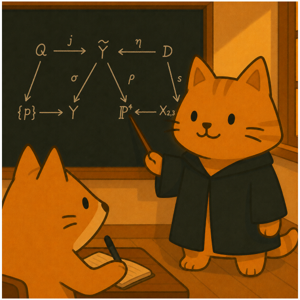


Stability conditions via Verdier localisation:

Nodal cubic 4-folds and K3 surfaces of genus 4

Peize Liu

(*joint with Chunyi Li*)

University of Warwick
Peize.Liu@warwick.ac.uk

14 May 2026

Stability conditions

Let X be a projective variety (always over $\mathbb{C}$ in this talk).
Let $\mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}^{\mathrm{b}}(\operatorname{Coh} X)$ be its bounded derived category of coherent sheaves.

Theorem. (Chunyi Li '26,
Li–Z.Y. Liu–Z.Q. Liu–Macrì–Perry–Stellari–Zhao, *in prep.*)

$\mathrm{D}^{\mathrm{b}}(X)$ *admits a Bridgeland stability condition.*

- Stability manifold?
- Moduli spaces?
- Kuznetsov component $\mathcal{K}u(X) \subseteq \mathrm{D}^{\mathrm{b}}(X)$?

Stability conditions on singular varieties

Recall that the Kuznetsov component $\mathcal{K}u(Y)$ of a hypersurface $Y \subseteq \mathbb{P}^{n+1}$ of degree d is the piece in a semi-orthogonal decomposition:

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \ldots, \mathcal{O}_Y(n-d+1) \rangle, \tag{1}$$

Let Y be a singular variety. Inducing stability conditions from categorical resolution $\mathrm{D}^{\mathrm{b}}(\widetilde{Y}) \to \mathrm{D}^{\mathrm{b}}(Y)$ or $\mathcal{D} \to \mathcal{K}u(Y)$:

- (T.-Y. Chou '24) Stability condition on $\mathrm{D}^{\mathrm{b}}(S)$ for S surface with ADE singularities;
- (T.-Y. Chou '25) Stability condition on $\mathrm{D}^{\mathrm{b}}(Y)$ for Y a nodal quadric 3-fold;
- (Vilches '26) Stability condition on $\mathcal{K}u(Y)$ for Y a nodal cubic 3-fold.

Categorical resolution of nodal singularities

Definition. Let $\mathcal{T}$ be a triangulated category. A *categorical resolution* of $\mathcal{T}$ is a triplet $(\widetilde{\mathcal{T}}, \pi_*, \pi^*)$, where

(a) $\widetilde{\mathcal{T}}$ is an admissible subcategory of $\mathrm{D^b}(Y)$ for some smooth projective variety Y;

(b) $\pi_* : \widetilde{\mathcal{T}} \to \mathcal{T}$ and $\pi^* : \mathcal{T}^{\mathrm{perf}} \to \widetilde{\mathcal{T}}$ are a pair of adjoint functors:

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}(\pi^* E, F) \cong \mathrm{Hom}_{\mathcal{T}}(E, \pi_* F), \quad E \in \mathcal{T}^{\mathrm{perf}},\ F \in \widetilde{\mathcal{T}};$$

(c) The natural transformation $\mathrm{id}_{\mathcal{T}^{\mathrm{perf}}} \to \pi_* \pi^*$ is an isomorphism.

Theorem. (Kuznetsov–Shinder '24,
Cattani–Giovenzana–S. Liu–Magni–Martinelli–Pertusi–Song '23.)

Let Y be a 1-nodal variety of $\dim \geqslant 2$. *Then* $\mathrm{D^b}(X)$ *admits a crepant categorical resolution* $\pi_* : \mathcal{D} \to \mathrm{D^b}(X)$ *such that*

(a) π_* *is a Verdier localisation;*

(b) $\ker \pi_*$ *is generated by a 2-spherical (resp. 3-spherical) object if* $\dim X$ *is even (resp. odd).*

Verdier quotients as boundary of stability manifold

Slogan. (Broomhead–Pauksztello–Ploog–Woolf '22, Bolognese '23)

- On the 'boundary' of $\mathrm{Stab}(\mathcal{T})$, the limiting stability condition σ acquires a set of 'massless objects' $\mathcal{N} \subseteq \ker Z_\sigma$;
- σ descends to a geniune stability condition $\overline{\sigma}$ on the Verdier quotient $\mathcal{T}/\mathcal{N}$ of $\mathcal{T}$ by the subcategory of massless objects.

Applications:

- Describe stability conditions on $\mathrm{D}^{\mathrm{b}}(Y)$ or $\mathcal{K}u(Y)$ by those on the resolution;
- Relate Bridgeland moduli spaces $M_{\overline{\sigma}}(Y, v)$ to $M_\sigma(\widetilde{Y}, \widetilde{v})$...
  Naïve guess: $M_\sigma(\widetilde{Y}, \widetilde{v}) \to M_{\overline{\sigma}}(Y, v)$ is birational (sometimes better, resolution of singularities!)
- Study stable objects in a family $\mathscr{Y} \to S$ degenerating to a nodal variety (Another appoarch: Z.Y. Liu–Mao, *Tilt stability in families for Fano 3-folds,* just posted yesterday.)

Kuznetsov component of nodal cubic 4-fold

If Y is a 1-nodal cubic 4-fold, then $\mathcal{K}u(Y)$ is the Verdier quotient:

$$\mathcal{K}u(Y) \cong \mathrm{D}^{\mathrm{b}}(X_{2,3})/\langle \Sigma \rangle ,$$

where

- $X_{2,3} \subseteq \mathbb{P}^4$ is a K3 surface of genus 4; and
- Σ is the spherical bundle on $X_{2,3}$ with Mukai vector $v = (2, -h, 2)$.

Theorem. Main results.

(1) *The weak stability condition*

$$\sigma = \sigma_{-\frac{1}{2},0} := \left(\mathrm{Coh}_h^{-\frac{1}{2}}(X_{2,3}), \quad Z_{-\frac{1}{2},0} := -\operatorname{ch}_2 + \mathrm{i}(h \operatorname{ch}_1 + 3\operatorname{rk}) \right)$$

on $\mathrm{D}^{\mathrm{b}}(X_{2,3})$ *induces a stability condition* $\overline{\sigma}$ *on* $\mathcal{K}u(Y)$.

(2) *For any primitive numerical character* $v \in \mathrm{K}_{\mathrm{num}}(\mathcal{K}u(Y))$*, there exists a lift* $\widetilde{v} \in \mathrm{K}_{\mathrm{num}}(X_{2,3})$ *and a birational map*

$$\varphi : \ M_h(\widetilde{v}) \dashrightarrow M_{\overline{\sigma}}(v)$$

where

- $M_h(\widetilde{v})$ *is the Gieseker moduli space with Mukai vector* v*; and*
- φ *is given by a sequence of flops followed by a divisorial contraction.*

Formal stuff on stability conditions

Categorical set-up

- $\mathcal{D}$ a $\mathbb{C}$-linear triangulated category, and $\mathrm{K}_0(\mathcal{D})$ its Grothendieck group.
- Finite-rank lattice Λ and a surjective group hom $\lambda\colon \mathrm{K}_0(\mathcal{D}) \to \Lambda$.
- $\mathcal{A}$ a **heart of bounded t-structure** on $\mathcal{D}$ (an Abelian subcategory of $\mathcal{D}$).

Definition. (Stability function / central charge.) A **weak stability function** is a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ such that, for any $E \in \mathcal{A} \setminus \{0\}$,

$$Z(\lambda(E)) \in \left\{z = m \cdot \mathrm{e}^{\mathrm{i}\pi\phi} \mid m \geqslant 0,\ \phi \in (0,1]\right\} = \mathbb{H} \cup \mathbb{R}_{\leqslant 0}.$$

$\phi = \phi(E)$ is called the phase of E. If we require further that $Z(\lambda(E)) \neq 0$ for $E \neq 0$, then Z is called a **stability function**.

$E \in \mathcal{A}$ is (semi)stable if for $0 \neq F \subsetneq E$, we have $\phi(F) <(\leqslant) \phi(E/F)$.

The slope of E with respect to Z:

$$\mu_Z(E) = -\cot(\pi\phi(E)) = \begin{cases} -\frac{\operatorname{Re} Z(\lambda(E))}{\operatorname{Im} Z(\lambda(E))}, & \operatorname{Im} Z(\nu(E)) > 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

(Weak) stability conditions

Definition. A **(weak) pre-stability condition** on $\mathcal{D}$ wth respect to Λ is a pair $\sigma = (\mathcal{A}, Z)$, where

- $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and
- $Z\colon \Lambda \to \mathbb{C}$ is a (weak) stability function, satisfying the
- **Harder–Narasimhan property**: for any $E \in \mathcal{A}$, there exists a filtration

  $$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell =: E$$

  by objects E_i in $\mathcal{A}$, such that the graded factors E_i/E_{i-1} are semi-stable of phase ϕ_i, and

  $$\phi^+(E) \coloneqq \phi_1 > \cdots > \phi_\ell =: \phi^-(E).$$

A (weak) pre-stability condition is called a (weak) stability condition if it satisfies the **support property**, namely, for any fixed norm $\|-\|$ on $(\Lambda/\Lambda_0) \otimes \mathbb{R}$, there exists $C > 0$ such that for any σ-semistable $E \in \mathcal{A}$, we have:

$$\|\overline{\lambda}(E)\| \leqslant C|Z(E)|.$$

Here $\Lambda_0 \subseteq \Lambda$ is the saturation of the sublattice generated by $\ker Z$, and $\overline{\lambda}\colon \mathrm{K}_0(\mathcal{D}) \to \Lambda/\Lambda_0$ is the induced homomorphism.

Bridgeland deformation theorem

Denote by $\mathrm{Stab}_\Lambda(\mathcal{D})$ the set of stability conditions on $\mathcal{D}$ with respect to Λ.

A celebrated result of Bridgeland:

Theorem. (Bridgeland '07)

If $\mathrm{Stab}_\Lambda(\mathcal{D})$ *is non-empty, then it can be equipped with a structure of complex manifold of dimension equal to* $\mathrm{rk}\,\Lambda$*, such that the forgetful map*

$$\begin{aligned} \mathrm{Forg}\colon\ \mathrm{Stab}_\Lambda(\mathcal{D}) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \\ \sigma = (\mathcal{A}, Z) &\longmapsto Z \end{aligned}$$

is a local homeomorphism.

Metric on $\mathrm{Stab}_\Lambda(\mathcal{D})$:

$$d(\sigma_1, \sigma_2) = \sup_{E \in \mathcal{D}} \max \left\{ |\phi_1^+(E) - \phi_2^+(E)|,\ |\phi_1^-(E) - \phi_2^-(E)|,\ \|Z_1 - Z_2\| \right\}.$$

Inducing stability conditions by Verdier quotient

Let $\pi : \widetilde{\mathcal{T}} \to \mathcal{T}$ be a Verdier localisation. Assume that $\sigma = (\widetilde{\mathcal{A}}, \widetilde{Z})$ be a weak (pre-)stability condition on $\widetilde{\mathcal{T}}$ satisfying

$$\ker\left(\pi|_{\widetilde{\mathcal{A}}}\right) = \ker\left(\widetilde{Z}|_{\widetilde{\mathcal{A}}}\right).$$

- $\mathcal{A} \coloneqq \pi(\widetilde{\mathcal{A}})$ is a heart of bounded t-structure of $\mathcal{T}$;
- $\pi|_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} \to \mathcal{A}$ is exact;
- There is a unique $Z \in \mathrm{Hom}(\mathrm{K}_0(\mathcal{T}), \mathbb{C})$ such that $Z = \widetilde{Z} \circ \pi_*$.

Proposition.

The data $\pi_\flat\sigma \coloneqq (\mathcal{A}, Z)$ defines a (pre-)stability condition on $\mathcal{T}$. Moreover,

(1) If $E \in \widetilde{\mathcal{A}}$ is σ-(semi)stable and $\pi(E) \neq 0$, then $\pi(E) \in \mathcal{A}$ is $\pi_\flat\sigma$-(semi)stable.

(2) If $F \in \mathcal{A}$ is $\pi_\flat\sigma$-semistable, then there exists some σ-semistable $E \in \widetilde{\mathcal{A}}$ such that $F = \pi(E)$.

Nodal cubic and K3

Geometric set-up

$$\begin{array}{ccc} Q & \xhookrightarrow{\;j\;} & \widetilde{Y} \\ \swarrow & & \swarrow \sigma \\ \{p\} & \hookrightarrow & Y \end{array}$$

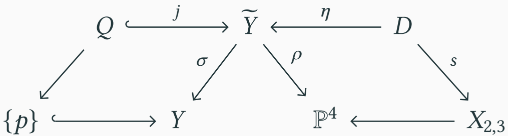
(Kuznetsov ’10)

- Let $Y \subseteq \mathbb{P}^5$ be a cubic 4-fold with a node $p \in Y$ and smooth otherwise. Choose coordinates such that Y is given by

$$Y : \; x_0 f(x_1, \ldots, x_5) + g(x_1, \ldots, x_5) = 0.$$

- $\sigma : \widetilde{Y} \to Y$ is the blow-up at $\{p\}$; expcetional divisor $Q \subseteq \widetilde{Y}$ is a smooth quadric.

Geometric set-up



(Kuznetsov '10)

- Let $Y \subseteq \mathbb{P}^5$ be a cubic 4-fold with a node $p \in Y$ and smooth otherwise. Choose coordinates such that Y is given by

$$Y: \; x_0 f(x_1, \ldots, x_5) + g(x_1, \ldots, x_5) = 0.$$

- $\sigma : \widetilde{Y} \longrightarrow Y$ is the blow-up at $\{p\}$; expcetional divisor $Q \subseteq \widetilde{Y}$ is a smooth quadric.
- The rational map $Y \dashrightarrow \mathbb{P}^4$ lifts to a morphism $\rho : \widetilde{Y} \longrightarrow \mathbb{P}^4$, and we have $\widetilde{Y} \cong \mathrm{Bl}_{X_{2,3}} \mathbb{P}^4$, where $X_{2,3}$ is the smooth K3 surface of genus 4 defined by the equations

$$X_{2,3}: \; f(x_1, \ldots, x_5) = g(x_1, \ldots, x_5) = 0.$$

- Geometrically, $X_{2,3}$ parametrises lines in Y passing through the node p, and $D \longrightarrow X_{2,3}$ is the 'universal family'.

Categorical resolution of the Kuznetsov component

$$\begin{array}{ccccccc} & Q & \xhookrightarrow{\ j\ } & \widetilde{Y} & \xleftarrow{\ \eta\ } & D & \\ \swarrow & & \sigma \swarrow & & \searrow \rho & & \searrow s \\ \{p\} & \hookrightarrow & Y & & \mathbb{P}^4 & \longleftarrow & X_{2,3} \end{array}$$

Proposition. (Kuznetsov '10, CGLMMPS '23.)

$\mathcal{K}u(Y)$ admits a crepant categorical resolution $\pi_* : \mathrm{D}^{\mathrm{b}}(X_{2,3}) \to \mathcal{K}u(Y)$ given by

$$\pi_* = \sigma_* \circ \Phi = \sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(-h)} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(-2h)} \circ (- \otimes \mathcal{O}_{\widetilde{Y}}(-H+h)) \circ \eta_* \circ s^*.$$

Moreover, the kernel $\ker \pi_*$ is generated by a 2-spherical object $\Sigma \in \mathrm{D}^{\mathrm{b}}(X_{2,3})$.

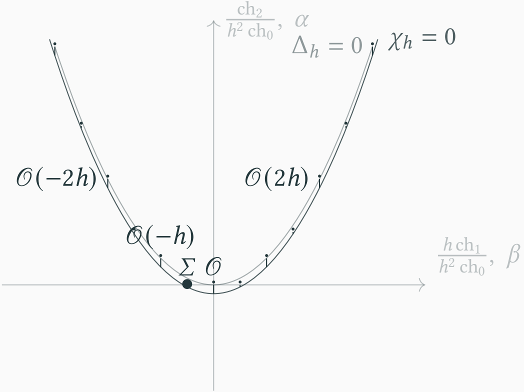
- $H \coloneqq \sigma^* \mathcal{O}_Y(1)$ and $h \coloneqq \rho^* \mathcal{O}_{\mathbb{P}^5}(1)$;
- $\Sigma \coloneqq \Sigma_Q|_{X_{2,3}}$, where Σ_Q is the rank-2 spinor bundle on the 3-dimensional smooth quadric $\rho \circ j(Q)$. Mukai vector $v(\Sigma) = (2, -H, 2)$.
- Verdier quotient: $\mathcal{K}u(Y) \cong \mathrm{D}^{\mathrm{b}}(X_{2,3}) / \langle \Sigma \rangle$.

Tilt stability on K3 surface

$(X_{2,3}, h)$ is a K3 surface with $h^2 = 6$. Tilt stability conditions cut out a 2-dimensional slice of $\mathrm{Stab}_h^*(X_{2,3})$:

$$\sigma_{\beta,\alpha} \coloneqq \left(\mathrm{Coh}_h^{\beta}(X_{2,3}), \quad Z_{\beta,\alpha} \coloneqq -(\mathrm{ch}_2 - \alpha h^2 \mathrm{ch}_0) + \mathrm{i}(h\,\mathrm{ch}_1 - \beta h^2 \mathrm{ch}_0)\right)$$

Think of each point (β, α) both as a tilt stability $\sigma_{\beta,\alpha}$ and a projectivised Chern character (via $\ker Z_{\beta,\alpha}$):



Induced morphisms between moduli spaces

Given $v \in \mathrm{K}_{\mathrm{num}}(\mathcal{K}u(Y))$, there is a lift $\widetilde{v} \in \mathrm{K}_{\mathrm{num}}(X_{2,3})$ such that $\pi_*(\widetilde{v}) = v$. What is the 'best lifting' that gives a good notion of morphism $M_\sigma(X_{2,3}, \widetilde{v}) \to M_{\overline{\sigma}}(\mathcal{K}u(Y), v)$?

- Numerical level: want $\chi(\Sigma, \widetilde{v}) = 0$ to get a largest $M_\sigma(X_{2,3}, \widetilde{v})$.
- Object level: we say that $F \in \mathrm{D}^{\mathrm{b}}(X_{2,3})$ is Σ-normal if $\mathrm{RHom}(\Sigma, F)$ is concentrated in degree 0.

Lemma.

Every object $F \in \mathrm{D}^{\mathrm{b}}(X_{2,3})$ has a unique Σ-normalisation.

Think of Σ-normalisation as a section of the Verdier quotient $\mathrm{D}^{\mathrm{b}}(X_{2,3}) \to \mathcal{K}u(Y)$.

- Which σ to choose?

MMP for moduli spaces on K3 surface

Let $\widetilde{v} \in [\Sigma]^{\perp}$ be a primitive class. Let L be the wall joining $\widetilde{v}$ and $[\Sigma]$, and σ_L be a stability condition in a neighbourhood of $\sigma_{-\frac{1}{2},0}$ in L.

Proposition. (Bayer–Macrì '14.)

There are birational maps:

$$M_h(\widetilde{v}) \overset{f}{\dashrightarrow} M_{\sigma_{-\frac{1}{2},+}}(\widetilde{v}) \xrightarrow{g} M_{\sigma_L}(\widetilde{v}) \tag{2}$$

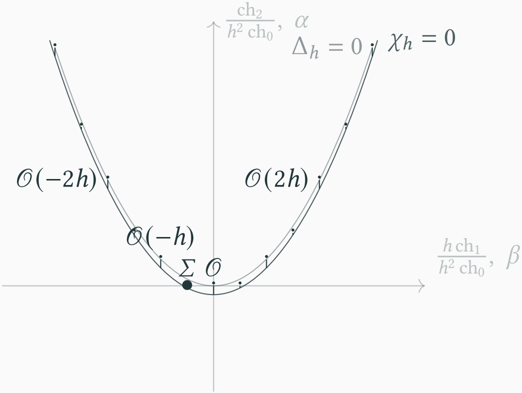
Here f is a sequence of flops, and g is a divisorial contraction with exceptional locus given by the σ_L-stable objects F satisfying $\mathrm{Hom}(\Sigma, F) \neq 0$.

Proposition.

There is an isomorphism

$$M_{\sigma_L}(\widetilde{v}) \cong M_{\overline{\sigma}}(\mathcal{K}u(Y), v).$$

MMP for moduli spaces on K3 surface



An example: Fano variety of lines

Hilbert scheme of 2 points on K3 surface

For rest of the talk, Y is a 1-nodal cubic 4-fold that does not contain a plane.

$$\begin{tikzcd} & Q \arrow[r, hook, "j"] \arrow[dl] & \widetilde{Y} \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & D \arrow[l, "\eta"'] \arrow[dr, "s"] & \\ \{p\} \arrow[r, hook] & Y & & \mathbb{P}^4 & X_{2,3} \arrow[l] \end{tikzcd}$$

Recall that $X_{2,3}$ parametrises lines in Y passing through p.
What does $\mathrm{Hilb}^2(X_{2,3})$ *parametrise?*

- $Z \in \mathrm{Hilb}^2(X_{2,3})$ defines either two lines ℓ_1, ℓ_2 in Y passing through p, or a double line $\boldsymbol{\ell}$ in Y passing through p.
- Either way, they span a plane Π in $\mathbb{P}^5$, and for degree reason, $\Pi \cap Y = \ell_1 \cup \ell_2 \cup \ell_Z$ or $\Pi \cap Y = \boldsymbol{\ell} \cup \ell_Z$.
- The assignment defines a map to the Fano variety of lines $\mathcal{F}_1(Y)$:

$$\varphi : \mathrm{Hilb}^2(X_{2,3}) \longrightarrow \mathcal{F}_1(Y),$$
$$Z \longmapsto \ell_Z.$$

Hilbert scheme of 2 points on K3 surface

Proposition. (Hassett '00.)

The map $\varphi : \operatorname{Hilb}^2(X_{2,3}) \to \mathcal{F}_1(Y)$ is a blow-up:

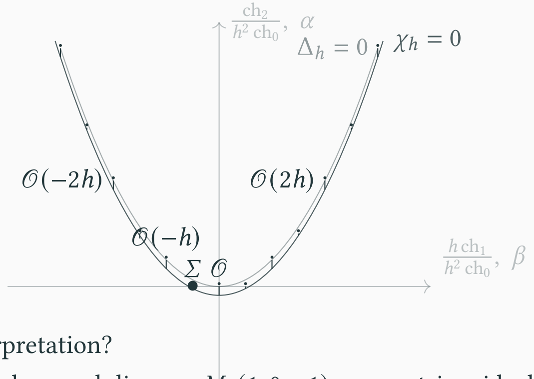
$$\begin{array}{ccc} \Psi & \longrightarrow & \operatorname{Hilb}^2(X_{2,3}) \\ \downarrow & & \downarrow \varphi \\ X_{2,3} & \longrightarrow & \mathcal{F}_1(Y) \end{array}$$

The exceptional divisor Ψ (the '*trident divisor*') has the following equivalent characterisations:

(a) $Z \in \Psi$;
(b) ℓ_Z passes through the node p;
(c) The line $\lambda_Z \subseteq \mathbb{P}^4$ defined by Z is contained in the quadric $\rho \circ j(Q)$;
(d) $\operatorname{length}(\lambda_Z \cap X_{2,3}) = 3$;
(e) $\operatorname{Hom}(\Sigma, \mathcal{I}_Z) \neq 0$.

Modular interpretation?

Wall-crossing for the Mukai vector $v = (1, 0, -1)$



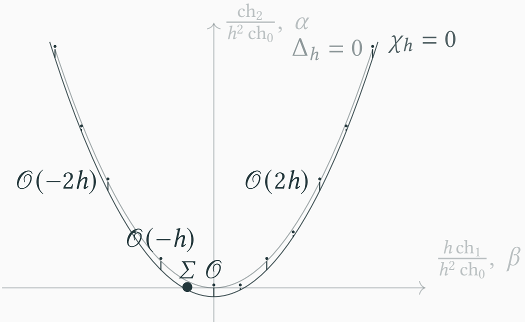
Modular interpretation?

- The Gieseker moduli space $M_h(1, 0, -1)$ parametrises ideal sheaves $\mathcal{I}_Z$ and is isomorphic to $\mathrm{Hilb}^2(X_{2,3})$.
- On the wall L joining v and $[\Sigma]$, if $Z \in \Psi$, the destabilising sequence for $\mathcal{I}_Z$ is given by

$$0 \longrightarrow \mathcal{I}_q(-h) \longrightarrow \Sigma \longrightarrow \mathcal{I}_Z \longrightarrow 0,$$

where $q \in X_{2,3}$ is the point corresponding to the residual line ℓ_Z.

Wall-crossing for the Mukai vector $v = (1, 0, -1)$



Modular interpretation?

- The divisorial contraction $M_h(1, 0, -1) \to M_{\sigma_L}(1, 0, -1)$ contracts $[\mathcal{I}_Z]$ to $[\mathcal{I}_q(-h) \oplus \Sigma]$, i.e., contracts Ψ to $X_{2,3}$.
- This is exactly φ! We get

$$\mathcal{F}_1(Y) \cong M_{\sigma_L}(1, 0, -1) \cong M_{\overline{\sigma}}(\mathcal{K}u(Y), \lambda_1).$$

Moduli space $M_{\overline{\sigma}}(\mathcal{K}u(Y), \lambda_1)$

What are the objects in $M_{\overline{\sigma}}(\mathcal{K}u(Y), \lambda_1)$? After a lengthy computation...

Theorem.

The moduli space $M_{\overline{\sigma}}(\mathcal{K}u(Y), \lambda_1)$ parametrises two types of objects:

1. *(perfect) $\mathcal{F}_\ell$, for ℓ not passing through the node;*
2. *(non-perfect) 3-term complexes $\pi_*\mathcal{I}_Z$, for ℓ passing through the node, with cohomology*

$$\mathcal{H}^i(\pi_*\mathcal{I}_Z) \cong \begin{cases} \mathcal{F}_\ell^\vee(-H), & i = -1; \\ \bigwedge^2 \mathcal{Q}_p(-H) \oplus \mathcal{O}_{\ell_Z}(-H), & i = 0; \\ \mathcal{O}_p, & i = 1, \end{cases}$$

where $\mathcal{Q}_p \coloneqq \operatorname{coker}(\mathcal{O}_Y(-H) \hookrightarrow \mathcal{O}_Y^{\oplus 5})$ is associated to the linear system $|\mathcal{I}_{p|X}(H)|$.

Here $\mathcal{F}_\ell \coloneqq \operatorname{pr}(\mathcal{O}_Y(H))$ is the rank 3 syzygy sheaf:

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow \mathcal{O}_Y^{\oplus 4} \longrightarrow \mathcal{I}_Y(H) \longrightarrow 0.$$

Question. Put this in a family $M_\sigma(\mathscr{Y}, \lambda_1)$, with smooth general fibres and nodal central fibre?