IMPROVED REGULARITY OF HARMONIC MAP FLOWS WITH HÖLDER CONTINUOUS ENERGY *

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Abstract

For a smooth harmonic map flow $u : \mathcal{M} \times [0, T) \to \mathcal{N}$ with blow-up as $t \uparrow T$, it has been asked ([6], [5], [7]) whether the weak limit $u(T) : \mathcal{M} \to \mathcal{N}$ is continuous. Recently, in [12], we showed that in general it need not be. Meanwhile, the energy function $E(u(\cdot)) : [0, T) \to \mathbb{R}$, being weakly positive, smooth and weakly decreasing, has a continuous extension to $[0, T]$. Here we show that if this extension is also Hölder continuous, then the weak limit $u(T)$ must also be Hölder continuous.

1 Introduction

Given a smooth map $v : \mathcal{M} \to \mathcal{N} \hookrightarrow \mathbb{R}^N$ from a compact boundaryless Riemannian surface $\mathcal{M}$ to a compact boundaryless Riemannian manifold $\mathcal{N}$ - which we assume without loss of generality to be isometrically embedded in $\mathbb{R}^N$ - we may define the harmonic map energy to be

$$E(v) := \frac{1}{2} \int_{\mathcal{M}} |\nabla v|^2. \tag{1.1}$$

The harmonic map flow is the L^2-gradient flow for this functional. Defining the tension $\tau(v)$ of v to be the L^2-gradient of E, we may check that

$$\tau(v) = (\Delta v)^T, \tag{1.2}$$

the projection of the laplacian of v (seen as a map into $\mathbb{R}^N$) onto the tangent space of $\mathcal{N}$. A smooth harmonic map flow $u : \mathcal{M} \times [0, T) \to \mathcal{N}$ is then a smooth solution to the nonlinear PDE

$$\frac{\partial u}{\partial t} = \tau(u(t)),$$

where $u(t) := u(\cdot, t)$, and the energy decreases according to

$$\frac{d}{dt} E(u(t)) = -\|\tau(u(t))\|^2_{L^2(\mathcal{M})}. \tag{1.3}$$

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Note, for later, that the function $E(u(\cdot)) : [0,T) \to \mathbb{R}$ is smooth, bounded below and weakly decreasing, and is thus extendable to a continuous function on $[0,T]$.

Any smooth initial map $u_0 : \mathcal{M} \to \mathcal{N}$ gives rise to a smooth flow over some time interval (Eells-Sampson [4]). In our situation, where the domain has dimension 2, Struwe [9] looked at the maximal time interval $[0,T)$ over which the flow can be smoothly extended (such an extension is necessarily unique) and found that the flow enjoys uniform derivative bounds as $t \uparrow T$, away from a finite set of points $\mathcal{S} \subset \mathcal{M}$ where energy concentrates. By virtue of this theory, we may extend the flow smoothly to $(\mathcal{M} \times [0,T]) \backslash (\mathcal{S} \times \{T\})$ by defining

$$u(x,T) := \lim_{t \uparrow T} u(x,t),$$

for all $x \in \mathcal{M} \backslash \mathcal{S}$. Combining this almost-everywhere smooth convergence with the uniform boundedness of $u(t)$ in $W^{1,2}(\mathcal{M},\mathcal{N})$ for $t \in [0,T)$, we can also be sure of the weak convergence $u(t) \rightharpoonup u(T)$ in $W^{1,2}(\mathcal{M},\mathcal{N})$ as $t \uparrow T$. In particular, we know that $u(T) \in W^{1,2}(\mathcal{M},\mathcal{N})$. The problem of whether $u(T)$ is continuous has been raised by Qing-Tian [6] and Lin-Wang [5], and considered recently by Qing [7]. Of course, the fact that $u(T)$ lies in $W^{1,2}$ just fails to be sufficient to immediately deduce continuity. Very recently, in [12], we have shown that in general, $u(T)$ need *not* be continuous. It is natural then to ask for additional hypotheses to guarantee continuity of $u(T)$. A good candidate in the light of [8] might be real analyticity of the target. Here we succeed assuming that the smooth bounded scalar energy function of time $E(u(\cdot)) : [0,T) \to \mathbb{R}$ has a *Hölder* continuous extension to $[0,T]$; the limit $u(T)$ is then also Hölder continuous.

Theorem 1.1 *Let $u : \mathcal{M} \times [0,T) \to \mathcal{N}$ be a smooth harmonic map flow on a maximal time interval, which we extend as above to time $t = T$. If $E(u(\cdot)) : [0,T) \to \mathbb{R}$ lies in $C^{0,\alpha}([0,T])$ for some $\alpha \in (0,1]$, then $u(T) \in C^{0,\frac{\alpha}{2}}(\mathcal{M},\mathcal{N})$.*

It is not clear at this stage whether there are simple hypotheses on the target which would control the regularity of the energy as we require here. In fact, although there are now many examples of finite time blow-up in this flow (see [12]) it is not obvious whether Hölder continuous energy is an unreasonably strong hypothesis; in particular, we shall see that it restricts the rate of blow-up of the flow. However, it turns out – see the nonrigorous but detailed calculations in [2] and the forthcoming work [1] – that this rate of blow-up is consistent with finite time blow-up examples of Chang-Ding-Ye [3].

Remark 1.2 In a 'bubbling analysis' (see [12]) our estimates are strong enough to prove that the union of the images of the bubbles and of $u(T)$ must be a connected set under our hypothesis. See [6] for the equivalent 'infinite time' result.

Remark 1.3 The assumption that $\mathcal{M}$ is boundaryless is mainly to simplify the exposition. Analogous results hold for flows on domains with boundary - in which case the flow is fixed on the boundary as time progresses - and if there is blow-up only in the interior of the domain, then the results and proof need no alteration.

2 Basic theory

In this section, we recall some finer properties of the blow-up of the harmonic map flow, and discuss what we can deduce from our Hölder energy hypothesis. We also prepare here a useful 'neck estimate' (Lemma 2.3) and a parabolic estimate (Lemma 2.4) from the recent work of Qing [7].

We will use two different notations for discs. If $x \in \mathcal{M}$ and $r > 0$, then $D_r(x)$ will denote the geodesic disc of radius r, centred at x. The notation $B_r(y)$, will imply a disc in $\mathbb{R}^2$, and we write $B_r := B_r(0)$ and $B := B_1$. The energy of a map v over some subdomain Ω will be written $E(v, \Omega)$. It is important to keep in mind that this energy is invariant under dilations of the map, since we are working on two dimensional domains.

The following lemma, which is a small part of Theorem 1.6 of [12] (see also Remark 1.10 of that paper) controls the rate of concentration of energy at a singularity.

Lemma 2.1 *Let $u : \mathcal{M} \times [0, T) \to \mathcal{N}$ be a smooth heat flow which we extend to time T away from the set of points $\mathcal{S} \subset \mathcal{M}$ as discussed in Section 1. Then for each $x \in \mathcal{S}$, if we define*

$$L_x := \lim_{\eta \downarrow 0} \limsup_{t \uparrow T} E(u(t), D_\eta(x)) > 0, \tag{2.1}$$

then

$$\lim_{t \uparrow T} E(u(t), D_{\sqrt{T-t}}(x)) = L_x. \tag{2.2}$$

(Note that the positivity of L_x characterises the points $x \in \mathcal{S}$ - see [9].) Estimate (2.2) constrains the blow-up rate for a general flow. When the global energy is assumed to be Hölder, the blow-up rate must be even quicker, local energy is unable to concentrate as quickly, and we have somewhat better control on the tension of the flow just prior to the singularity:

Lemma 2.2 *Suppose $u : \mathcal{M} \times [0, T) \to \mathcal{N}$ is smooth, with $E(u(0)) \leq E_0$, and that for some $\alpha > 0$, the Hölder α-seminorm of the energy is constrained by*

$$[E(u(\cdot))]_{C^{0,\alpha}([0,T])} \leq \kappa$$

for some $\kappa > 0$. Then the following statements are true.

(i) For $0 \leq t \leq s < T$, $y \in \mathcal{M}$ and $\mu > 0$, with μ less than the injectivity radius of $\mathcal{M}$, we have

$$E(u(s), D_{\mu/2}(y)) \leq E(u(t), D_\mu(y)) + \frac{C}{\mu}(s-t)^{(1+\alpha)/2},$$

for some $C = C(E_0, \kappa)$.

(ii) If u blows up at $x \in \mathcal{M}$ as $t \uparrow T$, then with L_x as defined in (2.1), we have

$$\lim_{\eta\to\infty} \liminf_{t\uparrow T} E(u(t), D_{\eta(T-t)^{(1+\alpha)/2}}(x)) = L_x.$$

(iii) For all $\varepsilon > 0$, there exists $t \in (T-2\varepsilon, T-\varepsilon)$ such that

$$\|\tau(u(t))\|_{L^2(\mathcal{M})} \leq \kappa^{1/2}\varepsilon^{-(1-\alpha)/2}.$$

Part (i) extends a useful estimate of Struwe [9, Lemma 3.6] valid for arbitrary flows. Part (ii) controls the rate of blow-up, allowing us to assume that any 'bubble scale' (see [12]) lies below $(T-t)^{(1+\alpha)/2}$. This is certainly not true for all flows ([12]) but *is* the case for flows constructed by Chang-Ding-Ye [3] as discussed in [2] and [1]. The control on the tension of part (iii) is a little better than we expect in general. In particular, we mention that there are flows for which the function $\|\tau(u(\cdot))\|_{L^2(\mathcal{M})}$ does not lie in $L^{2+\delta}(0,T)$ for any $\delta > 0$ (see [12]). In practice, this improved tension control means that we do not have to blow up the flow as much (in space) to obtain a map with small tension.

We will require a so-called neck estimate to control energy decay in annular regions of the flow. The following lemma is essentially identical (and equivalent) to [12, Lemma 4.4] and is a special case of Lemma 2.9 (see also Remark 2.8) from [11]. (See the latter of these references for a proof based on work of Lin-Wang [5] which followed earlier work of Qing-Tian [6].)

Lemma 2.3 *Suppose that $v : B \to \mathcal{N}$ is smooth and satisfies $E(v,B) \leq E_0$ for some E_0. Then there exist $\delta > 0$ (dependent only on $\mathcal{N}$) and $K > 0$ (dependent only on E_0 and $\mathcal{N}$) such that if*

$$E(v, B\backslash B_{r^2}) < \delta$$

for some $r \in (0, \frac{1}{2}]$, and

$$\|\tau(v)\|^2_{L^2(B)} \leq \delta,$$

then we have the estimate

$$E(v, B_{2r}\backslash B_{r/2}) \leq K\, r. \tag{2.3}$$

We also need a parabolic estimate for flows with locally small energy, giving pointwise control on the energy density. The following is equivalent to a recent result of Qing [7, Proposition 2.2], and may also be derived (along with analogous higher derivative bounds) more along the lines of [9], or via the theory in [10].

Lemma 2.4 *There exist constants $\varepsilon_0 > 0$ and $C > 0$ depending on $\mathcal{N}$ such that if $u : B_\nu \times [-\nu^2, 0] \to \mathcal{N}$ is a smooth heat flow satisfying $E(u(t), B_\nu) \leq \varepsilon_0$ for all $t \in [-\nu^2, 0]$ then*

$$|\nabla u|^2(0,0) \leq \frac{C}{\nu^4}\int_{-\nu^2}^0 E(u(t), B_\nu)dt.$$

3 Proof of Lemma 2.2

Each part of the lemma will use, directly or indirectly, the following estimate on the tension, obtained by integrating (1.3) and using the Hölder hypothesis:

$$0 < \int_t^s \|\tau(u(a))\|^2_{L^2(\mathcal{M})} da = E(u(t)) - E(u(s)) \leq \kappa(s-t)^\alpha, \tag{3.1}$$

for $0 \leq t \leq s < T$.

3.1 Proof of part (i)

We begin in the spirit of Struwe [9, Lemma 3.6] or [12, Section 2]. Let $\phi \in C^\infty(\mathcal{M}, [0,1])$ be a cut-off function supported in $D_\mu(y)$, with $\phi \equiv 1$ on $D_{\mu/2}(y)$ and $|\nabla\phi| \leq \frac{3}{\mu}$. For $t \in [0,T)$ the local energy

$$\Theta(t) := \frac{1}{2}\int_{\mathcal{M}} \phi^2 |\nabla u(t)|^2$$

differentiates to give

$$\begin{aligned} \frac{d\Theta(t)}{dt} &= \int_{\mathcal{M}} \phi^2 \nabla u.\nabla\tau = -\int_{\mathcal{M}} \phi^2|\tau|^2 - 2\int_{\mathcal{M}} \phi\tau.(\nabla\phi.\nabla)u \leq \frac{6}{\mu}\int_{\mathcal{M}} |\tau||\nabla u| && (3.2) \\ &\leq \frac{C}{\mu}\|\tau(u(t)\|_{L^2(\mathcal{M})}, && (3.3) \end{aligned}$$

with $C = C(E_0)$. Integrating between times t and s, we find that

$$\begin{aligned} E(u(s), D_{\mu/2}(y)) - E(u(t), D_\mu(y)) &\leq \Theta(s) - \Theta(t) \leq \frac{C}{\mu}\int_t^s \|\tau(u(a)\|_{L^2(\mathcal{M})} da && (3.4) \\ &\leq \frac{C(s-t)^{1/2}}{\mu}\left(\int_t^s \|\tau(u(a)\|^2_{L^2(\mathcal{M})} da\right)^{\frac{1}{2}} && (3.5) \\ &\leq \frac{C}{\mu}(s-t)^{(1+\alpha)/2} && (3.6) \end{aligned}$$

by (3.1), where $C = C(E_0, \kappa)$. ∎

3.2 Proof of part (ii)

We begin by applying part (i) with $y = x$, and taking the limit $s \uparrow T$, to give

$$L_x \leq \limsup_{s\uparrow T} E(u(s), D_{\mu/2}(x)) \leq E(u(t), D_\mu(x)) + \frac{C}{\mu}(T-t)^{(1+\alpha)/2}.$$

For any $\eta > 0$, we may then set $\mu = \eta(T-t)^{(1+\alpha)/2}$ - provided that t is sufficiently close to T - to give

$$L_x \leq E(u(t), D_{\eta(T-t)^{(1+\alpha)/2}}(x)) + \frac{C}{\eta}.$$

Combining this with (2.2) from Lemma 2.1, we find that

$$L_x = \lim_{t \uparrow T} E(u(t), D_{\sqrt{T-t}}(x)) \geq \liminf_{t \uparrow T} E(u(t), D_{\eta(T-t)^{(1+\alpha)/2}}(x)) \geq L_x - \frac{C}{\eta},$$

with C still dependent on E_0 and κ. It remains to take the limit $\eta \to \infty$. ∎

3.3 Proof of part (iii)

By applying (3.1) with $t = T - 2\varepsilon$ and $s = T - \varepsilon$ we have

$$\int_{T-2\varepsilon}^{T-\varepsilon} \|\tau(u(a))\|^2_{L^2(\mathcal{M})} da \leq \kappa \varepsilon^\alpha,$$

and hence we must be able to pick $a \in (T - 2\varepsilon, T - \varepsilon)$ with

$$\|\tau(u(a))\|^2_{L^2(\mathcal{M})} \leq \kappa \varepsilon^{\alpha - 1}$$

as desired. ∎

4 Proof of Theorem 1.1

For the flow u, let $E_0 := E(u(0))$ so that $E(u(t)) \leq E_0$ for all $t \in [0, T)$. We must prove that $u(T)$ is Hölder continuous near an arbitrary blow-up point $x \in \mathcal{M}$.

For simplicity of exposition, we will assume that $\mathcal{M}$ is flat in a neighbourhood of x. The general case is an easy (albeit messy) adaptation. By making a parabolic dilation about the point (x, T) in space-time, and translating variables, we may see the flow locally as being a flow $\overline{B} \times [T-1, T] \to \mathcal{N}$ with a single blow-up singularity at the origin of B as $t \uparrow T$. (You may prefer to assume without loss of generality that $T = 0$ in what follows.) Note that although u is not fixed on the boundary of B (and in particular, $E(u(\cdot), B)$ need not be a weakly decreasing function) by keeping in mind the entire flow on a dilated domain $\mathcal{M}$, we still have access to Lemma 2.2.

By the definition of L_x (see (2.1)) we may assume - after making a further parabolic dilation - that for all $t \in [T-1, T]$,

$$E(u(t), B) < L_x + \frac{\delta}{2}, \tag{4.1}$$

where $\delta > 0$ is taken to be as in Lemma 2.3 (and depends only on $\mathcal{N}$).

Next, by part (ii) of Lemma 2.2 we may choose $\eta > 0$ sufficiently large so that

$$\liminf_{t\uparrow T} E(u(t), B_{\eta(T-t)^{(1+\alpha)/2}}) \geq L_x - \frac{\delta}{4}.$$

In particular, for $t < T$ sufficiently close to T (partly to ensure that $\eta(T-t)^{(1+\alpha)/2} < 1$) we must have

$$E(u(t), B_{\eta(T-t)^{(1+\alpha)/2}}) \geq L_x - \frac{\delta}{2}.$$

After making a third and final parabolic dilation, about $(0, T)$, by an amount depending on η, we may then assume that

$$E(u(t), B_{(T-t)^{(1+\alpha)/2}}) \geq L_x - \frac{\delta}{2}, \tag{4.2}$$

for all $t \in [T-1, T)$. Note that when we dilate the flow parabolically, the Hölder seminorm of the energy decreases. In particular, it remains below κ, as originally hypothesised.

Our considerations above, and their conclusions (4.1) and (4.2), give us the annular energy bound

$$E(u(t), B\backslash B_{(T-t)^{(1+\alpha)/2}}) < \delta, \tag{4.3}$$

for $t \in [T-1, T)$.

The remainder of the proof of Theorem 1.1 will be devoted to the proof of the following claim which is easily seen to imply the Hölder continuity of the theorem. The claim, which cannot be true for general flows in the light of [12], should be compared to the similar result of Qing [7, Theorem 1.1] which does hold in general, but is not strong enough to deduce continuity.

Claim 4.1 *The flow u currently being considered satisfies*

$$|\nabla u(z, T)| \leq C|z|^{\frac{\alpha}{2}-1} \tag{4.4}$$

for all $z \in B\backslash\{0\}$, where C is a constant independent of z.

Proof of claim. We will use the shorthand $R := |z|$, and $\varepsilon := R^2 \frac{1}{4}\left(\frac{\kappa}{\delta}\right)^{1/2}$. We are free at this stage to increase κ, if necessary, to ensure that

$$\varepsilon \geq \left(\frac{R}{4}\right)^2. \tag{4.5}$$

Given that $u(T)$ is smooth on $\overline{B}\backslash\{0\}$, it suffices to prove the claim for z in some small neighbourhood of the origin. In particular, we may assume that ε is sufficently small to guarantee that

$$(a)\quad \varepsilon < \frac{1}{2}; \qquad (b)\quad \sigma := \delta^{1/2}\kappa^{-1/2}\varepsilon^{(1-\alpha)/2} < 1; \qquad (c)\quad r := 2\delta^{-1/4}\kappa^{1/4}\varepsilon^{\alpha/2} < \frac{1}{2}. \tag{4.6}$$

With this value of ε, we may now request from part (iii) of Lemma 2.2 a time $t \in (T-2\varepsilon, T-\varepsilon) \subset (T-1, T)$ at which

$$\|\tau(u(t))\|_{L^2}^2 \leq \kappa\varepsilon^{-(1-\alpha)}.$$

Rescaling $u(t)$ to a map $v : B \to \mathcal{N}$ defined by $v(y) = u(\sigma y, t)$ (with $\sigma < 1$ as in (4.6b)) scales the tension to give

$$\|\tau(v)\|^2_{L^2(B)} = \sigma^2 \|\tau(u(t))\|^2_{L^2(B)} \leq \delta. \tag{4.7}$$

Meanwhile, by (4.3) we must have

$$E(v, B \backslash B_{\sigma^{-1}(T-t)^{(1+\alpha)/2}}) < \delta,$$

and since

$$\sigma^{-1}(T-t)^{(1+\alpha)/2} \leq \delta^{-1/2}\kappa^{1/2}\varepsilon^{-(1-\alpha)/2}(2\varepsilon)^{(1+\alpha)/2} \leq 4\delta^{-1/2}\kappa^{1/2}\varepsilon^{\alpha} = r^2,$$

with r as in (4.6c), we have

$$E(v, B \backslash B_{r^2}) < \delta.$$

Coupled with (4.7) and the fact that $E(v, B) \leq E_0$ (and keeping in mind (4.6c)) we may then apply Lemma 2.3 to deduce that

$$E(v, B_{2r} \backslash B_{r/2}) \leq K\, r,$$

with $K = K(\mathcal{N}, E_0)$. Noting that

$$\sigma r = \left(\delta^{1/2}\kappa^{-1/2}\varepsilon^{(1-\alpha)/2}\right)\left(2\delta^{-1/4}\kappa^{1/4}\varepsilon^{\alpha/2}\right) = 2\delta^{1/4}\kappa^{-1/4}\varepsilon^{1/2} = R,$$

we may rescale back from v to $u(t)$ to find that

$$E(u(t), B_{2R} \backslash B_{R/2}) \leq K\, r = C(\mathcal{N}, E_0, \kappa)\varepsilon^{\alpha/2} \leq C(\mathcal{N}, E_0, \kappa) R^{\alpha}.$$

Consequently, for the z of the claim, since $B_{R/2}(z) \subset B_{2R} \backslash B_{R/2}$, we have

$$E(u(t), B_{R/2}(z)) \leq C(\mathcal{N}, E_0, \kappa) R^{\alpha}.$$

We wish to obtain such an estimate not just at the special time t, but over a time interval of length (of the order of) R^2, just prior to time T. To achieve this, we apply part (i) of Lemma 2.2 again, with $y = z$ and $\mu = \frac{R}{2}$. For $s \in [t, T) \supset [T - \varepsilon, T)$ we then have

$$\begin{aligned} E(u(s), B_{R/4}(z)) &\leq E(u(t), B_{R/2}(z)) + \frac{C(E_0, \kappa)}{R/2}(s-t)^{(1+\alpha)/2} \\ &\leq C(\mathcal{N}, E_0, \kappa)\, R^{\alpha} + C(E_0, \alpha, \kappa)\frac{1}{R}\varepsilon^{(1+\alpha)/2} \\ &\leq C(\mathcal{N}, E_0, \alpha, \kappa) R^{\alpha}. \end{aligned}$$

By (4.5), and the smoothness of u on $B_{R/4}(z) \times [T-1, T]$, this inequality holds, in particular, for $s \in [T - (R/4)^2, T]$, and therefore we may apply Lemma 2.4 with $\nu = R/4$ (over a translated domain) to deduce that

$$|\nabla u(z, T)|^2 \leq \frac{C}{R^4} \int_{T-(R/4)^2}^{T} E(u(s), B_{R/4}(z)) ds$$

provided that R is sufficiently small with respect to ε_0 to satisfy the hypothesis of Lemma 2.4. In particular, we may conclude that

$$|\nabla u(z, T)|^2 \leq \frac{C}{R^4}\left(\frac{R}{4}\right)^2 C\, R^{\alpha} \leq C\, R^{\alpha - 2},$$

where $C = C(\mathcal{N}, E_0, \alpha, \kappa)$, provided that $|z|$ is sufficiently small. ∎

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