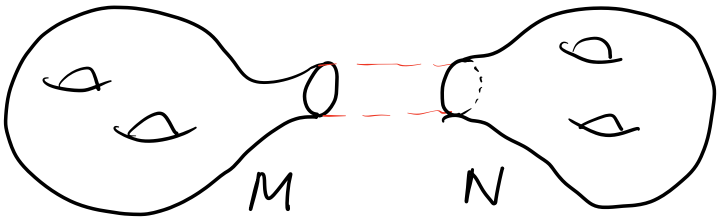
Connected sum of manifolds :

Let M, N be compact, connected n-manifolds.

Can assume smooth mfds so they have finitely generated homology groups.

$M \# N := (M \setminus U) \sqcup (N \setminus V) / \partial U \sim \partial V$

where $U, V \cong D^n$ are closed n-discs.



$\rightsquigarrow M \# N$ is compact connected n-manifold.

Q0 : Orientability ?

Recall : M is orientable $\Rightarrow$ $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$

nonorientable $\Rightarrow$ $H_n(M; \mathbb{Z}) = 0$, $H_n(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$

Q1 : Compute $H_\bullet(M \# N)$ in terms of $H_\bullet(M)$ and $H_\bullet(N)$.

(assuming both M & N orientable)

Good pair $(M \# N, S^{n-1}) \rightsquigarrow (M \# N)/S^{n-1} \cong M \vee N$

$\partial U \cong \partial V \cong \partial D^n \cong S^{n-1}$

Relative homology LES :

$$\cdots \longrightarrow \tilde{H}_i(S^{n-1}) \longrightarrow \tilde{H}_i(M \# N) \longrightarrow \underbrace{\tilde{H}_i(M) \oplus \tilde{H}_i(N)}_{= \tilde{H}_i(M \vee N)} \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \cdots$$

$\Rightarrow \tilde{H}_i(M \# N) \cong \tilde{H}_i(M) \oplus \tilde{H}_i(N)$ for $0 \le i \le n-2$

When $i = n-1, n$:

$$H_n(S^{n-1}) = 0 \longrightarrow H_n(M \# N) \longrightarrow \underbrace{H_n(M \# N, S^{n-1})}_{= \mathbb{Z}^{\oplus 2}} \xrightarrow{\delta} \underbrace{H_{n-1}(S^{n-1})}_{= \mathbb{Z}}$$

$$\longrightarrow H_{n-1}(M \# N) \longrightarrow H_{n-1}(M \# N, S^{n-1}) \longrightarrow 0$$

If both M, N orientable, then $H_n(M \# N, S^{n-1}) \cong H_n(M) \oplus H_n(N) \cong \mathbb{Z}^{\oplus 2}$

Take fundamental class $[M] \in H_n(M)$. So $[M] - [U] \in H_n(M \# N, S^{n-1})$

$\delta([M] - [U]) = [\partial U] = [S^{n-1}]$ is the fundamental class of S^{n-1}

$\Rightarrow \delta$ is surjective.

Clearly $\delta : \mathbb{Z}^{\oplus 2} \to \mathbb{Z}$ is not injective $\Rightarrow H_n(M \# N) \neq 0$

$\Rightarrow$ $M \# N$ is orientable $\Rightarrow H_n(M\#N) \cong \mathbb{Z}$.

Summary:

$$H_i(M\#N) = \begin{cases} \mathbb{Z}, & i = 0, n \\ H_i(M) \oplus H_i(N), & 1 \le i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

Q1′ : Do Q1 again if M is orientable and N is non-orientable.

Same sequence:

$$H_n(S^{n-1}) = 0 \longrightarrow H_n(M\#N) \longrightarrow \underset{\mathbb{Z}}{\underset{\|}{H_n(M\#N, S^{n-1})}} \xrightarrow{\delta} \underset{\mathbb{Z}}{\underset{\|}{H_{n-1}(S^{n-1})}}$$

$$\longrightarrow H_{n-1}(M\#N) \longrightarrow H_{n-1}(M\#N, S^{n-1}) \longrightarrow 0$$

Same argument: δ is surjective. $\Rightarrow$ δ is isomorphism

$\Rightarrow H_n(M\#N) = 0 \Rightarrow$ $M \# N$ is not orientable.

Summary:

$$H_i(M\#N) = \begin{cases} \mathbb{Z}, & i = 0 \\ H_i(M) \oplus H_i(N), & 1 \le i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

Q1″ : What if both M & N are non-orientable? [Harder]

Claim: M non-orientable $\Rightarrow H_{n-1}(M) = \underbrace{\mathbb{Z}/2}_{\text{torsion subgrp}} \oplus \underbrace{\overline{H}_{n-1}(M)}_{\text{free subgrp}}$

Universal coefficient theorem for homology: (off-syllabus?)

$H_n(M;R) \cong H_n(M) \otimes R \oplus \mathrm{Tor}_1^R(H_{n-1}(M), R)$

$\Rightarrow \mathrm{Tor}_1^R(H_{n-1}(M), R) = \{r \in R \mid 2r = 0\}$ [Hatcher Thm 3.26]

$\Rightarrow$ Torsion subgroup of $H_{n-1}(M) \cong \mathbb{Z}/2$

$$H_n(S^{n-1}) = 0 \longrightarrow H_n(M\#N) \longrightarrow \underset{0}{\underset{\|}{H_n(M\#N, S^{n-1})}} \xrightarrow{\delta} \underset{\mathbb{Z}}{\underset{\|}{H_{n-1}(S^{n-1})}}$$

$$\longrightarrow H_{n-1}(M\#N) \longrightarrow H_{n-1}(M\#N, S^{n-1}) \longrightarrow 0$$

$H_n(M\#N) = 0 \Rightarrow$ $M\#N$ not orientable

$\Rightarrow H_{n-1}(X) = \mathbb{Z}/2 \oplus \overline{H}_{n-1}(X)$ for $X = M, N, M\#N$

$$0 \to \mathbb{Z} \to \mathbb{Z}/2 \oplus \overline{H}_{n-1}(M\#N) \to (\mathbb{Z}/2)^{\oplus 2} \oplus \overline{H}_{n-1}(M) \oplus \overline{H}_{n-1}(N) \to 0$$

$\Rightarrow \tilde{H}_{n-1}(M\#N) \cong \mathbb{Z} \oplus \tilde{H}_{n-1}(M) \oplus \tilde{H}_{n-1}(N)$

Summary :

$$H_i(M\#N) = \begin{cases} \mathbb{Z}, & i=0 \\ H_i(M)\oplus H_i(N), & 1\le i\le n-2 \\ \underbrace{\mathbb{Z}/2\oplus \tilde{H}_{n-1}(M)\oplus\mathbb{Z}\oplus\tilde{H}_{n-1}(N)}, & i=n-1 \\ 0, & \text{otherwise} \end{cases}$$

$H_{n-1}(M)\oplus H_{n-1}(N)$ with one of the two $\mathbb{Z}/2$ summand replaced by $\mathbb{Z}$.

From now on assume both M and N are orientable.

Q2 : Compute $H^\bullet(M\#N)$ in terms of $H^\bullet(M)$ & $H^\bullet(N)$.

Cohomological relative LES / UCT for cohomology :

$$H^i(M\#N) = \begin{cases} \mathbb{Z}, & i=0,n \\ H^i(M)\oplus H^i(N), & 1\le i\le n-1 \\ 0, & \text{otherwise} \end{cases}$$

Cup product: $H^k(M\#N)\otimes H^\ell(M\#N) \longrightarrow H^{k+\ell}(M\#N)$

$$(\alpha_k,\beta_k)\cup(\alpha_\ell,\beta_\ell) = \begin{cases} (\alpha_k\cup\alpha_\ell,\ \beta_k\cup\beta_\ell), & k+\ell<n \\ \alpha_k\cup\alpha_\ell+\beta_k\cup\beta_\ell, & k+\ell=n \end{cases}$$

Applications !

Q3. Compute $H^\bullet(\Sigma_g)$, where $\Sigma_g = \overbrace{T^2\#\cdots\#T^2}^{g \text{ times}}$ is the orientable surface of genus g.

$$H^i(\Sigma_g;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0,2 \\ \mathbb{Z}^{\oplus 2g}, & i=1 \\ 0, & \text{otherwise}. \end{cases}$$

If $H^1(T_i^2,\mathbb{Z}) = \mathbb{Z}\alpha_i\oplus\mathbb{Z}\beta_i$, with :

$\alpha_i\cup\alpha_i=\beta_i\cup\beta_i=0$; $\alpha_i\cup\beta_i=1\in H^2(T_i^2;\mathbb{Z})$,

then $H^1(\Sigma_g)\otimes H^1(\Sigma_g)\to H^2(\Sigma_g)$ is given by :

$\alpha_i\cup\alpha_j=\beta_i\cup\beta_j=0$; $\alpha_i\cup\beta_j=\delta_{ij}\in H^2(\Sigma_g;\mathbb{Z})$.

Q4. Show that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are not homeomorphic.

Here $M \# \overline{N}$ is the oriented connected sum, where the orientation of N is reversed.

Both $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ have cohomology groups:

$$H^i = \begin{cases} \mathbb{Z}, & i = 0, 4 \\ \mathbb{Z}^{\oplus 2}, & i = 2 \\ 0, & \text{otherwise} \end{cases}$$

But they have different cup product at $H^2 \otimes H^2 \to H^4$ (this is called the intersection form on the 4-manifolds)

For $S^2 \times S^2$, by Künneth's formula:

$\alpha_i \in H^2(S^2_i) \Rightarrow \alpha_1 \cup \alpha_1 = \alpha_2 \cup \alpha_2 = 0$; $\alpha_1 \cup \alpha_2 = \alpha_2 \cup \alpha_1 = 1$.

Intersection form: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

For $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, by Q2:

$\beta_1 \in H^2(\mathbb{CP}^2)$, $\beta_2 \in H^2(\overline{\mathbb{CP}^2})$

$\Rightarrow \beta_1 \cup \beta_1 = 1$, $\beta_2 \cup \beta_2 = -1$, $\beta_1 \cup \beta_2 = \beta_2 \cup \beta_1 = 0$.

Intersection form: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

It is clear that these two matrices are not congruent over $\mathbb{Z}$.

So $H^\bullet(S^2 \times S^2) \not\cong H^\bullet(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ as graded rings.

Algebraic geometry remarks:

$$\begin{cases} S^2 \times S^2 \text{ is } \mathbb{P}^1 \times \mathbb{P}^1 \\ \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \text{ is } \mathrm{Bl}_x \mathbb{P}^2, \ \mathbb{P}^2 \text{ blown-up at a point.} \end{cases}$$

They are not isomorphic as varieties, because the exceptional divisor of $\mathrm{Bl}_x \mathbb{P}^2$ has self-intersection (-1) (this is β_2 in $H^2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$) but no curve in $\mathbb{P}^1 \times \mathbb{P}^1$ has self-intersection (-1).

Fun fact: $\mathrm{Bl}_{x,y} \mathbb{P}^2 \cong \mathrm{Bl}_x(\mathbb{P}^1 \times \mathbb{P}^1)$!