Week 6

Computation of Ext groups

Example 1.1

1. $\operatorname{Ext}^\bullet_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n)$;
2. $\operatorname{Ext}^\bullet_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$;
3. $\operatorname{Ext}^\bullet_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$. Why do the higher Ext groups not vanish?
4. $\operatorname{Ext}^\bullet_{\mathbb{Z}}(A, \mathbb{Q})$ for any $\mathbb{Z}$-module A.

1. $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) = 0$ for all $i > 0$ because $\mathbb{Z}$ is free.
2. $\mathbb{Z}/m$ admits the following free resolution:

$$\underset{2}{0} \longrightarrow \underset{1}{\mathbb{Z}} \xrightarrow{m} \underset{0}{\mathbb{Z}}$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/n)$, we have

$$\underset{0}{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n)} \xrightarrow{\varphi} \underset{1}{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n)} \longrightarrow \underset{2}{0}$$

where the homomorphism φ is given by $f \mapsto m \cdot f$. Note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) \cong \mathbb{Z}/n$ by the isomorphism $f \mapsto f(1)$.

- $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \ker \varphi \cong \{[k] \in \mathbb{Z}/n : n \mid mk\} \cong \mathbb{Z}/\gcd(m, n)$.
- $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \operatorname{coker} \varphi \cong \dfrac{\mathbb{Z}/n}{m \cdot \mathbb{Z}/n} \cong \mathbb{Z}/\gcd(m, n)$.
- $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = 0$ for $i \geqslant 2$.

(*Use Bezout's lemma to justify the claimed isomorphisms.*)

3. $\mathbb{Z}/2$ admits the following free resolution as a $\mathbb{Z}/4$-module:

$$\cdots \longrightarrow \underset{3}{\mathbb{Z}/4} \xrightarrow{2} \underset{2}{\mathbb{Z}/4} \xrightarrow{2} \underset{1}{\mathbb{Z}/4} \xrightarrow{2} \underset{0}{\mathbb{Z}/4}$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}/4}(-, \mathbb{Z}/2)$, we have

$$\cdots \longleftarrow \underset{3}{\operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2)} \xleftarrow{2} \underset{2}{\operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2)} \xleftarrow{2} \underset{1}{\operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2)} \xleftarrow{2} \underset{0}{\operatorname{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2)}$$

which is isomorphic to the cochain complex

$$\cdots \longleftarrow \underset{3}{\mathbb{Z}/2} \xleftarrow{0} \underset{2}{\mathbb{Z}/2} \xleftarrow{0} \underset{1}{\mathbb{Z}/2} \xleftarrow{0} \underset{0}{\mathbb{Z}/2}$$

Hence $\operatorname{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $n \geqslant 0$.

4. Since $\mathbb{Z}$ is a PID, A admits a free resolution:

$$\cdots \longrightarrow \underset{2}{0} \longrightarrow \underset{1}{F_1} \xrightarrow{\varphi} \underset{0}{F_0}$$

We claim that $\varphi^* : \operatorname{Hom}_{\mathbb{Z}}(F_0, \mathbb{Q}) \to \operatorname{Hom}_{\mathbb{Z}}(F_1, \mathbb{Q})$ is an surjective. Let $\{x_\alpha\}_{\alpha \in I}$ be a basis of F_1 and $\{y_\beta\}_{\beta \in J}$ a basis of F_0. Then $\operatorname{Hom}_{\mathbb{Z}}(F_0, \mathbb{Q}) \cong (\mathbb{Q}^{\oplus J})^*$ and $\operatorname{Hom}_{\mathbb{Z}}(F_1, \mathbb{Q}) \cong (\mathbb{Q}^{\oplus I})^*$ are $\mathbb{Q}$-vector spaces. Since φ is injective, we identify F_1 as a submodule of F_0, and hence $\mathbb{Q}^{\oplus I}$ as a $\mathbb{Q}$-vector subspace of $\mathbb{Q}^{\oplus J}$. Then there exists a splitting $\mathbb{Q}^{\oplus J} \cong V \oplus \mathbb{Q}^{\oplus I}$. For $f \in (Q^{\oplus I})^*$, we define g by extension by zero, that is,

$$g(v) := \begin{cases} f(v), & v \in \mathbb{Q}^{\oplus I} \\ 0, & v \in V \end{cases} \qquad \text{(extend by linearity.)}$$

Then $f = \varphi^*(g)$. Hence φ^* is surjective. As a result, $\operatorname{Ext}^1_{\mathbb{Z}}(A, \mathbb{Q}) = \operatorname{coker} \varphi = 0$.

Remark. In fact $\operatorname{Ext}^n_R(A, I) = 0$ for $n > 0$ if I is an **injective** R-module. For $\mathbb{Z}$-modules, this is equivalent to I being divisible.

Submodule of a free module over a PID

If R is a PID, then it admits a two-step free resolution, and hence $\operatorname{Ext}^i_R(A, B) = 0$ for all A, B and $i \geqslant 2$. This relies on the fact that a submodule of a free R-module is also free. If the module is finitely generated, we can use the structure theorem to deduce this. The general case needs the argument by Zorn's lemma.

Theorem. Every R-submodule of a free R-module M is free when R is a PID.

Proof. Let N be a R-submodule of M. Let X be a basis of M. We consider the set $\mathcal{S}$ of triplets (Y, Z, b), where

- $Z \subseteq Y \subseteq X$;
- $N_Y := N \cap \bigoplus_{y \in Y} Ry$ is free;
- $b : Z \to N$ is a map such that $\operatorname{im} b$ is a basis of N_Y.

Equip $\mathcal{S}$ with the partial order

$$(Y, Z, b) \leqslant_{\mathcal{S}} (Y', Z', b') \iff (Y \subseteq Y') \wedge (Z \subseteq Z') \wedge (b'\big|_Z = b)$$

$\mathcal{S}$ is non-empty, as $(\varnothing, \varnothing, \varnothing) \in \mathcal{S}$. Let $\{(Y_i, Z_i, b_i)\}_{i \in I}$ be a chain in $\mathcal{S}$. Let $Y := \bigcup_i Y_i$, $Z := \bigcup_i Z_i$ and $b = \bigcup_i b_i$. We claim that $(Y, Z, b) \in \mathcal{S}$. Indeed $Z \subseteq Y$. The union $\operatorname{im} b = \bigcup_i \operatorname{im} b_i$ is clearly linearly independent and spans N_Y. Hence N_Y is free.

Now by Zorn's Lemma, $\mathcal{S}$ has a maximal element, which will be denoted again by (Y, Z, b). Hopefully it does not cause any ambiguity in the subsequent discussions.

We claim that $Y = X$. Suppose for contradiction that it is not. Then we take $x \in X \setminus Y$. Consider the ideal

$$I := \left\{ a \in R : \left(ax + \bigoplus_{y \in Y} Ry \right) \cap N \neq \varnothing \right\}$$

If $I = \{0\}$, then $N_{Y \cup \{x\}} = N \cap \left(\bigoplus_{y \in Y} Ry \oplus Rx \right) = N_Y$. We have $(Y, Z, b) <_{\mathcal{S}} (Y \cap \{x\}, Z, b)$. This is a contradiction.

Suppose that $I \neq \{0\}$. Since R is a PID, $I = \langle c \rangle$ for some $c \in R$. Pick

$$m = cx + \sum_i a_i y_i \in \left(cx + \sum_{y \in Y} Ry \right) \cap N$$

We claim that $N_{Y \cup \{x\}} = N_Y \oplus Rm$. For $n \in N_{Y \cup \{x\}}$, $n = \sum_j b_j y'_j + rx$ for some $b, b_i \in R$ and $y'_j \in Y$. Then by definition $r \in \langle c \rangle$. Let $r = sc$ for some $s \in R$. Hence

$$n = \sum_j b_j y'_j + scx = sm + \left(\sum_j b_j y'_j - \sum_i a_i y_i \right) \in N_Y + Rm$$

It is clear that $N_Y \cap Rm = \{0\}$ as $Y \cup \{x\}$ is linearly independent. This proves our claim. Now we let $Z' = Z \cup \{x\}$, $Y' = Y \cup \{x\}$, and $b' : Z' \to N$ which satisfies $b'\big|_Z = b$ and $b'(x) = m$. We have $(Y, Z, b) <_{\mathcal{S}} (Y', Z', b')$. This is a contradiction.

In conclusion, we have $X = Y$. Hence $N_Y = N \cap \bigoplus_{y \in X} Rx = N$ is free. □