Low-Dimensional Moduli of Sheaves on K3 Surfaces

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We will roughly cover Section 6.1, Example 5.3.7–5.3.8 of [HL10] and some derived-categorical aspects. The whole story is essentially developed by Shigeru Mukai in the marvellous paper [Muk87].

K3 Surfaces: Recap

By K3 surface we always mean a **smooth projective complex K3 surface** S with a fixed polerisation $H \in \operatorname{Amp} S$. the degree of (S, H) is defined by $d := H^2 = 2g - 2$, where g is called the genus of S. Using Serre duality and Riemann–Roch theorem, it is easy to compute the Chern classes and Hodge numbers of a K3 surface S. The Hodge diamond is given by

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

To study vector bundles on the K3 surface, we define the Euler form for coherence sheaves E, F on S:

$$\chi(E, F) := \sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}^i(E, F).$$

By Serre duality, $\operatorname{Ext}^i(E, F) \cong \operatorname{Ext}^{2-i}(F, E)^{\vee}$. So the Euler form $\chi(-,-)$ is symmetric on $\mathsf{Coh}(S)$. The Hirzebruch–Riemann–Roch theorem for Euler form is given by

$$\chi(E, F) = \int_S \operatorname{ch}^{\vee}(E) \operatorname{ch}(F) \operatorname{td}(\mathcal{T}_S),$$

where $\operatorname{ch}^{\vee}{}_i := (-1)^i \operatorname{ch}$. Note that we have for the Euler characteristic,

$$\chi(E) = \chi(\mathcal{O}_S, E) = 2\operatorname{rk}(E) + \operatorname{ch}_2(E).$$

The **Mukai vector** of $E \in \mathsf{Coh}(S)$ is defined to be:

$$v(E) := \operatorname{ch}(E)\sqrt{\operatorname{td}(\mathcal{T}_S)} = (\operatorname{rk}(E), \operatorname{c}_1(E), \operatorname{ch}_2(E) + \operatorname{rk}(E)) = (\operatorname{rk}(E), \operatorname{c}_1(E), \chi(E) - \operatorname{rk}(E)),$$

which is considered as element of $\mathbb{Z} \oplus \operatorname{Pic} S \oplus \mathbb{Z} \subseteq \operatorname{H}^{\bullet}(S; \mathbb{Z})$ or the numerical Grothendieck group $\operatorname{K}_{\text{num}}(S) := \operatorname{K}_0(S)/\ker \chi(-,-)$.

For $v = (r, \ell, s)$ and $v' = (r', \ell', s')$, we define the **Mukai pairing** as the symmetric bilinear form:

$$\langle v, v' \rangle := \ell \cdot \ell' - rs' - sr',$$

so that $\chi(E, F) = -\langle v(E), v(F) \rangle$. Note that the Mukai vector v of a coherent sheaf E determines the Hilbert polynomial of E as

$$P(E, m) = \chi(E \otimes \mathcal{O}_S(mH)) = -\langle v(E), v(\mathcal{O}_S(-mH)) \rangle.$$

So we may consider the moduli space $\operatorname{M}_H^{\text{s}}(v)$ (*resp.* $\operatorname{M}_H^{\text{ss}}(v)$) of stable (*resp.* semi-stable) sheaves on S with Mukai vector $v(E) = v$.

Proposition 0.1

The moduli space $\mathrm{M}^{\mathrm{s}}_H(v)$ of stable sheaves on a K3 surface is either empty, or a non-singular quasi-projective variety of dimension $(\langle v, v\rangle + 2)$.

Proof. We quote Theorem 4.5.4 for a local computation of the dimension of $M := \mathrm{M}^{\mathrm{s}}_h(v)$. Let E be a stable sheaf corresponding to a closed point $[E] \in M$. Then the tangent space $\mathrm{T}_{[E]}M \cong \mathrm{Ext}^1(E, E)$ naturally. And if the trace map

$$\mathrm{tr}^2 : \mathrm{Ext}^2(E, E) \longrightarrow \mathrm{H}^2(S, \mathcal{O}_S) \cong \mathrm{Ext}^2(\det E, \det E)$$

is injective, then M is smooth at $[E] \in M$.

Since S is a K3 surface, for a stable sheaf $[E] \in M$, the trace map $\mathrm{Ext}^2(E, E) \longrightarrow \mathrm{H}^2(S, \mathcal{O}_S)$ is Serre-dual to the map $\mathrm{H}^0(S, \mathcal{O}_S) \longrightarrow \mathrm{Hom}(E, E)$ given by $\lambda \mapsto \lambda\,\mathrm{id}$, which is an isomorphism. Therefore M is non-singular and

$$\dim M = \dim \mathrm{Ext}^1(E, E) = 2 - \chi(E, E) = 2 + \langle v, v\rangle . \qquad \square$$

Remark. There is no stable sheaf E with Mukai vector $v(E) = v$ such that $\langle v, v\rangle < -2$. This translates to an inequality on the Chern classes of E:

$$\Delta(E) - 2(\mathrm{rk}(E)^2 - 1) \geqslant 0.$$

This is much stronger than the Bogomolov's inequality $\Delta(E) \geqslant 0$ for semi-stable sheaves.

Proposition 0.2

Suppose that the Mukai vector $v = (r, \ell, s)$ is **primitive**, i.e. not an integer multiple of another Mukai vector. For a generic polarisation $H \in \mathrm{Amp}(S)$, every μ-semi-stable sheaf is μ-stable. In particular, $\mathrm{M}^{\mathrm{s}}_H(v) = \mathrm{M}^{\mathrm{ss}}_H(v)$.

Proof. This is essentially Theorem 4.C.3. $\square$

Proposition 0.3

Suppose that the Mukai vector $v = (r, \ell, s)$ is such that $\gcd(r, \ell \cdot H, s) = 1$, then $\mathrm{M}^{\mathrm{s}}_H(v)$ is a fine moduli space, and there exists a universal family $\mathscr{E}$ on $\mathrm{M}^{\mathrm{s}}_H(v) \times S$.

Proof. This is Corollary 4.6.7. $\square$

Existence / Non-Emptiness

- By elementary transformation (Theorem 5.2.5), for $v = (r, \ell, s)$ with $r > 0$ and $s \ll 0$, there exists a μ-stable vector bundle E with $v(E) = v$. This result holds for general surfaces.
- For $\langle v, v\rangle = -2$, Kuleshov [Kul90] shows that, there exists a μ-semi-stable, simple ($\mathrm{Hom}(E, E) \cong \mathbb{C}$), rigid ($\mathrm{Ext}^1(E, E) = 0$) vector bundle E with $v(E) = v$. If in addition $\rho(S) = 1$, then E is stable. The vector bundle is first constructed on an elliptic K3 surface and then deformed to arbitrary K3 surfaces.
- For $\langle v, v\rangle = 0$ with $r > 0$, Mukai [Muk87, §5] shows that, there exists a μ_H-semi-stable vector bundle E with $v(E) = v$. If in addition $\ell = H$, then E can be chosen to be μ_H-stable.
- A stronger result: For $\langle v, v\rangle \geqslant -2$ with either $r > 0$ or ℓ ample, there exists a semi-stable sheaf E with $v(E) = v$.

  This follows from a result of Yoshioka [Yos01] that $\mathrm{M}^{\mathrm{s}}_H(v)$ is deformation equivalent to $\mathsf{Hilb}^{\frac{1}{2}\langle v, v\rangle + 1}(S)$ for primitive v. It is also first proved for elliptic K3 surfaces and then deformed to arbitrary K3 surfaces. A thorough exposition is [Vog].

Uniqueness in Dimension 0

The simplest case is when $\langle v, v\rangle = -2$, where the expected dimension of $\mathrm{M}_H^{\mathrm{ss}}(v)$ is 0.

Theorem 0.4

Suppose that $r > 0$ and $\langle v, v\rangle + 2 = 0$. If $\mathrm{M}_H^{\mathrm{s}}(v) \neq \varnothing$, then $\mathrm{M}_H^{\mathrm{ss}}(v)$ consists of a single reduced point which represents a stable vector bundle. In particular $\mathrm{M}_H^{\mathrm{s}}(v) = \mathrm{M}_H^{\mathrm{ss}}(v)$.

Proof. Suppose that $[E] \in \mathrm{M}_h^{\mathrm{s}}(v)$ and $[F] \in \mathrm{M}_h^{\mathrm{ss}}(v)$. Note that

$$\chi(E, F) = \dim \mathrm{Hom}\,(E, F) + \dim \mathrm{Hom}\,(F, E) - \mathrm{Ext}^1\,(E, F) = 2.$$

Therefore either $\mathrm{Hom}\,(E, F) \neq 0$ or $\mathrm{Hom}\,(F, E) \neq 0$. Then $E \cong F$. So $\mathrm{M}_h^{\mathrm{s}}(v) = \mathrm{M}_h^{\mathrm{ss}}(v) = \{[E]\}$.

It remains to prove that E is locally free. It is clear that E is torsion-free, so it embeds into its double dual $E^{\vee\vee}$. Let $Z := E^{\vee\vee}/E$. Since the singular locus of E has codimension $\geqslant 2$, Z is supported on a 0-dimensional subscheme of S. Denote by $\ell(Z)$ the length of $\mathrm{Supp}\,(Z)$. Some computation of homological algebra ([Muk87, Proposition 2.14]) shows that, for any torsion-free E,

$$\dim \mathrm{Ext}^1\,(E^{\vee\vee}, E^{\vee\vee}) + 2\ell(Z) \leqslant \dim \mathrm{Ext}^1\,(E, E).$$

In particular, for simple E with $\chi(E, E) = 2$, we have $\mathrm{Ext}^1\,(E, E) = 0$ and hence $\ell(Z) = 0$. This proves the claim. □

Remark. One can also exploit the fact that $\mathrm{Quot}(E^{\vee\vee}, \ell(Z))$ is irreducible, which implies that $E^{\vee\vee} \twoheadrightarrow Z$ can be deformed to $E^{\vee\vee} \twoheadrightarrow Z_t$ with $\mathrm{Supp}\, Z \neq \mathrm{Supp}\, Z_t$. Hence E deforms non-trivially to $E_t := \ker(E^{\vee\vee} \twoheadrightarrow Z_t)$, contradicting that $\mathrm{Ext}^1(E, E) = 0$.

Suppose that $E \in \mathrm{M}_H^{\mathrm{s}}(v)$ is not only (Gieseker) stable but μ-stable. It is much easier to prove that E is locally free. Indeed, suppose that it is not locally free. Let $E^{\vee\vee}$ be its reflexive hull and then $Z := E^{\vee\vee}/E$ is a 0-dimensional subscheme of length $\ell > 0$. Since E is μ-stable, so is $E^{\vee\vee}$. But

$$\langle v(E^{\vee\vee}), v(E^{\vee\vee})\rangle = \langle v(E), v(E)\rangle - 2\,\mathrm{rk}(E)\ell = -2 - 2\,\mathrm{rk}(E)\ell < 0,$$

which is impossible by Proposition 0.1. This type of argument is very useful in proving the local-freeness of a μ-stable sheaf in a low-dimensional moduli space. It is used in Example 0.10.

Dimension 2

The next simplest case is when $\langle v, v\rangle = 0$, i.e. v is an isotropic vector. The expected dimension of $\mathrm{M}_H^{\mathrm{ss}}(v)$ is 2. The moduli space has a nice geometric description when it is fine.

Theorem 0.5

Suppose that $\langle v, v\rangle = 0$, $\mathrm{M}_H^{\mathrm{ss}}(v) = \mathrm{M}_H^{\mathrm{s}}(v)$, and there exists a universal family $\mathscr{E}$ over $\mathrm{M}_H^{\mathrm{s}}(v) \times S$. Then $\mathrm{M}_H^{\mathrm{s}}(v)$ is a K3 surface.

Remark. Suppose that $\langle v, v\rangle = 0$ and there exists $v' \in \widetilde{\mathrm{H}}^{1,1}(S, \mathbb{Z})$ such that $\langle v, v'\rangle = 1$. Then one can find $H \in \mathrm{Amp}(S)$ such that $\gcd(r, \ell \cdot H, s) = 1$, so that $\mathrm{M}_H^{\mathrm{s}}(v)$ is a fine moduli space. For a generic choice of such H we further have $\mathrm{M}_H^{\mathrm{s}}(v) = \mathrm{M}_H^{\mathrm{ss}}(v)$. So the conditions are satisfied and $\mathrm{M}_H^{\mathrm{s}}(v)$ is a K3 surface.

The proof of this theorem is much clearer from the derived-categorical perspective. We are following [Huy16] instead of [HL10]. The strategy is the following:

1. Construct a non-degenerate 2-form $\omega \in \mathrm{H}^0(M, \Omega_M^2)$, i.e. $M := \mathrm{M}_H^{\mathrm{s}}(v)$ is an algebraic symplectic variety. This shows $\omega_M \cong \mathscr{O}_M$.

2. Using the fact that both S and M has trivial canonical bundle, the Fourier–Mukai transform $\Phi_{\mathscr{E}}$ with the kernel of the universal family $\mathscr{E}$ induces a derived equivalence between $\mathsf{D}^{\mathrm{b}}(S)$ and $\mathsf{D}^{\mathrm{b}}(M)$.
3. Any smooth projective variety derived-equivalent to a K3 surface is also a K3 surface.

Remark. In fact the assumption on the existence of universal family is redundant. One can use the so-called twisted universal sheaf, which is obtained by gluing the universal sheaves $\mathscr{E}_i$ over $U_i \times S$ on an étale covering $\bigcup_i U_i \subseteq R^{\mathrm{s}}$ of M by Luna's étale slice theorem. See [Huy16, Section **10**.2.2] for details.

Step 1

The symplectic structure on M is the content of Chapter 10. We may sketch the idea of construction:

Let $\mathscr{E}$ be the universal family over $M \times S$. Denote the projections by $p: M \times S \to M$ and $q: M \times S \to S$. The **Kodaira–Spencer map** $\mathrm{KS}: \mathscr{T}_M \to \mathscr{E}xt^1_p(\mathscr{E}, \mathscr{E})$ is an isomorphism, where $\mathscr{E}xt^1_p(\mathscr{E}, -)$ is the derived functor $\mathsf{R}^1(p_* \circ \mathscr{H}om(\mathscr{E}, -))$.

Remark. Note that $\mathscr{E}xt^1_p(\mathscr{E}, \mathscr{E})$ and the isomorphism still makes sense even if the universal family does not exist (cf. Section 10.2).

Intuitively, the Kodaira–Spencer map can be thought of the globalisation of the local identification $\mathrm{T}_{[E]}M \cong \mathrm{Ext}^1(E, E)$ of the tangent spaces. The globalisation of the local isomorphism gives the globalised Serre duality:

$$\mathscr{T}_M \times \mathscr{T}_M \xrightarrow{\mathrm{KS}\times\mathrm{KS}} \mathscr{E}xt^1_p(\mathscr{E}, \mathscr{E}) \times \mathscr{E}xt^1_p(\mathscr{E}, \mathscr{E}) \longrightarrow \mathscr{O}_M$$

which is non-degenerate. In particular it defines a nowhere vanishing 2-form $\omega \in \mathrm{H}^0(M, \Omega^2_M)$. On the other hand, this also means an isomorphism $\mathscr{T}_M \cong \Omega_M$, i.e. $\omega_M^{\otimes 2} \cong \mathscr{O}_M$. Using the Pfaffian one can in fact show that $\omega_M \cong \mathscr{O}_M$. So M is either a K3 surface or an Abelian surface by the Enriques' classification.

Step 2

The universal family $\mathscr{E}$ over $M \times S$ defines a Fourier–Mukai transform

$$\Phi_{\mathscr{E}}: \mathsf{D}^{\mathrm{b}}(M) \to \mathsf{D}^{\mathrm{b}}(S), \qquad F \mapsto \mathsf{R}q_*\left(p^*F \overset{\mathsf{L}}{\otimes} \mathscr{E}\right).$$

Lemma 0.6

A Fourier–Mukai transform $\Phi: \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ is fully faithful if and only if for any closed points $x, y \in X$,

$$\mathrm{Hom}(\Phi(\mathscr{O}_x), \Phi(\mathscr{O}_y)[i]) = \begin{cases} \mathbb{C}, & \text{if } x = y \text{ and } i = 0; \\ 0, & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim X. \end{cases}$$

Proof. [Huy06, Proposition 7.1]. □

For $s, t \in M$ with $s \neq t$, $\Phi_{\mathscr{E}}(\mathscr{O}_s) = E_s$ and $\Phi_{\mathscr{E}}(\mathscr{O}_t) = E_t$ are non-isomorphic stable sheaves on S. Hence $\mathrm{Hom}(E_s, E_t) = \mathrm{Hom}(E_t, E_s) = 0$. By Serre duality $\mathrm{Ext}^2(E_s, E_t) = \mathrm{Ext}^2(E_t, E_s) = 0$. Since $\langle v, v\rangle = 0$, we have $\mathrm{Ext}^1(E_s, E_t) = \mathrm{Ext}^1(E_t, E_s) = 0$. For $s = t \in M$, $\Phi_{\mathscr{E}}(\mathscr{O}_s)$ corresponds to a stable sheaf E on S and we know that $\mathrm{Hom}(E, E) = \mathbb{C}$. Hence by the lemma $\Phi_{\mathscr{E}}$ is fully faithful.

Lemma 0.7

Let $\Phi: \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ be a fully faithful Fourier–Mukai transform. Suppose that Φ commutes with the Serre functors, i.e. $\Phi \circ S_X \simeq S_Y \circ \Phi$. Then Φ is an equivalence.

Proof. [Huy06, Corollary 1.56]. □

Since both S and M have trivial canonical bundles, their Serre functors are shift by [2]. In conclusion we have the remarkable result:

Theorem 0.8

The universal family $\mathscr{E}$ over $\mathrm{M}^{\mathrm{s}}_H(v) \times S$ induces an exact equivalence $\Phi_{\mathscr{E}} : \mathsf{D}^{\mathrm{b}}(\mathrm{M}^{\mathrm{s}}_H(v)) \to \mathsf{D}^{\mathrm{b}}(S)$.

Step 3

We will use the derived equivalence $\mathsf{D}^{\mathrm{b}}(M) \simeq \mathsf{D}^{\mathrm{b}}(S)$ to prove that $\mathrm{H}^1(M, \mathscr{O}_M) = 0$ and hence M is a K3 surface. Consider the Leray spectral sequences:

$$E_2^{ij} := \mathrm{H}^i(S, \mathscr{E}xt^j_q(\mathscr{E}, \mathscr{E})) \Longrightarrow \mathrm{Ext}^{i+j}_{M\times S}(\mathscr{E}, \mathscr{E})$$
$$E_2'^{ij} := \mathrm{H}^i(M, \mathscr{E}xt^j_p(\mathscr{E}, \mathscr{E})) \Longrightarrow \mathrm{Ext}^{i+j}_{M\times S}(\mathscr{E}, \mathscr{E})$$

They provide the long exact sequences

$$0 \longrightarrow \mathrm{H}^1(S, \mathscr{E}xt^0_q(\mathscr{E}, \mathscr{E})) \longrightarrow \mathrm{Ext}^1_{M\times S}(\mathscr{E}, \mathscr{E}) \longrightarrow \mathrm{H}^0(S, \mathscr{E}xt^1_q(\mathscr{E}, \mathscr{E})) \longrightarrow \cdots$$
$$0 \longrightarrow \mathrm{H}^1(M, \mathscr{E}xt^0_p(\mathscr{E}, \mathscr{E})) \longrightarrow \mathrm{Ext}^1_{M\times S}(\mathscr{E}, \mathscr{E}) \longrightarrow \mathrm{H}^0(M, \mathscr{E}xt^1_p(\mathscr{E}, \mathscr{E})) \longrightarrow \cdots$$

Note that $\mathrm{H}^1(S, \mathscr{E}xt^0_q(\mathscr{E}, \mathscr{E})) \cong \mathrm{H}^1(S, \mathscr{O}_S) = 0$ and $\mathrm{H}^0(S, \mathscr{E}xt^1_q(\mathscr{E}, \mathscr{E})) \cong \mathrm{H}^0(S, \mathscr{T}_S) = 0$. From the first sequence we have $\mathrm{Ext}^1_{M\times S}(\mathscr{E}, \mathscr{E}) = 0$. The second sequence then implies that $\mathrm{H}^1(M, \mathscr{O}_M) \cong \mathrm{H}^1(M, \mathscr{E}xt^0_p(\mathscr{E}, \mathscr{E})) = 0$.

In fact, the Fourier–Mukai transform tells much more than that:

Theorem 0.9. Derived Torelli's Theorem

If $\Phi_{\mathscr{E}} : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ is an equivalence between the derived categories of K3 surfaces X and Y, then the induced map on the cohomology defines a **Hodge isometry**:

$$\Phi^{\mathrm{H}}_{\mathscr{E}} : \widetilde{\mathrm{H}}(X; \mathbb{Z}) \to \widetilde{\mathrm{H}}(Y; \mathbb{Z}), \qquad \alpha \mapsto q_*(p^*\alpha \cdot v(\mathscr{E})).$$

In fact, $\mathsf{D}^{\mathrm{b}}(X) \simeq \mathsf{D}^{\mathrm{b}}(Y)$ if and only if there exists a Hodge isometry $\widetilde{\mathrm{H}}(X; \mathbb{Z}) \cong \widetilde{\mathrm{H}}(Y; \mathbb{Z})$.

Remark. $\widetilde{\mathrm{H}}(X; \mathbb{Z})$ is the wedge-two Hodge structure with grading given by

$$\widetilde{\mathrm{H}}^{2,0}(X; \mathbb{C}) = \mathrm{H}^{2,0}(X; \mathbb{C}); \qquad \widetilde{\mathrm{H}}^{1,1}(X; \mathbb{C}) = \mathrm{H}^0(X; \mathbb{C}) \oplus \mathrm{H}^{1,1}(X; \mathbb{C}) \oplus \mathrm{H}^4(X; \mathbb{C}); \qquad \widetilde{\mathrm{H}}^{0,2}(X; \mathbb{C}) = \mathrm{H}^{0,2}(X; \mathbb{C}).$$

This can be compared with the classical Torelli's theorem: two K3 surfaces $X \cong Y$ if and only if there exists a Hodge isometry $\mathrm{H}^2(X; \mathbb{Z}) \cong \mathrm{H}^2(Y; \mathbb{Z})$.

Isomorphism as K3 Surfaces

In some special cases the moduli space of sheaves of a K3 surface is isomorphic to the original surface.

Example 0.10

Let $S \subseteq \mathbb{P}^3$ be a general quartic surface. The Picard group of S is generated by the hyperplane divisor H. We have that $M := \mathrm{M}^{\mathrm{s}}_H(2, -H, 1)$ is a K3 surface isomorphic to S.

Remark. Beware of the notation here — $\mathrm{M}^{\mathrm{s}}_H(2,-H,1)$ means the Mukai vector $v = (2,-H,1)$. In other words, the Chern class $(r, \mathrm{c}_1, \mathrm{c}_2) = (2, -H, 3)$.

Proof. The idea is that the points $x \in S$ has a bijective correspondence with stable sheaves $F \in M$ via

$$F := \ker\left(\mathrm{H}^0(S, \mathscr{I}_x(H)) \otimes \mathscr{O}_S \twoheadrightarrow \mathscr{I}_x(H)\right).$$

We start with the opposite direction: take $F \in M$ to be a μ-stable sheaf. If F is not locally free then $F^{\vee\vee}$ is μ-stable with $c_2(F^{\vee\vee}) < 3$. But the $\mathrm{M}^{\mathrm{s}}_H(v(F^{\vee\vee}))$ has expected dimension

$$2 + \langle v(F^{\vee\vee}), v(F^{\vee\vee})\rangle = 4\,\mathrm{c}_2(F^{\vee\vee}) - 10 \geqslant 0.$$

Hence $c_2(F^{\vee\vee}) \geqslant 3$, contradiction. By Hirzebruch–Riemann-Roch, we have $\chi(F) = 3$. Since F is a stable sheaf of negative slope, we have that $\mathrm{h}^0(F) = 0$. Hence by Serre duality $\dim \mathrm{Hom}(F, \mathscr{O}_S) = \mathrm{h}^2(F) \geqslant 3$. A choice of three linearly independent homomorphisms $F \to \mathscr{O}_S$ combines to give $\varphi : F \to \mathscr{O}_S^{\oplus 3}$.

We claim that φ is injective. If not, then then $\mathrm{im}(\varphi)$ would be of the form $\mathscr{I}_Z(aH)$ for some codimension two subscheme Z. Since $\mathscr{I}_V(aH) \subseteq \mathscr{O}_S^3$, one has $a \leqslant 0$. On the other hand, as a quotient of the stable sheaf F the rank one sheaf $\mathscr{I}_Z(aH)$ has non-negative degree. Therefore, $a = 0$. But then

$$\varphi : F \to \mathscr{I}_Z \subseteq \mathscr{O}_S \subseteq \mathscr{O}_S^{\oplus 3}$$

and hence the φ_i would only span a one-dimensional subspace of $\mathrm{Hom}\,(F, \mathscr{O}_S)$, which contradicts our choice. Therefore φ is injective.

A Chern class calculation shows that its cokernel has $(r, \mathrm{c}_1, \mathrm{c}_2) = (1, H, 1)$. We claim that it is torsion-free and hence of the form $\mathscr{I}_x(H)$ for some $x \in S$. If not, let F' be the saturation of F in $\mathscr{O}_S^{\oplus 3}$. Then F' is a rank two vector bundle as well and

$$\det(F) \subseteq \det(F') \cong \mathscr{O}_S(bH) \subseteq \textstyle\bigwedge^2 \mathscr{O}_S^{\oplus 3}.$$

for some $-1 \leqslant b \leqslant 0$. Since both F and F' are locally free, $\det(F') \not\cong \det(F)$; hence $b = 0$. The quotient $\mathscr{O}_S^{\oplus 3}/F'$ then is necessarily of the form $\mathscr{I}_Z$ for a codimension two subscheme Z. But $\mathrm{Hom}\,(\mathscr{O}_S, \mathscr{I}_Z) = 0$ unless $Z = \varnothing$, which then implies that $F' \cong \mathscr{O}_S^{\oplus 2}$, contradicting again the linear independence of the φ_i. Eventually, we see that indeed any $F \in M$ is part of a short exact sequence of the form

$$0 \to F \to \mathscr{O}_S^{\oplus 3} \to \mathscr{I}_x(H) \to 0$$

Since $\mathrm{H}^0(S, F) = 0$, $\mathrm{H}^0(S, \mathscr{O}_S^{\oplus 3}) \to \mathrm{H}^0(S, \mathscr{I}_x(H)) \cong \mathbb{C}^3$ is bijective.[1] Hence $\mathrm{Ext}^1(F, \mathscr{O}_S) \cong \mathrm{H}^1(S, F) = 0$. By Hirzebruch–Riemann–Roch we conclude that $\dim \mathrm{Hom}(F, \mathscr{O}_S) = 3$. That is, the short exact sequence is uniquely determined by F up to the action of $\mathrm{GL}(3)$.

On the other hand, we start with a point $x \in S$ and let $F_{(x)}$ be the sheaf in the exact sequence

$$0 \to F_{(x)} \to \mathrm{H}^0(S, \mathscr{I}_x(H)) \otimes \mathscr{O}_S \to \mathscr{I}_x(H) \to 0 \tag{$\star$}$$

It is clear that $F_{(x)}$ is locally free and has no global section. Suppose that $F_{(x)}$ has a destabilising line bundle F' then $\mu(F') > \mu(F_{(x)}) = -2$. So $F' \cong \mathscr{O}_S$. But it contradicts that $\mathrm{h}^0(F') \leqslant \mathrm{h}^0(F_{(x)}) = 0$. Hence $F_{(x)}$ is stable.

In order to globalize this construction let $\Delta \subseteq S \times S$ denote the diagonal, $\mathscr{I}_\Delta$ its ideal sheaf, and let p and q be the two projections to S. Define a sheaf $\mathscr{F}$ by means of the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow p^* p_*(\mathscr{I}_\Delta \otimes q^* \mathscr{O}_S(H)) \longrightarrow \mathscr{I}_\Delta \otimes q^* \mathscr{O}_S(H) \longrightarrow 0 \tag{$*$}$$

$\mathscr{F}$ is p-flat and $\mathscr{F}_x := \mathscr{F}|_{p^{-1}(x)} \cong F_{(x)}$. Thus $\mathscr{F}$ defines a morphism $S \to M$, $x \longmapsto [F_{(x)}]$. The considerations above show that this map is surjective, because any F is part of an exact sequence of this

[1] To compute $\mathrm{h}^0(S, \mathscr{I}_x(H))$, use the long exact sequence of $0 \to \mathscr{O}_{\mathbb{P}^3}(-3) \to \mathscr{O}_{\mathbb{P}^3}(1) \to \mathscr{O}_S(H) \to 0$ and $0 \to \mathscr{I}_x(H) \to \mathscr{O}_S(H) \to \mathscr{O}_x \to 0$.

form, and injective, because φ is uniquely determined by F. Since both spaces are smooth, $S \to M$ is an isomorphism. □

Note that we have the following commutative diagram where each row and column are exact.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{(x)} & \longrightarrow & \mathrm{H}^0(S, \mathcal{I}_x(H)) \otimes \mathcal{O}_S & \longrightarrow & \mathcal{I}_x(H) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\mathbb{P}^3}(1)\big|_S & \longrightarrow & \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_S & \longrightarrow & \mathcal{O}_S(H) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_x & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Restricting $(*)$ to $\{x\} \times S$ yields $0 \to F_{(x)} \to \mathrm{H}^0(S, \mathcal{I}_x(H)) \otimes \mathcal{O}_S \to \mathcal{I}_x(H) \to 0$, whereas restricting $(*)$ to $S \times \{x\}$ yields $0 \to F_{(x)} \to \Omega_{\mathbb{P}^3}(1)|_S \to \mathcal{I}_x \to 0$.[2] Thus the vector bundle $\mathcal{F}$ on $S \times S$ identifies each factor as the moduli space of the other.

The isomorphism $S \cong M$ can be interpreted as a shift of a spherical twist.

Definition 0.11. Let S be a K3 surface. A **spherical object** in $\mathsf{D}^{\mathrm{b}}(S)$ is an object $E \in \mathrm{Obj}\,\mathsf{D}^{\mathrm{b}}(S)$ such that

$$\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbb{C}, & i = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

A **spherical twist** $T_E : \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}^{\mathrm{b}}(S)$ by a spherical object $E \in \mathrm{Obj}\,\mathsf{D}^{\mathrm{b}}(S)$ is a Fourier–Mukai transform with kernel

$$\mathcal{P}_E := \mathrm{Cone}\left(E^\vee \boxtimes E \longrightarrow (E^\vee \boxtimes E)\big|_\Delta \xrightarrow{\mathrm{tr}} \mathcal{O}_\Delta\right) \in \mathrm{Obj}\,\mathsf{D}^{\mathrm{b}}(S \times S).$$

The spherical twist can be computed as follows:

$$T_E(G) \cong \mathrm{Cone}\left(\bigoplus_i \mathrm{Hom}(E, G[i]) \otimes E[-i] \xrightarrow{\mathrm{ev}} G\right).$$

In other words, the spherical twist induces the exact triangle

$$\bigoplus_i \mathrm{Hom}(E, G[i]) \otimes E[-i] \xrightarrow{\mathrm{ev}} G \longrightarrow T_E(G) \xrightarrow{+1}$$

Lemma 0.12. Seidel–Thomas

Let $E \in \mathrm{Obj}\,\mathsf{D}^{\mathrm{b}}(S)$ be a spherical object. The spherical twist $T_E : \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}^{\mathrm{b}}(S)$ is an auto-equivalence.

For K3 surface S, $\mathcal{O}_S$ is a spherical object. Back to the example, the short exact sequence $(\star)$ implies that $F_{(x)} \cong T_{\mathcal{O}_S}(\mathcal{I}_x(H))[-1]$.

Definition 0.13. If X, Y are K3 surfaces such that there exists a $\mathbb{C}$-linear exact equivalence $\mathsf{D}^{\mathrm{b}}(X) \simeq \mathsf{D}^{\mathrm{b}}(Y)$, then Y is called a **Fourier–Mukai partner** of X. The set of Fourier–Mukai partners of X up to isomorphism is denoted by $\mathrm{FM}(X)$. In fact, every Fourier–Mukai partner of X can be realised as a 2-dimensional Moduli space of stable sheaves on X.

An example of K3 surface without non-trivial Fourier–Mukai partners is an elliptic K3 surface $S \to \mathbb{P}^1$ that admits a section. So any 2-dimension (fine) moduli space of S is isomorphic to S itself.

[2] Use the short exact sequence $0 \to \mathcal{I}_\Delta \otimes q^*\mathcal{O}(1) \to q^*\mathcal{O}(1) \to \mathcal{O}(1)|_\Delta \to 0$ to see that $p_*(\mathcal{I}_\Delta \otimes q^*\mathcal{O}_S(H)) \cong \Omega_{\mathbb{P}^3}(1)|_S$.

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