Constructions of μ-Stable Bundles

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We are going to construct some μ-stable bundles!

Serre's Construction

Slogan. On a surface S, *points (0-dimensional subschemes) on S $\longrightarrow$ rank 2 vector bundles.*

We start with a variety X and a torsion-free sheaf F of rank 1. Then the reflexive hull $L := F^{\vee\vee}$ is a line bundle, and $\mathcal{I}_Z := F \otimes L^\vee$ is an ideal sheaf of a subscheme $Z \subseteq X$ of codimension $\geqslant 2$. That is, $F = L \otimes \mathcal{I}_Z$.

Note that every torsion-free sheaf has a filtration by torsion-free sheaves of rank 1. In particular, every rank 2 torsion-free sheaf E admits an extension:

$$0 \longrightarrow L_1 \otimes \mathcal{I}_{Z_1} \longrightarrow E \longrightarrow L_2 \otimes \mathcal{I}_{Z_2} \longrightarrow 0$$

Question. *When is E a vector bundle?*

It is clear that E cannot be locally free unless $L_1 \otimes \mathcal{I}_{Z_1}$ is a line bundle, i.e. Z_1 is empty. Therefore we are interested in the pair (L, Z), where $L \in \operatorname{Pic} X$ and $Z \subseteq X$ is a subscheme of codimension $\geqslant 2$, such that the extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \otimes \mathcal{I}_Z \longrightarrow 0 \qquad (*)$$

produces a vector bundle E or rank 2. We will have a satisfactory result for the case of surface S.

Definition 0.1. Let $K \in \operatorname{Pic} S$ and $Z \subseteq S$ a 0-dimensional local complete intersection subscheme. We say that the pair (K, Z) satisfies the **Cayley–Bacharach property**, if, for any subscheme $Z' \subseteq Z$ with $\ell(Z') = \ell(Z) - 1$, and a section $s \in \mathrm{H}^0(S, K)$, $s|_{Z'} = 0$ implies that $s|_Z = 0$.

Remark.
1. If $\mathrm{H}^0(S, K) = 0$, then (CB) is satisfied for any Z;
2. If $\mathrm{H}^0(S, K) = \ell$, for a generic Z with $\ell(Z) > \ell$, the sheaf $K \otimes \mathcal{I}_Z$ has no non-trivial sections, and hence (K, Z) satisfies (CB).

Proposition 0.2

Let S be a surface and $Z \subseteq S$ a 0-dimensional local complete intersection subscheme. The sheaf E in $(*)$ is locally free if and only if $(L \otimes \omega_S, Z)$ satisfies the Cayley–Bacharach property.

Proof. " $\Longleftarrow$ ": Assume the Cayley–Bacharach property does not hold, i.e. there exist a subscheme $Z' \subseteq Z$ and a section $s \in H^0(S, L \otimes \omega_S)$ such that $\ell(Z') = \ell(Z) - 1$ and $s|_{Z'} = 0$ but $s|_Z \neq 0$. We have to show that given any extension $\xi : 0 \to L \to E \to M \otimes \mathcal{I}_Z \to 0$ the sheaf E is not locally free.

Use the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{I}_{Z'} \to \mathcal{O}_x \to 0$ induced by the inclusion $Z' \subseteq Z$ and the assumption to show that $\mathrm{H}^1(S, L \otimes \omega_S \otimes \mathcal{I}_Z) \to \mathrm{H}^1(S, L \otimes \omega_S \otimes \mathcal{I}_{Z'})$ is injective. The dual of this map is the natural homomorphism $\operatorname{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_S) \to \operatorname{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)$ which is, therefore, surjective. Hence any extension ξ fits into a commutative diagram of the form

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & E & \longrightarrow & L \otimes \mathcal{I}_Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & E' & \longrightarrow & L \otimes \mathcal{I}_{Z'} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_x & = & \mathcal{O}_x & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\mathcal{O}_S$ and $L \otimes \mathcal{I}_{Z'}$ are torsion-free, so is E'. Hence the sequence $0 \to E \to E' \to \mathcal{O}_x \to 0$ is non-split and E cannot be locally free.

" $\implies$ ": Using that Z is a local complete intersection, we can show that there are only finitely many $Z' \subseteq Z$ such that $\ell(Z') = \ell(Z) - 1$. Suppose now that

$$\xi : 0 \to \mathcal{O}_S \to E \to L \otimes \mathcal{I}_Z \to 0$$

is a non-locally free extension. Then there exists a non-split exact sequence $0 \to E \to E' \to \mathcal{O}_x \to 0$ where $x \in S$ is a singular point of E. The saturation of $\mathcal{O}_S$ in E' can differ from $\mathcal{O}_S$ only in the point x. Since $\mathcal{O}_S$ is locally free, it is saturated in E' as well. Thus we get a commutative diagram of the above form. Hence, the extension class ξ is contained in the image of the homomorphism $\mathrm{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_S) \to \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)$.

Since the Cayley–Bacharach property ensures that the map $\mathrm{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_S) \to \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_S)$ is not surjective, we can choose ξ such that it is not contained in the image of this map for any of the finitely many Z' that could occur. The corresponding E will be locally free. □

Example 0.3. Every point on $\mathbb{P}^2$ gives rise to a μ-semi-stable vector bundle E_x of rank 2.

Proof. Let $S = \mathbb{P}^2$ and $x \in \mathbb{P}^2$. **Claim:** $\mathrm{Ext}^1(\mathcal{I}_x, \mathcal{O}_S) \cong k$.

Short exact sequence: $0 \to \mathcal{I}_x(-3) \to \mathcal{O}_S(-3) \to \mathcal{O}_x \to 0$. Long exact sequence:

$$0 = \mathrm{H}^0(\mathcal{O}_S(-3)) \longrightarrow \mathrm{H}^0(\mathcal{O}_x) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{I}_x(-3)) \longrightarrow \mathrm{H}^1(\mathcal{O}_S(-3)) = 0$$

Hence $k \cong \mathrm{H}^0(\mathcal{O}_x) \cong \mathrm{H}^1(\mathcal{I}_x(-3)) \cong \mathrm{Ext}^1(\mathcal{I}_x, \mathcal{O}_S)$ by Serre duality.

Therefore there exists a unique non-split extension up to scalars:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E_x \longrightarrow \mathcal{I}_x \longrightarrow 0$$

Note that $\mathrm{H}^0(\omega_S) = \mathrm{H}^0(\mathcal{O}_S(-3)) = 0$, so E_x is a vector bundle. E_x is μ-semi-stable because $\mu(E_x) = 0$. □

Serre's construction can produce μ-stable rank 2 vector bundles with prescribed c_1 and large c_2.

Consider the extension $(*)$. The Chern class of E can be computed by the product formula:

$$c_1(E) = c_1(L); \qquad c_2(E) = \ell(Z).$$

Proposition 0.4

Let (S, H) be a smooth polarised surface. For $L \in \mathrm{Pic}\, S$, there exists an integer ℓ_0 such that, for Z a generic ℓ-tuple of points with $\ell > \ell_0$, the extension $(*)$ produces a μ-stable rank 2 vector bundle E with $c_1(E) = c_1(L)$ and $c_2(E) = \ell$.

Proof. We will take $\ell_0 := \max\{\ell_1, \ell_2\}$ where ℓ_1, ℓ_2 will be defined below.

1. Twisting L by $\mathcal{O}_S(nH)$ we may assume that L is sufficiently positive. Let $\ell_1 := \mathrm{h}^0(L \otimes \omega_S)$. If $\ell(Z) > \ell_1$ then a generic Z satisfies (CB) by the remark above. Hence E is a vector bundle.
2. Suppose $M \subseteq E$ were a destabilizing line bundle. It follows from the inequality

$$\mu(M) \geqslant \mu(E) = \frac{1}{2} c_1(L) \cdot H > 0 = \mu(\mathcal{O}_S)$$

   that $M \not\subseteq \mathcal{O}_S$. Thus the composite homomorphism $M \to E \to L \otimes \mathcal{I}_Z$ is non-zero. It vanishes along a divisor D with $Z \subseteq D$ and

$$\deg D = \mu(L) - \mu(M) \leqslant \frac{1}{2} c_1(L) \cdot H =: d.$$

   Let $\ell_2 := \dim Y$, where Y is the Hilbert scheme that parametrizes effective divisors on S of degree $\leqslant d$. An argument of Hilbert schemes shows that if $\ell(Z) > \ell_2$ then for a generic Z, a divisor D containing Z and having degree $\leqslant d$ does not exist. This leads to a contradiction and hence E is μ-stable. □

Now we look at a 3-fold example.

Example 0.5. Fano 3-fold X_{14} of index 1 and genus 8.

First there exists a smooth elliptic curve $C \subseteq X_{14}$ of degree 5 (Lemma 4.9.5 of [IP99]). Let $H = -K_X$ be the ample generator of $\operatorname{Pic} X_{14}$. Consider the Serre's construction:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{O}_X(H) \otimes \mathcal{I}_C \longrightarrow 0$$

E is a rank 2 vector bundle, globally generated, μ-stable, with $\mathrm{h}^0(E) = 6$, $c_1(E) = H$ and $c_2(E) = C$. (*For the proof we might need to give a geometric description of E more than just Serre's construction...See [Gus83].*) It is called the **Mukai bundle** on X_{14} (as a vector bundle with these properties is in fact unique).

The Mukai bundle E on X_{14} determines a morphism $\Phi_E : X_{14} \to \mathrm{Gr}(2,6)$. Here $\mathrm{Gr}(2,6)$ is a 8-dimensional Fano variety of coindex 3 (i.e. index 6) and degree 14 in the Plücker embedding $\mathbb{P}^{14}$. The diagram

$$\begin{array}{ccc} X_{14} & \xrightarrow{\Phi_E} & \mathrm{Gr}(2,6) \\ {\scriptstyle \Phi_{|H|}} \big\downarrow & & \big\downarrow {\scriptstyle \text{Plücker}} \\ \mathbb{P}^9 & \xhookrightarrow{\mathbb{P}^* \lambda_2} & \mathbb{P}^{14} \end{array}$$

is Cartesian ([Muk92]). Therefore X_{14} is a transversal linear section of $\mathrm{Gr}(2,6)$ in $\mathbb{P}^{14}$.

Elementary Transformation

Let S be a smooth surface and $C \subseteq S$ an effective divisor, with $\iota : C \hookrightarrow S$ the inclusion map. Let $F \in \mathsf{Vect}(S)$ and $G \in \mathsf{Vect}(C)$. We say that $E \in \mathsf{Vect}(S)$ is obtained by an **elementary transformation** of F along G if there exists an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \iota_* G \longrightarrow 0$$

The local-freeness of E is much easier than that in the Serre's construction:

Lemma 0.6

Suppose that $F \in \mathsf{Vect}(S)$ and $G \in \mathsf{Vect}(C)$ are locally free. Then $E := \ker(F \twoheadrightarrow \iota_* G)$ is also locally free.

Proof. Recall from Section 1.1 that it suffices to prove that the projective dimension of E is

$$\mathrm{pd}(E) := \max\{\mathrm{pd}(E_x) \mid x \in S\} = 0.$$

Since F is locally free, $\mathrm{pd}(E) = \max\{0, \mathrm{pd}(\iota_* G) - 1\}$, so it suffices to prove that $\mathrm{pd}(\iota_* G) \leqslant 1$. Indeed, we have a locally free resolution

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \iota_*\mathcal{O}_C \longrightarrow 0$$

So $\operatorname{pd}(\iota_*\mathcal{O}_C) \leqslant 1$. Locally we have $\iota_*G = \iota_*\mathcal{O}_C^{\oplus(\operatorname{rk} G)}$. So $\operatorname{pd}(\iota_*G) \leqslant 1$. □

By product formula and Hirzebruch–Riemann–Roch, the Chern class of E is given by

$$\begin{aligned} c_1(E) &= c_1(F) - (\operatorname{rk} G)C; \\ c_2(E) &= c_2(F) - (\operatorname{rk} G)C \cdot c_1(F) + \frac{1}{2}(\operatorname{rk} G)C \cdot ((\operatorname{rk} G)C + K_X) + \chi(G) \\ &= c_2(F) + \deg G - (\operatorname{rk} G)C \cdot c_1(F) + \frac{1}{2}(\operatorname{rk} G)(\operatorname{rk} G - 1)C^2. \end{aligned}$$

Proposition 0.7

Every vector bundle $E \in \mathsf{Vect}(S)$ of rank r it obtained by an elementary transformation of $\mathcal{O}_S^{\oplus r}(nH)$ with $n \gg 0$ along a line bundle on a smooth curve $C \subseteq X$.

Proof. For $n \gg 0$, $E' := E^\vee(nH)$ is globally generated of rank r. Let V be a linear subspace of $\mathrm{H}^0(E')$ of dimension r. The evaluation morphism $V \otimes \mathcal{O}_S \to E'$ has full rank away from a closed subscheme C of codimension 1, which is a smooth curve. Therefore for generic choice of V, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_S^{\oplus r} \longrightarrow E' \longrightarrow \iota_*L \longrightarrow 0$$

where L is a line bundle on C. Dualising the sequence and twisting by $\mathcal{O}_S(nH)$ yields

$$0 \longrightarrow E \longrightarrow \mathcal{O}_S^{\oplus r}(nH) \longrightarrow L' \longrightarrow 0$$

where $L' := \mathscr{E}xt^1(L, \mathcal{O}_S(nH)) \cong \iota_*(L^\vee \otimes \mathcal{O}_C(C + nH))$ is a line bundle on C. Therefore E is obtained by an elementary transformation of $\mathcal{O}_S^{\oplus r}(nH)$ along $L^\vee \otimes \mathcal{O}_C(C + nH)$. □

Proposition 0.8

Let (S, H) be a smooth polarised surface. For $L \in \operatorname{Pic} S$, $r \geqslant 2$, and $c_0 \in \mathbb{Z}$, there exists a μ-stable vector bundle E with $\operatorname{rk}(E) = r$, $c_1(E) = c_1(L)$, and $c_2(E) > c_0$.

Proof. Let C be a smooth curve. According to the Grothendieck Lemma 1.7.9, the torsion free quotients F of $\mathcal{O}_S^{\oplus r}$ with $\mu(F) \leqslant \frac{r-1}{r} C \cdot H$ and $\operatorname{rk}(F) < r$ form a bounded family $\mathcal{C}$.

Now $\dim \operatorname{Hom}\left(\mathcal{O}_S^{\oplus r}, \mathcal{O}_C(nH)\right)$ grows much faster than $\dim \operatorname{Hom}(F, \mathcal{O}_C(nH))$ for any F in the family $\mathcal{C}$. Thus, if n is sufficiently large, a general homomorphism $\varphi : \mathcal{O}_S^{\oplus r} \to \mathcal{O}_C(nH)$ is surjective and does not factor through any $F \in \mathcal{C}$. Let E be the kernel of φ. Then E is locally free with $\det(E) = \mathcal{O}_X(-C)$ and $c_2(E) = nH \cdot C \gg 0$.

In order to see that E is μ-stable, let $E' \subseteq E$ be a saturated proper subsheaf, let F' be the saturation of E' in $\mathcal{O}_S^{\oplus r}$ and consider the subsheaf $F'/E' \subseteq \mathcal{O}_C(nH)$. If F'/E' is nonzero, then $\det(E') = \det(F') \otimes \mathcal{O}_S(-C)$, hence

$$\mu(E') = \mu(F') - \frac{C \cdot H}{\operatorname{rk}(E')} < 0 - \frac{C \cdot H}{\operatorname{rk}(E)} = \mu(E),$$

and we are done. If on the other hand $F'/E' = 0$ then $F := \mathcal{O}_S^{\oplus r}/E'$ is torsion free and φ factors through F.

By construction F cannot be contained in $\mathcal{C}$, hence $\mu(F) > \frac{r-1}{r} C \cdot H$. It follows that

$$\mu(E') = -\frac{r - \operatorname{rk}(E')}{\operatorname{rk}(E')}\mu(F) < -\frac{r - \operatorname{rk}(E')}{\operatorname{rk}(E')} \cdot \frac{r-1}{r} C \cdot H < -\frac{C \cdot H}{r} = \mu(E).$$

So E is indeed μ-stable.

For arbitrary $L \in \operatorname{Pic} S$, pick $m \gg 0$ such that $L^\vee(rmH)$ is very ample and let $C \in |L^\vee(rmH)|$. If E is a μ-stable bundle constructed as above then $c_1(E) = -C = c_1(L) - rmH$. Then $E(mH)$ is μ-stable with $c_1(E) = c_1(L)$ and large c_2. □

Lazarsfeld–Mukai Bundles

We shall use elementary transformations to produce a special type of vector bundle on a K3 surface, following the beautiful paper [Laz86].

Theorem 0.9. Lazarsfeld–Mukai Bundles

Suppose that (S, H) is a K3 surface of genus $g = rs$, where $r, s \in \mathbb{Z}_+$, and $\operatorname{Pic} S = \mathbb{Z}H$. There exists a unique stable vector bundle E on S with

$$\operatorname{rk}(E) = r, \qquad c_1(E) = H, \qquad \operatorname{ch}_2(E) = s - r.$$

Moreover, E is globally generated, μ-stable, $h^0(S, E) = r + s$, and $h^1(S, E) = h^2(S, E) = 0$.

Spoiler: the uniqueness will be treated in Section 6.1! We only show existence this time.

Remark. The Lazarsfeld–Mukai bundle E on the K3 surface S_{2g-2} of genus g therefore determines a morphism $\Phi_E : S_{2g-2} \to \operatorname{Gr}(r, r+s)$. This plays an important rôle in the classification of Fano 3-folds.

Let C be a non-singular curve on the K3 surface S. Suppose that L is a globally generated line bundle on C with $\deg L = d$ and $h^0(C, L) = t + 1$. The surjection $H^0(C, L) \otimes_{\mathbb{C}} \mathcal{O}_C \twoheadrightarrow L$ lifts to S:

$$0 \longrightarrow F \longrightarrow H^0(C, L) \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow \iota_* L \longrightarrow 0 \tag{0.1}$$

where $\iota : C \hookrightarrow S$ is the inclusion and $F := \ker\left(H^0(C, L) \otimes_{\mathbb{C}} \mathcal{O}_S \twoheadrightarrow \iota_* L\right)$. Note that F is obtained by the elementary transformation of $\mathcal{O}_S^{\oplus(t+1)}$ along L, and hence is locally free. We call the dual bundle $E := F^\vee$ the **Lazarsfeld–Mukai bundle** associated to (C, L).

Lemma 0.10

The Lazarsfeld–Mukai bundle E is locally free, with

$$\operatorname{rk}(E) = h^0(L), \quad c_1(E) = C, \quad c_2(E) = \deg L, \quad h^0(E) = h^0(L) + h^1(L), \quad h^1(E) = h^2(E) = 0.$$

If moreover $L^\vee \otimes \omega_C$ is globally generated, then E is also globally generated.

Proof. E is locally free because F is. Note that E fits into the dual sequence of (0.1):

$$0 \longrightarrow H^0(C, L)^\vee \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow E \longrightarrow \iota_*(L^\vee \otimes \omega_C) \longrightarrow 0 \tag{0.2}$$

The Chern class and cohomology groups of E can be computed by that. □

The Lazarsfeld–Mukai bundles are closely related to Brill–Noether theory on K3 surfaces. We recall Brill–Noether theory on curves.

Definition 0.11. Suppose that C is a non-singular projective curve of genus g, and L is a line bundle on a with $\deg L = d$ and $\mathrm{h}^0(C, L) = t + 1$. We define the **Brill–Noether number** to be

$$\rho(g, t, d) := g - (t+1)(g - d + t) = g - \mathrm{h}^0(C, L)\,\mathrm{h}^1(C, L) = \mathrm{h}^0(C, \omega_C) - \mathrm{h}^0(C, L)\,\mathrm{h}^0(C, \omega_C \otimes L^{\check{}}).$$

It is the expected dimension of the **Brill–Noether loci**

$$W_d^t(C) := \{L \in \operatorname{Pic} C \mid \mathrm{h}^0(C, L) \geqslant t + 1,\ \deg L = d\},$$

which is a subvariety of $\operatorname{Pic}^d C$. It is proven by Kempf and Kleiman–Laksov that $W_d^t(C) \neq \varnothing$ for $\rho(g, t, d) \geqslant 0$. If $W_d^t(C) = \varnothing$ whenever $\rho(g, t, d) < 0$, then C is said to be **Brill–Noether general**.

Proposition 0.12. [Laz86]

Suppose that C is a non-singular curve on a K3 surface S such that all curves in $|C|$ are integral. Then C is Brill–Noether general.

Proof. It suffices to prove that for all line bundle L on C, we have $\rho(L) := g - \mathrm{h}^0(C, L)\,\mathrm{h}^1(C, L) \geqslant 0$. First we assume that both L and $L^\vee \otimes \omega_C$ are globally generated. Let E be the Lazarsfeld–Mukai bundle associated to (C, L). We claim that E is simple, i.e. $\operatorname{Hom}(E, E) \cong \mathbb{C}$.

Suppose that E is not simple. Then we pick any $v_0 : E \to E$ which is not a scalar multiple of id. Let λ be the eigenvalue of $v_0(x)$ for some $x \in E$ and consider $v := v_0 - \lambda\,\mathrm{id}$. Then

$$\det v \in \mathrm{H}^0\left(\det E^\vee \otimes \det E\right) \cong \mathrm{H}^0(\mathcal{O}_X)$$

vanishes at x, and hence is identically zero. It follows that $\operatorname{rk}(\ker v) \geqslant 1$. Let $N := \operatorname{im} v$ and $M_0 := \operatorname{coker} v$. Put $M := M_0/\mathrm{T}_1(M_0)$, where $\mathrm{T}_1(M_0)$ is the maximal torsion subsheaf of M_0. We have

$$C = \mathrm{c}_1(E) = \mathrm{c}_1(N) + \mathrm{c}_1(M) + \mathrm{c}_1(\mathrm{T}_1(M_0))$$

in the Picard group of S. Since E is globally generated by Lemma 0.10, N and M are also globally generated being quotients of E. Then $\det N = \mathcal{O}_S(C_1)$ and $\det M = \mathcal{O}_S(C_2)$ for some effective curves C_1, C_2. The torsion-free sheaves N, M are trivial if and only if they have vanishing first Chern class, and they cannot be trivial, as $\mathrm{H}^0(S, E^\vee) = 0$. Hence $C_1, C_2 \neq 0$. Then $C = C_1 + C_2 + \mathrm{c}_1(T(M_0))$ is not integral, which is a contradiction.

Since E is simple, we have $\chi(E, E) = 2 - \dim \operatorname{Ext}^1(E, E) \leqslant 2$. On the other hand, by Lemma 0.10 we have

$$\chi(E, E) = 2\operatorname{rk}(E)\left(\frac{1}{2}\mathrm{c}_1(E)^2 - \mathrm{c}_2(E) + \operatorname{rk}(E)\right) - \mathrm{c}_1(E)^2 = 2(t+1)(g - 1 - d + t) - (2g - 2) = 2 - 2\rho(L).$$

Therefore $\rho(L) \geqslant 0$ as desired.

Return to the general case. Let D_1 be the base locus of L and D_2 the base locus of $(L(-D_1))^\vee \otimes \omega_C$ respectively. Then $L' := L(D_2 - D_1)$ and $L'^\vee \otimes \omega_C$ are globally generated. And we have

$$\rho(L) \geqslant \rho(L(-D_1)) = \rho\big((L(-D_1))^\vee \otimes \omega_C\big) \geqslant \rho\big((L(-D_1))^\vee \otimes \omega_C \otimes \mathcal{O}_C(-D_2)\big) = \rho(L(D_2 - D_1)) \geqslant 0. \quad \square$$

The proposition particularly applies to the case when h generates the Picard group of S:

Corollary 0.13

Suppose that (S, H) is a polarised K3 surface with $\operatorname{Pic} S = \mathbb{Z}H$. Then every curve in $|H|$ is Brill–Noether general.

Proof of Theorem 0.9. Let L be a line bundle on $C \in |H|$ with $\mathrm{h}^0(L) = r$ and $\mathrm{h}^1(L) = s$. Such line bundle exists because

$$\rho(L) = g - \mathrm{h}^0(C, L)\,\mathrm{h}^1(C, L) = 0.$$

By Corollary 0.13, C is Brill–Noether general, which implies that both L and $L^\vee \otimes \omega_C$ are globally generated. Indeed, if D is the base locus of L, then $\mathrm{h}^0(L) = \mathrm{h}^0(L(-D))$ and $\deg L(-D) < \deg L$, which implies that $\rho(L(-D)) < 0$. This is a contradiction. Similar for $L^\vee \otimes \omega_C$.

By Lemma 0.10, the Lazarsfeld–Mukai bundle E associated to (C, L) is globally generated, with numerical invariants

$$\mathrm{rk}(E) = r, \quad \mathrm{c}_1(E) = h, \quad \mathrm{ch}_2(E) = \frac{1}{2}h^2 - \deg L = s - r, \quad \mathrm{h}^0(S, E) = r + s, \quad \mathrm{H}^1(S, E) = \mathrm{H}^2(S, E) = 0.$$

It remains to prove that E is μ-stable. We will prove the μ-stability of $F := E^\vee$. Suppose that $F' \subseteq F$ is a locally free subsheaf of F of rank $r' < r$. As in the proof of Proposition 0.12, we can show that $F'^\vee$ is globally generated and $(\det F')^\vee \cong \mathcal{O}_S(C_1)$ for some non-trivial curve $C_1 \subseteq X$. Since $\mathrm{Pic}\, S = \mathbb{Z}h$, $C_1 = kh$ for some $k \in \mathbb{Z}_+$. We have

$$\mu(F') = \frac{\deg F'}{\mathrm{rk}(F')} = -\frac{kh^2}{r'} < -\frac{h^2}{r} = \frac{\deg F}{\mathrm{rk}(F)} = \mu(F). \qquad \square$$

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