Week 4

Recall that the short exact sequence in the Universal Coefficient Theorem is **natural**: suppose that $f: X \to Y$ is a continuous map. Then there is a chain map of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(Y);Q) & \longrightarrow & \mathrm{H}^n(Y;Q) & \xrightarrow{h_Y} & \operatorname{Hom}_R(\mathrm{H}_n(Y),Q) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle (f_*)^\vee} & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle (f_*)^\vee} & & \\
0 & \longrightarrow & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(X);Q) & \longrightarrow & \mathrm{H}^n(X;Q) & \xrightarrow{h_X} & \operatorname{Hom}_R(\mathrm{H}_n(X),Q) & \longrightarrow & 0
\end{array}
$$

However, the splitting of the short exact sequence is **not natural**. The dotted lines do not give a commutative diagram in general. We illustrate this by an example.

$$
\begin{array}{ccccccccc}
 & & & & \overset{s_Y}{\longleftarrow\cdots\cdots} & & & & \\
0 & \longrightarrow & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(Y);Q) & \longrightarrow & \mathrm{H}^n(Y;Q) & \xrightarrow{h_Y} & \operatorname{Hom}_R(\mathrm{H}_n(Y),Q) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle (f_*)^\vee} & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle (f_*)^\vee} & & \\
0 & \longrightarrow & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(X);Q) & \longrightarrow & \mathrm{H}^n(X;Q) & \xrightarrow{h_X} & \operatorname{Hom}_R(\mathrm{H}_n(X),Q) & \longrightarrow & 0 \\
 & & & & \underset{s_X}{\longleftarrow\cdots\cdots} & & & &
\end{array}
$$

Moore Spaces

A Moore space $M(G,m)$ is a topological space with reduced homology groups

$$
\widetilde{\mathrm{H}}_n(M(G,m)) = \begin{cases} G, & n = m \\ 0, & \text{otherwise.} \end{cases}
$$

If G is finitely generated, by the structure theorem we have $G \cong \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_s$. If $m > 0$, we can construct $M(G,m)$ as the wedge sum

$$
M(G,m) = \bigvee^r M(\mathbb{Z},m) \vee \bigvee_{i=1}^s M(\mathbb{Z}/p_i,m).
$$

$M(\mathbb{Z},m)$ can be modelled as the m-sphere S^m. For $M(\mathbb{Z}/p,m)$, we can consider the following construction: let X be the CW-complex $S^m \cup_\varphi \mathbb{D}^{m+1}$, where the attaching map $\varphi: \partial\mathbb{D}^{m+1} \cong S^m \to S^m$ has degree p. Therefore the cellular complex of X is given by

$$
\cdots \longrightarrow 0 \longrightarrow \underset{m+1}{\mathbb{Z}} \xrightarrow{p} \underset{m}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots
$$

Hence $\widetilde{\mathrm{H}}_n(X) = \begin{cases} \mathbb{Z}/p, & n = m \\ 0, & \text{otherwise.} \end{cases}$ So X is a model of $M(\mathbb{Z}/p,m)$. By dualising the chain we obtain the reduced cellular cohomology $\widetilde{\mathrm{H}}^n(X) = \begin{cases} \mathbb{Z}/p, & n = m+1 \\ 0, & \text{otherwise.} \end{cases}$

Now we consider the quotient map $q: X \to X/S^m \cong S^{m+1}$. **Claim:** q induces zero maps on $\widetilde{\mathrm{H}}_\bullet$, but non-zero on $\widetilde{\mathrm{H}}^{m+1}$. To show it, we consider the relative homology LES and cohomology LES of the good pair (X, S^m). The relative homology LES gives the homomorphism

$$
q_*: \mathrm{H}_n(X) \to \mathrm{H}_n(X, S^m) \cong \mathrm{H}_n(S^{m+1}).
$$

Note that $\mathrm{H}_n(X) = 0$ for $n \neq 0, m$ and $\mathrm{H}_n(S^{m+1}) = 0$ for $n \neq 0, m+1$. Hence q_* is zero on H_n for $n > 0$.

From the relative cohomology LES, we have

$$\begin{array}{ccccccccccccc} & & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0 & & \\ & & \| & & \| & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & \mathrm{H}^m(X) & \xrightarrow{i^*} & \mathrm{H}^m(S^m) & \xrightarrow{\delta} & \mathrm{H}^{m+1}(S^{m+1}) & \xrightarrow{q^*} & \mathrm{H}^{m+1}(X) & \xrightarrow{i^*} & \mathrm{H}^{m+1}(S^m) & \longrightarrow & \cdots \end{array}$$

The map $q_* : \mathrm{H}^{m+1}(S^{m+1}) \to \mathrm{H}^{m+1}(X)$ is the quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$, which is non-zero. **Claim:** q induces a chain map between the SESs of UCT of H^{m+1}, but not on the splitting of the SESs.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_m(S^{m+1}), \mathbb{Z}) & \longrightarrow & \mathrm{H}^{m+1}(S^{m+1}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{m+1}(S^{m+1}), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (q_*)^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q_*)^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_m(X), \mathbb{Z}) & \longrightarrow & \mathrm{H}^{m+1}(X) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{m+1}(X), \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{\sim} & \mathbb{Z}/p & \xrightarrow{0} & 0 & \longrightarrow & 0 \end{array}$$

Note that the solid lines form a commutative diagram. But it is impossible to construct splittings such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z} & \overset{\sim}{\longleftarrow} & \mathbb{Z} \\ {\scriptstyle q^*}\downarrow & & \downarrow \\ \mathbb{Z}/p & \longleftarrow & 0 \end{array}$$

Back to the general discussion of Moore spaces. The question is how to construct $M(G, m)$ if G **is not finitely generated**. The idea is similar to above but involves infinitely many cells. First we take a free resolution of G:

$$0 \longrightarrow \bigoplus_{\alpha \in I} \mathbb{Z} x_\alpha \xrightarrow{\sigma} \bigoplus_{\beta \in J} \mathbb{Z} y_\beta \longrightarrow G \longrightarrow 0$$

where $\sigma(x_\alpha) = \sum_\beta d_{\alpha\beta} y_\beta$ and $G \cong \operatorname{coker} \varphi$. We can realise this construction as a cellular chain complex. Let $X^m = \bigvee_{\beta \in J} S^m$. For each $\alpha \in I$, we attach an $(m+1)$-cell e^{m+1} to X^m, with the attaching map $\varphi_\alpha : S^m_\alpha \to X^m$ is such that the composition $S^m_\alpha \xrightarrow{\varphi_\alpha} X^m \xrightarrow{q_\beta} S^m_\beta$ has degree $d_{\alpha\beta}$. The resulting cellular chain complex is exactly given by

$$\cdots \longrightarrow 0 \longrightarrow \underset{m+1}{\bigoplus_{\alpha \in I} \mathbb{Z} x_\alpha} \xrightarrow{\sigma} \underset{m}{\bigoplus_{\beta \in J} \mathbb{Z} y_\beta} \longrightarrow 0 \longrightarrow \cdots$$

Hence $\widetilde{\mathrm{H}}_n(X) = \begin{cases} \operatorname{coker} \sigma \cong G, & n = m \\ 0, & \text{otherwise.} \end{cases}$

Using Moore spaces, we can construct a CW-complex X such that $\mathrm{H}_n(X) \cong G_n$ for $n > 0$ and any prescribed Abelian groups G_n.

Eilenberg–MacLane Spaces

An Eilenberg–MacLane space $K(G, m)$ is a topological space with homotopy groups

$$\pi_n(K(G, m)) = \begin{cases} G, & n = m \\ 0, & \text{otherwise.} \end{cases}$$

When $m = 1$, the Eilenberg–MacLane space $K(G, 1)$ can also be characterised as a topological space with fundamental group G and contractible universal cover. We shall see that, in some sense, $K(G, 1)$ 'represents' the first cohomology functor $\mathrm{H}^1(-, G)$.