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Pressure estimates in two dimensional incompressible fluid flow ☆

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Abstract

We derive sharp pressure estimates for two dimensional incompressible fluid flow, in terms of natural quantities such as enstrophy, energy and angular momentum. We cover both the Euler and Navier–Stokes equations, and both periodic planar flows and spherical flows. ©2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a two dimensional incompressible fluid with bounded enstrophy, can we control the oscillation of pressure? A first inspection of the equations, armed with classical elliptic regularity theory and Sobolev inequalities, would suggest that this is impossible and that for fluids with vorticity concentrated in an appropriate manner, the ratio of the oscillation of pressure and the enstrophy could be arbitrarily large.

However, in this paper, we harness recent developments in the nonlinear theory of partial differential equations concerning the compensation properties of Jacobian determinants, to show that on the contrary, universal bounds are possible and that we can give many of them in optimal form. For our treatment of spherical flows, we must develop this compensation theory further, to handle Jacobian determinants of *vector fields* on surfaces rather than simply determinants of maps into surfaces supporting isoperimetric inequalities as in [7]. Our analysis gives estimates for the fluid viewed at a given instant of time. Estimates of a more dynamical nature then require only the computation of the evolution of enstrophy in addition.

Let us begin by detailing the Navier–Stokes equations. The planar equations for the velocity vector field $v : [0,T]\times\mathbb{R}^2\to\mathbb{R}^2$ and pressure $p:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are

$$\begin{aligned} &\partial_t v + (v\cdot\nabla)v - \nu\Delta v + \nabla p = 0,\\ &\nabla\cdot v = 0, \qquad\qquad .\\ &v(0,\cdot) = v_0, \end{aligned} \tag{1}$$

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where Δ is the Laplacian, and $\nu > 0$ is the viscosity. With $\nu = 0$, these equation become the Euler equations.

Meanwhile, on the sphere $S^2 \hookrightarrow \mathbb{R}^3$, the velocity is a tangent vector field $v : [0, T] \times S^2 \to \mathbb{R}^3$. We postulate the equations

$$\begin{aligned} &\partial_t v + (v \cdot \nabla)v + |v|^2 u - \nu(\Delta v + 2v) + \nabla p = 0, \\ &\mathrm{div}_{s^2} v = 0, \\ &v(0, \cdot) = v_0, \end{aligned} \qquad (2)$$

where $u \in S^2 \hookrightarrow \mathbb{R}^3$ is the space coordinate (also the unit normal vector) and $p : [0, T] \times S^2 \to \mathbb{R}$ is again the pressure. The surface divergence div_{s^2} at a point $u \in S^2$ is defined by

$$\mathrm{div}_{s^2} v = e_1 \cdot \nabla(v \cdot e_1) + e_2 \cdot \nabla(v \cdot e_2),$$

for any orthonormal basis e_1, e_2 of the tangent space $T_u S^2$, and Δ is now the Laplace–Beltrami operator, which is applied to the three components of v independently.

Note that in the spherical equations, the normal vector $|v|^2 u$ cancels the normal component of $(v \cdot \nabla)v$. (Indeed, $(v \cdot \nabla)v + |v|^2 u$ corresponds to the covariant derivative $\nabla_v v$.) The other additional term, $-2\nu v$, prevents the viscosity term from dissipating angular momentum, as we shall see. Of course, $(\Delta v + 2v)$ is simply the ordinary Laplacian in $\mathbb{R}^3$ of the linear radial extension of v to the ambient $\mathbb{R}^3$; this extension will be useful in some of our calculations. Note that the incompressibility condition is equivalent to the Δv term having no normal component (i.e. $u \cdot \Delta v = 0$). A remark on the physically correct viscosity term for the Navier–Stokes equations on curved manifolds may be found in ([4], Added in proof).

Two dimensional incompressible spherical flow equations such as those given have been used to study basic dynamical questions concerning the evolution of large scale atmospheric vortices. Typically, the viscosity term is small, and is adapted to suit numerical calculations; in the case of zero viscosity, the vorticity (see below) is often restricted to take only a finite number of values, with each region of constant vorticity referred to as a *vortex patch*. A starting point for further references is [5].

Let us define various relevant physical quantities for spherical flows. The vorticity $\omega : S^2 \to \mathbb{R}$ of v is taken to be

$$\omega(u) = u \cdot (\nabla \times v),$$

where v is extended arbitrarily to a neighbourhood of S^2 in $\mathbb{R}^3$. Ubiquitous in this work are the global physical quantities

$$\text{Enstrophy} := \int_{S^2} \omega^2, \quad \text{and} \quad \text{Kinetic energy} := \int_{S^2} |v|^2.$$

These will normally refer implicitly to the velocity field at time t, in other words to $v(t) := v(t, \cdot)$, for some flow v. The normalised angular momentum is simply

$$\Omega(v) = \frac{3}{8\pi} \int_{S^2} u \times v.$$

Finally, we define two renormalised quantities, the *excess* kinetic energy

$$K(v) = \int_{S^2} |v|^2 - \frac{8\pi}{3} |\Omega(v)|^2, \qquad (3)$$

and the *excess* enstrophy

$$W(v) = \int_{S^2} \omega^2 - \frac{16\pi}{3} |\Omega(v)|^2. \tag{4}$$

As we shall see, the excess kinetic energy and enstrophy are always positive; they correspond to the same physical quantity computed for the velocity field obtained by projecting v onto the orthogonal complement of the first eigenspace of Δ.

For planar flows we shall require merely the vorticity $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}$ (occasionally written $\hat{\omega}_v$) which is given analogously by

$$\hat{\omega}(x, y) = \partial_x v^2 - \partial_y v^1,$$

where $v = (v^1, v^2)$.

In the sequel, we talk of 'admissible' solutions. These may be taken to be *smooth* solutions for simplicity; there is a classical theory providing smooth solutions given smooth initial data v_0 (in contrast to the three-dimensional case) even in the case $\nu = 0$ of inviscid flows.

Theorem 1. *Suppose that* (v, p) *is an admissible solution to the spherical equations* (2). *Then at each time* t, *denoting the average value of* $p(t, \cdot)$ by $\overline{p_t}$, *the pressure is controlled in terms of the enstrophy and kinetic energy according to*

$$-\frac{1}{2\pi}\int_{S^2} \omega^2 + \frac{1}{4\pi}\int_{S^2} |v|^2 \leq p(t, u) - \overline{p_t} \leq \frac{1}{2\pi}\int_{S^2} \omega^2 + \frac{1}{4\pi}\int_{S^2} |v|^2, \tag{5}$$

for all $u \in S^2$, *and hence*

$$\operatorname{osc}(p(t, \cdot)) \leq \frac{1}{\pi}\int_{S^2} \omega^2. \tag{6}$$

Here, the oscillation of a function f is defined to be

$$\operatorname{osc}(f) = \operatorname*{ess\,sup}_{x,y} |f(x) - f(y)|.$$

The enstrophy and kinetic energy are calculated at time t in Theorem 1. However, in the zero viscosity case $\nu = 0$, these quantities do not depend on time, as one would expect (see Section 3) so we may calculate them for v_0 instead of $v(t, \cdot)$.

Looked at upside-down, inequality (6) of Theorem 1 tells us that if we take two local measurements of pressure, then any discrepancy gives us an explicit lower bound on the global enstrophy.

The senses in which Theorem 1 is sharp will be discussed in Section 4. Roughly speaking, if the vorticity is suitably concentrated, the lower pressure bound is achieved in the limit.

As we shall justify later on (see Section 3) the Poincaré inequality tells us that the kinetic energy may be controlled optimally in terms of the enstrophy:

Proposition 1. *For any velocity field* v *on* S^2, *we have*

$$\int_{S^2} |v|^2 \leq \frac{1}{2}\int_{S^2} \omega^2. \tag{7}$$

Therefore, (5) may be simplified (and weakened) to an inequality involving no more than the pressure and enstrophy

$$-\frac{1}{2\pi}\int_{S^2}\omega^2 \le p(t,u) - \overline{p_t} \le \frac{5}{8\pi}\int_{S^2}\omega^2. \tag{8}$$

More useful when the excess enstrophy is small, is the following result.

Theorem 2. *Suppose that* (v, p) *is an admissible solution to the spherical equations* (2). *Then at each time* t *we have*

$$\operatorname{osc}(p(t,\cdot)) \le \frac{1}{2}|\Omega(v_0)|^2 + \frac{1}{\pi}W(v(t)) + 2\sqrt{\frac{3}{\pi}}\,|\Omega(v_0)|\,W(v(t))^{1/2}. \tag{9}$$

This result will turn out to be more useful for $t \gg (1/\nu)$ because the excess enstrophy decays over that timescale; we prove the following proposition in Section 3.

Proposition 2. *The excess enstrophy* $W(v(t))$ *decays according to*

$$W(v(t)) \le W(v_0)\mathrm{e}^{-8\nu t}.$$

Combining Theorem 2 and Proposition 2 gives

$$\operatorname{osc}(p(t,\cdot)) \le \frac{1}{2}|\Omega(v_0)|^2 + \frac{1}{\pi}W(v_0)\mathrm{e}^{-8\nu t} + 2\sqrt{\frac{3}{\pi}}\,|\Omega(v_0)|\,W(v_0)^{1/2}\mathrm{e}^{-4\nu t}. \tag{10}$$

Theorem 2 and its consequence (10) are optimal in rather different ways to Theorem 1. As clarified in Section 4, we have equality for ‘uniformly rotating’ flows.

Let us now turn our attention to planar flows, and in particular, to those which are doubly periodic. In what follows, Γ is some lattice in $\mathbb{R}^2$, and $\mathbb{T}$ is defined to be the torus $\mathbb{R}^2/\Gamma$.

Theorem 3. *Suppose that* (v, p) *is an admissible solution to the planar equations* (2) *which is periodic with respect to* Γ. *Then at each time* t, *we have*

$$\operatorname{osc}(p(t,\cdot)) \le \frac{1}{2\pi}\int_{\mathbb{T}}\hat{\omega}^2. \tag{11}$$

This estimate is sharp for suitable velocity fields with highly-concentrated vorticity (see Section 4).

In (11) the enstrophy is calculated at time t. However, it will become clear in Section 3 (although we suppress the proof) that the enstrophy decays exponentially according to

$$\int_{\mathbb{T}}\hat{\omega}^2_{v(t)} \le \mathrm{e}^{-2\nu\lambda_1(\mathbb{T})t}\int_{\mathbb{T}}\hat{\omega}^2_{v_0},$$

where $\lambda_1(\mathbb{T})$ is the first eigenvalue of the Laplacian on $\mathbb{T}$.

2. Instantaneous pressure estimates

In this section we will derive the equations for the pressure, put them into a suggestive form, and then exploit the hidden regularity properties of Jacobian determinants to obtain the desired pressure estimates. For spherical flows, we will be led by the philosophy of the theory, whilst for planar flows, we apply the current theory directly. It has already been observed by Tartar [6] that the pressure equation for planar flows enjoys a geometric structure which may be exploited to obtain unexpected estimates.

Let us recall one of the results we proved in [7] which in fact holds for any compact Riemannian surface once we have made sense of the quantities involved.

Theorem 4. *Suppose that* $u \in H^1(\mathbb{T}, \mathbb{R}^2)$*, and that* φ *is a solution in* $W_0^{1,1}(\mathbb{T}, \mathbb{R})$ *to*

$$-\Delta\varphi = \det(\nabla u), \tag{12}$$

(which is unique up to the addition of a constant). Then we have the estimate

$$\mathrm{osc}(\varphi) \leq \frac{1}{4\pi}\|\nabla u\|^2_{L^2(\mathbb{T})}. \tag{13}$$

Note that at first glance, the Jacobian determinant $\det(\nabla u)$ appears to be controllable only in L^1 which in itself is too weak to conclude that φ lies in L^∞. A broader discussion of the geometry behind this theorem may be found in [7]. A Hardy space interpretation of this type of result on $\mathbb{R}^n$ was given in [3]. Previous estimates of this form have been developed by Wente [8], Brezis and Coron [2] and others, although these results are not applicable in the generality required here (see [7] for a survey).

Proof (Theorem 3). Let us turn our attention to the system of equations (1). Writing the velocity field in coordinates $v(x, y) = (a(x, y), b(x, y))$, the incompressibility condition is $a_x + b_y = 0$, and taking the divergence of the principal equation yields

$$\partial_t(\nabla\cdot v) + \nabla\cdot((v\cdot\nabla)v) - \nu\Delta(\nabla\cdot v) + \Delta p = 0.$$

Therefore using incompressibility,

$$\begin{aligned}-\Delta p &= \nabla\cdot((v\cdot\nabla)v) = (aa_x + ba_y)_x + (ab_x + bb_y)_y\\ &= 2(b_xa_y - a_xb_y) - ab_{xy} + ba_{xy} + ab_{xy} - ba_{xy} = -2\det(\nabla v),\end{aligned} \tag{14}$$

where we use the shorthand $a_x = \partial_x a$. Applying Theorem 4 directly we obtain

$$\mathrm{osc}(p) \leq \frac{1}{2\pi}\|\nabla v\|^2_{L^2(\mathbb{T})},$$

but a short calculation, using incompressibility, reveals that

$$\hat{\omega}^2 = |\nabla v|^2 + 2\det(\nabla v), \tag{15}$$

which after integration allows us to conclude that

$$\mathrm{osc}(p) \leq \frac{1}{2\pi}\int_{\mathbb{T}}\hat{\omega}^2. \qquad \square$$

Fluid flow in the plane is blessed with a velocity field taking values in $\mathbb{R}^2$; the fact that $\mathbb{R}^2$ is equipped with an isoperimetric inequality is then used directly to achieve the unexpected regularity of Theorem 4. For spherical fluid flow, the situation is more complicated with the velocity a section of a nontrivial tangent bundle. However, we are able to make progress by following the philosophy of the proof of Theorem 4, and our task is made easier by the symmetry of the Green function on S^2; in this sense, the proof has points in common with previous work of Baraket [1].

Before proving Theorem 1, let us record the spherical equivalent of (15) which is a simple calculation appealing implicitly to the fact that the Gauss curvature of the standard 2-sphere is identically equal to one.

Proposition 3. *Given a velocity field* v *on* S^2, *we may define a 1-form*

$$\alpha(u) = u \cdot (v \times \mathrm{d}v).$$

Then

$$\mathrm{d}\alpha = (2u \cdot (v_x \times v_y) - \rho^2 |v|^2)\,\mathrm{d}x \wedge \mathrm{d}y = (\omega^2 - |\nabla v|^2)\rho^2\,\mathrm{d}x \wedge \mathrm{d}y, \tag{16}$$

where x *and* y *are local isothermal coordinates for the position* $u \in S^2 \hookrightarrow \mathbb{R}^3$, *and* $\rho^2 = |u_x|^2 = |u_y|^2$, *and therefore*

$$\int_{S^2} \omega^2 - \int_{S^2} |\nabla v|^2 = \int_{S^2} (2u \cdot (v_x \times v_y) - \rho^2 |v|^2)\,\mathrm{d}x \wedge \mathrm{d}y = 0.$$

It is worth clarifying that $|\nabla v|^2$ in (16) above refers to the *coordinate independent* harmonic energy density $|\nabla v|^2 = (1/\rho^2)(|v_x|^2 + |v_y|^2)$.

We may now give the spherical analogue to Theorem 4.

Theorem 5. *Suppose that* $v \in H^1(S^2, \mathbb{R}^3)$ *is a tangent vector field on the 2-sphere, and that* φ *is the unique solution in* $W_0^{1,1}(S^2, \mathbb{R})$ *to*

$$-\Delta\varphi = \frac{1}{\rho^2} u \cdot (v_x \times v_y) - \frac{1}{2}|v|^2, \tag{17}$$

having an average value of zero (where x *and* y *are again local isothermal coordinates, and* ρ *is defined as before). Then we have the pointwise estimate*

$$-\frac{1}{4\pi}\int_{S^2} |\nabla v|^2 - \frac{1}{8\pi}\int_{S^2} |v|^2 \le \varphi \le \frac{1}{4\pi}\int_{S^2} |\nabla v|^2 - \frac{1}{8\pi}\int_{S^2} |v|^2. \tag{18}$$

Note that with x, y and ρ as in the preceeding theorem, the Laplace–Beltrami operator is simply

$$\Delta = \frac{1}{\rho^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Theorem 5 is the main ingredient in the proofs of Theorems 1 and 2.

Proof (Theorem 1). Analogously to the proof of Theorem 3, we begin by taking the surface divergence of the flow equations (2). Using the incompressibility condition carefully, we find that $\mathrm{div}_{s^2}((v \cdot \nabla)v) = -2(1/\rho^2)u \cdot (v_x \times v_y) - |v|^2$, and $\mathrm{div}_{s^2}(|v|^2 u) = 2|v|^2$, and therefore the pressure satisfies the equation

$$-\Delta p = -2\left(\frac{1}{\rho^2} u \cdot (v_x \times v_y) - \frac{1}{2}|v|^2\right). \tag{19}$$

Applying Theorem 5 gives

$$-\frac{1}{2\pi}\int_{S^2} |\nabla v|^2 + \frac{1}{4\pi}\int_{S^2} |v|^2 \le p \le \frac{1}{2\pi}\int_{S^2} |\nabla v|^2 + \frac{1}{4\pi}\int_{S^2} |v|^2,$$

and we conclude by applying the final part of Proposition 3. □

Our proof of Theorem 5 is fairly direct, but hidden within are the seeds of an ‘isoperimetric inequality for vector fields’. If we were to generalise the result to handle vector fields on more general surfaces, it might be more appropriate to mimic the theory developed in [7] more closely, and work explicitly with some such isoperimetric inequality.

Proof (Theorem 5). We may assume, via an approximation argument, that v (and hence φ) are smooth. Recall that the Green function on the sphere

$$G(\hat{u}, u) = -\frac{1}{2\pi}\ln\left(\frac{|u-\hat{u}|}{2}\right),$$

is the solution of the equation

$$-\Delta G(\hat{u}, u) = \delta_{u,\hat{u}} - \frac{1}{4\pi}.$$

Let us stereographically project the sphere onto the plane, sending $\hat{u}$ to the origin. This gives us stereographic cartesian and polar coordinates (x, y) and (r, θ) respectively, corresponding to the point $u \in S^2$.

In these coordinates, $G(u) = (1/4\pi)\ln(1 + (1/r^2))$, and so

$$\frac{\mathrm{d}G}{\mathrm{d}r} = -\frac{1}{4\pi}\frac{\rho}{r}, \tag{20}$$

where ρ, defined as before, may now be given explicitly as $\rho(r) = 2/(1 + r^2)$. Since the volume form on S^2 is given by $\rho^2 \mathrm{d}x \wedge \mathrm{d}y$, Green's representation for φ is

$$\varphi(\hat{u}) = \int_{S^2} G(\hat{u}, u)\left(\frac{1}{\rho^2} u\cdot(v_x \times v_y) - \frac{1}{2}|v|^2\right)\rho^2 \mathrm{d}x \wedge \mathrm{d}y,$$

and using Proposition 3 and (20) we progress with

$$\varphi(\hat{u}) = \int_{S^2} G(\hat{u}, u)\frac{1}{2}\mathrm{d}(u\cdot(v\times \mathrm{d}v)) = -\frac{1}{2}\int_{S^2}\mathrm{d}G \wedge (u\cdot(v\times\mathrm{d}v)) = \frac{1}{8\pi}\int_0^\infty \frac{\rho}{r}\left(\int_0^{2\pi} u\cdot(v\times v_\theta)\,\mathrm{d}\theta\right)\mathrm{d}r. \tag{21}$$

Let us write v in terms of the orthogonal basis of the tangent space of S^2 suggested by the cartesian coordinates (x, y). Explicitly,

$$v = \alpha u_x + \beta u_y.$$

Therefore the magnitude of v is given by $|v|^2 = \rho^2(\alpha^2 + \beta^2)$. Differentiating with respect to y, say, we find

$$v_y = \alpha_y u_x + \alpha u_{xy} + \beta_y u_y + \beta u_{yy}, \tag{22}$$

and therefore

$$\begin{aligned} u\cdot(v\times v_y) &= (u\times v)\cdot v_y = (\alpha u_y - \beta u_x)\cdot(\alpha_y u_x + \alpha u_{xy} + \beta_y u_y + \beta u_{yy}) \\ &= \alpha\beta_y\rho^2 + \alpha^2(u_x\cdot u_{xy}) + \alpha\beta(u_y\cdot u_{yy}) - \beta\alpha_y\rho^2 - \alpha\beta(u_x\cdot u_{xy}) - \beta^2(u_x\cdot u_{yy}). \end{aligned}$$

Exploiting the conformality of stereographic projection gives us the 'connection coefficients'

$$\begin{aligned} &u_y\cdot u_{xy} = \tfrac{1}{2}(|u_y|^2)_x = \rho\rho_x, \quad u_y\cdot u_{yy} = \tfrac{1}{2}(|u_y|^2)_y = \rho\rho_y, \\ &u_x\cdot u_{xy} = \tfrac{1}{2}(|u_x|^2)_y = \rho\rho_y, \quad u_x\cdot u_{yy} = (u_x\cdot u_y)_y - u_{xy}\cdot u_y = -\rho\rho_x, \end{aligned} \tag{23}$$

and hence

$$u \cdot (v \times v_y) = (\alpha\beta_y - \beta\alpha_y)\rho^2 + \rho\rho_x(\alpha^2 + \beta^2).$$

Likewise, we see that

$$u \cdot (v \times v_x) = (\alpha\beta_x - \beta\alpha_x)\rho^2 - \rho\rho_y(\alpha^2 + \beta^2),$$

and hence

$$u \cdot (v \times v_\theta) = (\alpha\beta_\theta - \beta\alpha_\theta)\rho^2 + r\rho\rho_r(\alpha^2 + \beta^2) = (\alpha\beta_\theta - \beta\alpha_\theta)\rho^2 - r^2\rho|v|^2,$$

which we integrate to

$$\int_0^{2\pi} u \cdot (v \times v_\theta)\,\mathrm{d}\theta = \rho^2 \int_0^{2\pi} (\alpha\beta_\theta - \beta\alpha_\theta)\,\mathrm{d}\theta - r^2\rho \int_0^{2\pi} |v|^2\,\mathrm{d}\theta, \tag{24}$$

since we are hoping to estimate the θ-integral in (21).

We can control the first term on the right-hand side of (24) using the isoperimetric inequality on $\mathbb{R}^2$, or simply by direct calculation using Wirtinger's inequality. Denoting the average value of α for each fixed r by $\overline{\alpha} = \overline{\alpha}(r)$, and likewise for β, we have

$$\begin{aligned}\left|\rho^2 \int_0^{2\pi} (\alpha\beta_\theta - \beta\alpha_\theta)\,\mathrm{d}\theta\right| &= \left|\rho^2 \int_0^{2\pi} ((\alpha - \overline{\alpha})\beta_\theta - (\beta - \overline{\beta})\alpha_\theta)\,\mathrm{d}\theta\right| \\ &\le \left|\frac{\rho^2}{2} \int_0^{2\pi} ((\alpha - \overline{\alpha})^2 + \beta_\theta^2 + (\beta - \overline{\beta})^2 + \alpha_\theta^2)\,\mathrm{d}\theta\right| \\ &\le \left|\frac{\rho^2}{2} \int_0^{2\pi} (\alpha_\theta^2 + \beta_\theta^2 + \beta_\theta^2 + \alpha_\theta^2)\,\mathrm{d}\theta\right| = \rho^2 \int_0^{2\pi} (\alpha_\theta^2 + \beta_\theta^2)\,\mathrm{d}\theta. \end{aligned} \tag{25}$$

In order to continue with this estimate, we return to (22). Taking the scalar product with the unit vector $(1/\rho)u_x$ and appealing to (23), we find that

$$v_y \cdot \frac{u_x}{\rho} = \rho\alpha_y - \rho_x\beta,$$

and so by utilizing Young's inequality we may estimate

$$\rho^2\alpha_y^2 = \left(v_y \cdot \frac{u_x}{\rho} - x\rho^2\beta\right)^2 \le \left(v_y \cdot \frac{u_x}{\rho}\right)^2 (1 + r^2) + x^2\rho^4\beta^2 \left(1 + \frac{1}{r^2}\right) = \frac{2}{\rho}\left(v_y \cdot \frac{u_x}{\rho}\right)^2 + 2\frac{x^2}{r^2}\rho^3\beta^2.$$

Summing the analogous expressions for $\rho^2\alpha_x^2$, $\rho^2\beta_x^2$ and $\rho^2\beta_y^2$ gives

$$\rho^2(\alpha_x^2 + \alpha_y^2 + \beta_x^2 + \beta_y^2) \le \frac{2}{\rho}\left(\left(v_y \cdot \frac{u_x}{\rho}\right)^2 + \left(v_y \cdot \frac{u_y}{\rho}\right)^2 + \left(v_x \cdot \frac{u_x}{\rho}\right)^2 + \left(v_x \cdot \frac{u_y}{\rho}\right)^2\right) + 2\rho^3(\alpha^2 + \beta^2),$$

and by observing (using the fact that v is a tangent vector field on the sphere) that

$$\left(v_y \cdot \frac{u_x}{\rho}\right)^2 + \left(v_y \cdot \frac{u_y}{\rho}\right)^2 + \left(v_x \cdot \frac{u_x}{\rho}\right)^2 + \left(v_x \cdot \frac{u_y}{\rho}\right)^2 = |\hat{\nabla} v|^2 - |v|^2\rho^2,$$

(where the notation $|\hat{\nabla} v|^2 = |v_x|^2 + |v_y|^2$ avoids confusion with the $|\nabla v|^2 = (1/\rho^2)(|v_x|^2 + |v_y|^2)$ used previously) we reduce to

$$\frac{1}{r^2}\rho^2(\alpha_\theta^2 + \beta_\theta^2) \le \rho^2(\alpha_x^2 + \alpha_y^2 + \beta_x^2 + \beta_y^2) \le \frac{2}{\rho}(|\hat{\nabla} v|^2 - |v|^2\rho^2) + 2\rho|v|^2,$$

or simply

$$\rho^2(\alpha_\theta^2 + \beta_\theta^2) \le \frac{2r^2}{\rho}|\hat{\nabla} v|^2.$$

Blending this with (24) and (25) yields

$$-\frac{2r^2}{\rho}\int_0^{2\pi} |\hat{\nabla} v|^2\,\mathrm{d}\theta - r^2\rho\int_0^{2\pi} |v|^2\,\mathrm{d}\theta \le \int_0^{2\pi} u\cdot(v\times v_\theta)\,\mathrm{d}\theta \le \frac{2r^2}{\rho}\int_0^{2\pi} |\hat{\nabla} v|^2\,\mathrm{d}\theta - r^2\rho\int_0^{2\pi} |v|^2\,\mathrm{d}\theta. \quad (26)$$

In order to use the representation (21) we compute

$$\frac{1}{8\pi}\int_0^\infty \frac{\rho}{r}\left(\frac{2r^2}{\rho}\int_0^{2\pi} |\hat{\nabla} v|^2\,\mathrm{d}\theta\right)\mathrm{d}r = \frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} r|\hat{\nabla} v|^2\,\mathrm{d}\theta\,\mathrm{d}r = \frac{1}{4\pi}\int_{\mathbb{R}^2}|\hat{\nabla} v|^2 = \frac{1}{4\pi}\int_{S^2}|\nabla v|^2,$$

(where the final equality alludes to the conformal invariance of the Dirichlet energy on two-dimensional domains) and

$$\frac{1}{8\pi}\int_0^\infty \frac{\rho}{r}\left(r^2\rho\int_0^{2\pi} |v|^2\,\mathrm{d}\theta\right)\mathrm{d}r = \frac{1}{8\pi}\int_0^\infty\int_0^{2\pi} r|v|^2\rho^2\,\mathrm{d}\theta\,\mathrm{d}r = \frac{1}{8\pi}\int_{\mathbb{R}^2}|v|^2\rho^2 = \frac{1}{8\pi}\int_{S^2}|v|^2.$$

Combining these calculations with (26) and (21) gives the conclusion

$$-\frac{1}{4\pi}\int_{S^2}|\nabla v|^2 - \frac{1}{8\pi}\int_{S^2}|v|^2 \le \varphi(\hat{u}) \le \frac{1}{4\pi}\int_{S^2}|\nabla v|^2 - \frac{1}{8\pi}\int_{S^2}|v|^2. \qquad \square$$

Let us now turn our attention to Theorem 2. The basic supplementary principle behind this result is that the velocity field may be decomposed into a ‘purely rotational’ part and an ‘orthogonal’ part, the latter having zero angular momentum. The contribution to pressure variation from each component is then considered separately — the variation due to the purely rotational component being calculated explicitly. As we shall see in the next section, the effect of viscosity is to kill the orthogonal part leaving the purely rotational part intact. We are therefore guaranteed a sharp estimate in the limit of this process.

We define a ‘purely rotational’ velocity field to be a field of the form

$$v(u) = e \times u,$$

for some fixed $e \in \mathbb{R}^3$, and we may refer to the corresponding flow as ‘uniformly rotating’. The angular momentum, vorticity and enstrophy are then given by

$$\Omega(v) = e, \quad \omega(u) = 2(u\cdot e), \quad \int_{S^2}\omega^2 = \frac{16\pi}{3}|e|^2. \quad (27)$$

Moreover, the pressure is precisely

$$p(u) = \tfrac{1}{6}(|e|^2 - 3(u\cdot e)^2),$$

and hence

$$\mathrm{osc}(p) = \frac{|e|^2}{2}. \tag{28}$$

Given any other field v, we decompose it into divergence free components

$$v = v^{\mathrm{rot}} + v^{\perp},$$

where

$$v^{\mathrm{rot}}(u) = \Omega(v) \times u, \quad \text{and} \quad v^{\perp} = v - v^{\mathrm{rot}}.$$

This is simply taking the L^2 projection onto the first eigenspace of the Laplacian, and the orthogonal component. Consequently, since each component of $v^{\perp}$ is perpendicular to linear functions (a fact which may be verified directly) we have simple decomposition of most of our physical quantities into the sum of the quantities for each component of v. Explicitly,

$$\int_{S^2} \omega^2 = \frac{16\pi}{3} |\Omega(v)|^2 + \int_{S^2} (\omega^{\perp})^2,$$

where $\omega^{\perp}(u) = u \cdot (\nabla \times v^{\perp})$, and

$$\int_{S^2} |v|^2 = \frac{8\pi}{3} |\Omega(v)|^2 + \int_{S^2} |v^{\perp}|^2,$$

which explains our definitions of the renormalised quantities in (3) and (4). A short calculation confirms that

$$\Omega(v^{\mathrm{rot}}) = \Omega(v). \tag{29}$$

The proof of Theorem 2 will also require a rather coarser estimate than (18) for the oscillation of solutions to Poisson's equation. Since we are insisting upon explicit constants, we state:

Proposition 4. *Suppose $\varphi : S^2 \to \mathbb{R}$ is a solution to the equation*

$$-\Delta\varphi = f,$$

where $f \in L^2(S^2, \mathbb{R})$. Then

$$\mathrm{osc}(\varphi) \leq \frac{1}{\sqrt{\pi}} \left(\int_{S^2} f^2 \right)^{1/2}.$$

Proof. For any $\hat{u} \in S^2 \hookrightarrow \mathbb{R}^3$, explicit calculations confirm that

$$\int_{S^2} \left(\ln \frac{|u - \hat{u}|}{2} \right)^2 = 2\pi, \quad \text{and} \quad c := \frac{1}{4\pi} \int_{S^2} \ln \frac{|u - \hat{u}|}{2} = \frac{1}{2}.$$

Therefore, using Green's representation

$$\varphi(\hat{u}) = \int_{S^2} \left(-\frac{1}{2\pi} \ln \frac{|u - \hat{u}|}{2} \right) f(u) = -\frac{1}{2\pi} \int_{S^2} \left(\ln \frac{|u - \hat{u}|}{2} - c \right) f(u),$$

we may estimate directly

$$|\varphi(\hat{u})| \leq \frac{1}{2\pi} \left(\int_{S^2} \left(\ln \frac{|u - \hat{u}|}{2} - c \right)^2 \right)^{1/2} \left(\int_{S^2} f^2 \right)^{1/2} = \frac{1}{2\sqrt{\pi}} \left(\int_{S^2} f^2 \right)^{1/2},$$

and hence

$$\operatorname{osc}(\varphi) \le \frac{1}{\sqrt{\pi}} \left(\int_{S^2} f^2 \right)^{1/2}. \qquad \square$$

Proof (Theorem 2). In this proof we only concern ourselves with optimal coefficients for the quadratic angular momentum term in (9) which dominates in the limit of large time (assuming $\nu > 0$). If the other terms are large, then Theorem 1 may be more appropriate.

We begin by decomposing Eq. (19) for the pressure into its components. A short calculation, using (29) yields

$$-\Delta p = -\Delta p^{\mathrm{rot}} - \Delta p^{\perp} - \Delta \gamma,$$

where

$$-\Delta p^{\mathrm{rot}} = -2\left(\frac{1}{\rho^2} u \cdot (v_x^{\mathrm{rot}} \times v_y^{\mathrm{rot}}) - \frac{1}{2} |v^{\mathrm{rot}}|^2 \right), \quad -\Delta p^{\perp} = -2\left(\frac{1}{\rho^2} u \cdot (v_x^{\perp} \times v_y^{\perp}) - \frac{1}{2} |v^{\perp}|^2 \right),$$

and

$$-\Delta \gamma = -2((u \cdot \Omega(v))\omega^{\perp} - v^{\mathrm{rot}} \cdot v^{\perp}).$$

By (28) and (29) we have

$$\operatorname{osc}(p^{\mathrm{rot}}) = \frac{|\Omega(v)|^2}{2};$$

using Theorem 5 (c.f. Theorem 1) we have

$$\operatorname{osc}(p^{\perp}) \le \frac{1}{\pi} \int_{S^2} (\omega^{\perp})^2 = \frac{1}{\pi} W(v).$$

Once we have observed, using the inequality $|v^{\mathrm{rot}}| \le |\Omega(v)|$ and Proposition 1, that

$$\int_{S^2} (-\Delta \gamma)^2 \le 8|\Omega(v)|^2 \int_{S^2} ((\omega^{\perp})^2 + |v^{\perp}|^2) \le 12|\Omega(v)|^2 \int_{S^2} (\omega^{\perp})^2 = 12|\Omega(v)|^2 \, W(v),$$

we may invoke Proposition 4 to get

$$\operatorname{osc}(\gamma) \le 2\sqrt{\frac{3}{\pi}} \, |\Omega(v)| W(v)^{1/2}.$$

Combining all three oscillation estimates yields

$$\operatorname{osc}(p) \le \frac{|\Omega(v)|^2}{2} + \frac{1}{\pi} W(v) + 2\sqrt{\frac{3}{\pi}} \, |\Omega(v)| \, W(v)^{1/2},$$

which is our destination, modulo conservation of angular momentum which will be established mathematically in the next section. □

3. Evolution of physical quantities

In this section we give mathematical proofs of the evolution properties of physical quantities which we have referred to elsewhere in this paper.

In our calculations, we will use repeatedly the following consequence of the divergence theorem (or equivalently the first variation formula).

Proposition 5. *If $w : S^2 \to \mathbb{R}^3$ is a divergence-free tangent vector field on the sphere, then*

$$\int_{S^2} w \cdot \nabla f = 0,$$

for any function $f : S^2 \to \mathbb{R}$.

We begin by proving conservation of angular momentum — a fact used already in the proof of Theorem 2. We need only concern ourselves with an arbitrary component

$$e \cdot \Omega(v) = \frac{3}{8\pi}\int_{S^2} e \cdot (u \times v) = \frac{3}{8\pi}\int_{S^2}(e \times u)\cdot v$$

of Ω, where $e \in \mathbb{R}^3$ is any unit vector, and this component does not vary in time according to the calculation

$$\begin{aligned}\frac{\partial}{\partial t}\int_{S^2}(e\times u)\cdot v &= \int_{S^2}(e\times u)\cdot(\nu(\Delta v + 2v) - (v\cdot\nabla)v - |v|^2 u - \nabla p)\\ &= \int_{S^2}[\nu(\Delta(e\times u) + 2(e\times u))\cdot v - (v\cdot\nabla((e\times u)\cdot v) - (e\times v)\cdot v)\\ &\quad -|v|^2(e\times u)\cdot u - (e\times u)\cdot\nabla p] = 0,\end{aligned}$$

where we have used the fact that the components of $(e \times u)$ are linear functions, and are therefore eigenfunctions on the sphere with eigenvalue 2, and also that both v and $(e \times u)$ are divergence-free, and may therefore be fed into Proposition 5.

Conservation of momentum requires only the incompressibility condition. Using Proposition 5 we have, for an arbitrary vector $e \in \mathbb{R}^3$,

$$e \cdot \int_{S^2} v = \int_{S^2} v \cdot \nabla(u \cdot e) = 0.$$

It is this fact, together with the final part of 3 which we require in order to prove Proposition 1 by applying the Poincaré inequality

$$2\int_{S^2} f^2 \le \int_{S^2}|\nabla f|^2, \quad \text{if} \quad \int_{S^2} f = 0, \tag{30}$$

to each component of v, and summing. The coefficient 2 in (30) is, of course, the first eigenvalue of the Laplacian on S^2. Indeed for functions orthogonal to the first eigenspace, this coefficient may be improved to 6, the second eigenvalue, and hence for $v^\perp$ we may improve Proposition 1 to

$$\int_{S^2}|v^\perp|^2 \le \frac{1}{6}\int_{S^2}|\nabla v^\perp|^2 = \frac{1}{6}\int_{S^2}(\omega^\perp)^2. \tag{31}$$

Of course, we have equality in (7) for purely rotational vector fields such as v^{rot}, or equivalently we have $W(v^{\mathrm{rot}}) = K(v^{\mathrm{rot}}) = 0$.

It remains to handle the decay of kinetic energy and enstrophy, both of which are conserved if $\nu = 0$. If not, these quantities decrease, with the *excess* kinetic energy and *excess* enstrophy decaying exponentially.

To handle the kinetic energy, we take the scaler product of (2) with $2v$ and integrate, using Proposition 5 and (31) to find that

$$\begin{aligned}\frac{\partial K(v)}{\partial t} &= \frac{\partial}{\partial t}\int_{S^2}|v|^2 = -\int_{S^2} v\cdot\nabla|v|^2 - 2\int_{S^2} v\cdot\nabla p - 2\nu\left(\int_{S^2}|\nabla v|^2 - 2\int_{S^2}|v|^2\right)\\ &= -2\nu\left(\int_{S^2}|\nabla v^\perp|^2 - 2\int_{S^2}|v^\perp|^2\right) \le -8\nu\int_{S^2}|v^\perp|^2 = -8\nu K(v).\end{aligned}$$

Therefore the excess kinetic energy decays exponentially.

It is well known that for planar flows, the vorticity evolves according to

$$\partial_t\hat{\omega} + (v\cdot\nabla)\hat{\omega} = \nu\Delta\hat{\omega}.$$

For spherical flows, we may extend the velocity field to the ambient $\mathbb{R}^3$ and take the curl of the extention of equations (2) — or proceed intrinsically — to derive the evolution equation for vorticity

$$\partial_t\omega + (v\cdot\nabla)\omega = \nu(\Delta\omega + 2\omega).$$

By analogy to the kinetic energy, we multiply this equation by 2ω and integrate to obtain

$$\begin{aligned}\frac{\partial W(v)}{\partial t} &= \frac{\partial}{\partial t}\int_{S^2}|\omega|^2 = -\int_{S^2} v\cdot\nabla|\omega|^2 - 2\nu\left(\int_{S^2}|\nabla\omega|^2 - 2\int_{S^2}|\omega|^2\right)\\ &= -2\nu\left(\int_{S^2}|\nabla\omega^\perp|^2 - 2\int_{S^2}|\omega^\perp|^2\right) \le -8\nu\int_{S^2}|\omega^\perp|^2 = -8\nu W(v),\end{aligned}$$

where we have used the fact that $\omega^\perp$ is orthogonal to both constant and linear functions (which is easily verified) and hence (cf. (31)) that

$$\int_{S^2}|\omega^\perp|^2 \le \frac{1}{6}\int_{S^2}|\nabla\omega^\perp|^2.$$

(We are also using implicitly the fact that the rotational component $u\cdot(\nabla\times v^{\text{rot}})$ of ω is a linear function, in order to decompose the vorticity as shown.) We see, therefore, that the excess enstrophy decays exponentially as described in Proposition 2.

4. Optimality of the estimates

Most of our estimates are sharp, as we detail below, though different estimates are sharpest for different distributions of vorticity. Most of the velocity fields we describe below are modifications of examples of Baraket [1] who was interested in proving limits on the strength of Wente-type inequalities, before the optimal inequalities were given in [7].

For Theorem 1, the first inequality in (5) (bounding from below the variation of pressure from its average value) is optimal in the weak sense that we may find a sequence of non-zero velocity fields such that the ratio of the right-hand side and the left-hand side converges to 1 in the limit. Consequently, (6) must be sharp up to a possible factor of two.

Explicitly, using spherical polar coordinates (α,φ) for u and the notation $e = (0,0,1)$ (so that $\cos\alpha = u\cdot e$) we take, for small ϵ, the velocity fields

$$v_\epsilon(u) = \begin{cases}(\epsilon^{(1/2)-\epsilon}\alpha^{\epsilon-1})e\times u & \alpha<\epsilon,\\ (\epsilon^{(1/2)}\alpha^{-1})e\times u & \alpha\ge\epsilon.\end{cases}$$

Clearly we have the pointwise estimate $|v|\le\epsilon^{1/2}$, and so

$$\int_{S^2}|v|^2 = \mathrm{O}(\epsilon).$$

Working a little harder, we have, with the aid of Proposition 3,

$$\int_{S^2}\omega^2 = \int_{S^2}|\nabla v|^2 = \pi + \mathrm{o}(1),$$

where the contribution from 'radial' (harmonic) energy is in fact merely $O(\epsilon)$. Finally, it is easy to calculate that

$$\int_0^{2\pi} u\cdot(v\times v_\theta)\,\mathrm{d}\theta = \begin{cases} 2\pi\epsilon^{1-2\epsilon}\alpha^{2\epsilon-2}\sin^2\alpha\cos\alpha & \alpha<\epsilon, \\ 2\pi\epsilon\alpha^{-2}\sin^2\alpha\cos\alpha & \alpha\geq\epsilon, \end{cases}$$

and hence using Green's representation (cf. (21)) we find that

$$p(e) = -\tfrac{1}{2} + \mathrm{o}(1),$$

where the average value of p is 0. Given the above, the first inequality in (5) is clearly seen to be optimal by taking the limit $\epsilon \to 0$.

Note that these considerations do not pin the coefficient of the kinetic energy. However, it is worth pointing out that the energy with the coefficient given arises naturally in the proof.

In contrast, we may actually achieve equality in (9) of Theorem 2. However instead of concentrating the vorticity, it now pays to distribute it more evenly. Indeed, it is easy to see that for any purely rotational velocity field $v(u) = e\times u$, for some fixed vector e, we must have equality – see (27) and (28). Moreover, since the excess enstrophy decays in the limit to leave such a flow, we must have an optimal estimate in the limit of large time, provided that $\nu > 0$.

Theorem 3 is completely sharp, as we see by considering more examples of the type used by Baraket, or by inspecting the proof of Theorem 4 given in [7]. It suffices to consider a velocity field which is zero outside a small ball $B(0, d)$ in the torus $\mathbb{T}$. Inside, we may set

$$v_\epsilon(x, y) = \begin{cases} (\epsilon^{(1/2)-\epsilon}r^{\epsilon-1})(-y, x) & r<\epsilon, \\ \epsilon^{1/2}\dfrac{1}{d-\epsilon}\left(\dfrac{d}{r}-1\right)(-y, x) & \epsilon\leq r<d, \end{cases}.$$

where $r^2 = x^2 + y^2$, in which case

$$\int_{\mathbb{T}} \hat{\omega}^2 = \pi + \mathrm{o}(1),$$

and, after some thought,

$$\mathrm{osc}(p) = \tfrac{1}{2} + \mathrm{o}(1),$$

as $\epsilon \to 0$.

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