isomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \simeq} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & A \oplus C & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

where $\iota : A \hookrightarrow A \oplus C$ is the inclusion and $\pi : A \oplus C \twoheadrightarrow C$ is the projection.

(2) There exists a *retraction* $r : B \longrightarrow A$ (i.e. $r \circ f = \mathrm{id}_A$).

(3) There exists a *section* $s : C \longrightarrow B$ (i.e. $g \circ s = \mathrm{id}_C$).

Back to the question, by Exercise 1.3 we know that $0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$ is exact. Since $r \circ f = \mathrm{id}_A$, dualisation gives $f^* \circ r^* = \mathrm{id}_{A^*}$. Hence f^* is surjective. Therefore we have the short exact sequence $0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \longrightarrow 0$. It is split, because $r^* : A^* \longrightarrow B^*$ provides a section for $f^* : B^* \longrightarrow A^*$.