An approach to the Willmore conjecture *

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Abstract

We highlight one of the methods developed in [6] to give partial answers to the Willmore conjecture, and discuss how it might be used to prove the complete conjecture.

The Willmore conjecture, dating from around 1965, asserts that the integral of the square of the mean curvature H over a torus immersed in $\mathbb{R}^3$ (with the induced metric) should always be at least $2\pi^2$:

$$W := \int_{T^2} H^2 \geq 2\pi^2.$$

A weaker lower bound of 4π for this 'Willmore energy' is rather easy to obtain (see [6]) but this bound is known not to be sharp (see [5]) unless we change the problem to allow integrals over spheres instead of tori.

It is well known that this problem has an equivalent formulation in S^3. Indeed, if we transport the torus from $\mathbb{R}^3$ to S^3 via inverse stereographic projection, the Willmore energy may be written in terms of the new induced metric on the torus, and the new mean curvature, as

$$W = \int_{T^2} (1 + H^2).$$

This energy is preserved if we move the torus via conformal transformations of the ambient S^3 (see [6]).

Once formulated within S^3, we see that the conjecture asserts that the Clifford torus, which has area $2\pi^2$ and satisfies $H = 0$, should be a minimiser for W. Of course, the Clifford torus is the case $r = \frac{1}{\sqrt{2}}$ of the family of flat tori

$$\mathbb{T}_r := \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1| = r, \ |z_2| = \sqrt{1 - r^2}\} \hookrightarrow S^3 \hookrightarrow \mathbb{C}^2$$

which, as r varies within $(0, 1)$, foliate S^3 minus the great circles $\mathbb{T}_0$ and $\mathbb{T}_1$.

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Given a general torus $\mathcal{M} \hookrightarrow S^3$, we can use the degenerate foliation $\{\mathbb{T}_r\}$ to define a 'tangency map'

$$\Phi : G_{2,4} \to \mathbb{N} \cup \{0, \infty\}$$

from the Grassmannian of oriented 2-planes in $\mathbb{R}^4 \hookleftarrow S^3$. More precisely, given a plane $p \in G_{2,4}$, we may rotate the degenerate foliation $\{\mathbb{T}_r\}$ within S^3 so that the great circle $\mathbb{T}_0$ lies within the plane p, and then define $\Phi(p)$ to count the number of points of tangency between this rotated foliation and the original torus $\mathcal{M}$.

It is possible to compute (see the proofs in [6]) that in the case that $\mathcal{M}$ is the Clifford torus, we have $\Phi(p) \equiv 8$ for almost all planes $p \in G_{2,4}$. When $\mathcal{M}$ is a general torus, a Morse theory argument, explained in [6], provides the lower bound $\Phi \geq 4$ almost everywhere. This information about Φ should be compared to the main result we wish to discuss in this article:

Theorem 1 *(Taken from [6, Theorem 4].)*
Given an immersed torus in S^3, its Willmore energy is bound by

$$W \geq \frac{\pi^2}{4} \left(\fint_{G_{2,4}} \Phi \right).$$

We see that the general Morse theoretic bound $\Phi \geq 4$ a.e. coupled with Theorem 1 is only enough to establish that $W \geq \pi^2$ which is already implied by the trivial bound $W \geq 4\pi$ mentioned earlier. However, if our torus $\mathcal{M}$ is invariant under the antipodal map in S^3, then we showed in [6] that by performing Morse theory on the quotient of $\mathcal{M}$ under the antipodal map, and then lifting back to the covering $\mathcal{M}$, we have the stronger bound $\Phi \geq 8$, and hence

Corollary 1 *(Taken from [6, Corollary 5]. Proved with different methods by Ros [4].)*

For any torus $\mathcal{M}$ immersed in S^3 which is invariant under the antipodal map, the Willmore conjecture $W \geq 2\pi^2$ holds.

Combining with a result of Kitagawa [2] which states that any flat torus in S^3 is invariant under the antipodal map, we recover the following well-known result of Chen [1] (which has extensions given in [3] and [6]).

Corollary 2 *For any flat torus $\mathcal{M}$ immersed in S^3, the Willmore conjecture $W \geq 2\pi^2$ holds.*

We would also like to give another corollary of Theorem 1 which is much more general than the two given above. Before doing so, we make the following observation of the consequences of antipodal symmetry.

Proposition 1 *If $\mathcal{M}$ is a torus immersed in S^3 which is invariant under the antipodal map, then:*

(P1) *Given any great circle C in S^3 disjoint from $\mathcal{M}$, the distance function $\mathcal{M} \to \mathbb{R}$ given by $x \to dist_{S^3}(x, C)$ has at least 2 local minima.*

(P2) *Given any great circle C intersecting $\mathcal{M}$, if all of the intersections are transverse, then there must be at least four of them.*

Proof. To establish property P1, take any point $x \in \mathcal{M} \hookrightarrow S^3 \hookrightarrow \mathbb{R}^4$ where the given distance function is minimised. Then $-x$ is also a minimum.

Property P2 follows from the fact that $\mathcal{M}$ separates S^3 into two components Σ_1 and Σ_2, and the fact, proved by Ros in [4], that the antipodal map *preserves* each of these components, rather than switches them, since $\mathcal{M}$ has odd genus. (Note, of course, that if $x \in \mathcal{M} \cap C$ then $-x \in \mathcal{M} \cap C$.) ∎

This proposition shows that the following corollary of Theorem 1 is a strict generalisation of the other two.

Corollary 3 *For any torus $\mathcal{M}$ immersed in S^3 which satisfies both properties P1 and P2, the Willmore conjecture $W \geq 2\pi^2$ holds.*

Proof. The proof of this final corollary follows along the same lines as the proof of the Morse theoretic bound $\Phi \geq 4$ mentioned above (see [6]). By Theorem 1, all we must show is that $\Phi(p) \geq 8$ for almost all planes $p \in G_{2,4}$. Given a plane $p \in G_{2,4}$ spanned by orthonormal vectors e_1 and e_2 in $\mathbb{R}^4$, consider the function $f : \mathcal{M} \to [0, 1]$ defined by

$$f(x) = \langle x, e_1 \rangle^2 + \langle x, e_2 \rangle^2,$$

and representing the square of the length of x once projected in $\mathbb{R}^4$ onto the plane p. We propose to apply Morse theory to f, for p within the set of full measure for which f is a Morse function *and* for which any intersection between either great circle $p \cap S^3$ or $p^\perp \cap S^3$ and $\mathcal{M}$ occurs transversely.

The critcial points of f with critical value within $(0, 1)$ - i.e. neither zero nor one - are then the points of tangency counted by $\Phi(p)$. Therefore, all we must do is to use properties P1 and P2 to establish the existence of at least eight of these.

After reflecting on the similarity between f and the distance function appearing in property P1, we make the following observations.

First, if $p^\perp \cap \mathcal{M} = \emptyset$ then we may apply P1, with $C = p^\perp \cap S^3$, to find that f has at least two local minima with critical value greater than zero (i.e. both of these critical points are counted by $\Phi(p)$).

Second, if $p \cap \mathcal{M} = \emptyset$ then we may apply P1, with $C = p \cap S^3$, to find that f has at least two local maxima with critical value less than one (i.e. both of these critical points are counted by $\Phi(p)$).

Third, if $p^\perp \cap \mathcal{M} \neq \emptyset$ then we may apply P2, with $C = p^\perp \cap S^3$, to find that f has at least four local minima with a critical value of zero (i.e. none of these critical points are counted by $\Phi(p)$).

Fourth, if $p \cap \mathcal{M} \neq \emptyset$ then we may apply P2, with $C = p \cap S^3$, to find that f has at least four local maxima with a critical value of one (i.e. none of these critical points are counted by $\Phi(p)$).

Finally, since Morse theory implies that the number of saddle points of f is at least the total number of maxima and minima (since $\mathcal{M}$ is a torus! - see [6, Section 4.3]) the observations above provide us with at least eight critical points with critical value within $(0, 1)$. ■

With a view to proving the entire Willmore conjecture, we are then led to ask:

Question Can every torus $\mathcal{M} \hookrightarrow S^3$ be moved by a conformal transformation of S^3 so that properties P1 and P2 hold?

Strategy of the proof of Theorem 1

A general principle of which several incarnations may be found in [6] is that the Willmore energy is related to the harmonic map energy of various forms of Gauss map. In this proof, we consider the usual Gauss map $\mathcal{G} : \mathcal{M} \to G_{2,4}$ which assigns, to each point on the torus $\mathcal{M} \hookrightarrow S^3 \hookrightarrow \mathbb{R}^4$, the oriented tangent plane to the surface. It is well known, by representing elements of $G_{2,4}$ by unit simple 2-vectors in $\bigwedge^2(\mathbb{R}^4)$ and projecting onto components of equal length in the self-dual and antiself-dual subspaces $\bigwedge^2_+(\mathbb{R}^4)$ and $\bigwedge^2_-(\mathbb{R}^4)$, that $G_{2,4}$ may be identified with $S^2 \times S^2$ (see [6]). This enables us to see the Gauss map as a map $\sigma : \mathcal{M} \to S^2 \times S^2$, and decompose into components $\sigma = (a, b)$ with $a : \mathcal{M} \to S^2$ and $b : \mathcal{M} \to S^2$, and our strategy is to perform an integral geometric construction on the target $S^2 \times S^2$.

In order to do this, we assign to each point $(r, s) \in S^2 \times S^2$, the associated torus of points in $S^2 \times S^2$ defined by

$$T_{r,s} := \{(k, l) \in S^2 \times S^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle r, k \rangle = \langle s, l \rangle = 0\}.$$

For example, when r and s are the north poles in their respective spheres, $T_{r,s}$ is the product of the equators.

Our construction is then to assign to each point $x \in \mathcal{M}$ the torus $T_{a(x),b(x)} \hookrightarrow S^2 \times S^2$. As we move x around the two dimensional surface $\mathcal{M}$, the two dimensional torus $T_{a(x),b(x)}$ then sweeps out *four* dimensional volume in $S^2 \times S^2$. We are then able to relate this total swept volume V both to the Willmore energy of $\mathcal{M}$, and the average of the tangency map Φ.

Indeed, local considerations suggest that if $\mathcal{M}$ has large curvature in some region then a and b will move substantially within that region, and thus so will $T_{a(x),b(x)}$, sweeping out a large volume in

$S^2 \times S^2$. A calculation along these lines (see [6]) tells us that

$$W \geq \frac{1}{64} V. \tag{1}$$

Meanwhile we can count the number of times the sweeping tori $T_{a(x),b(x)}$ hit an arbitrary point $(k,l) \in S^2 \times S^2$. First we observe the pretty duality, typical of integral geometry, that $(k,l) \in T_{a,b}$ is exactly equivalent to $(a,b) \in T_{k,l}$. A calculation (see [6]) then shows that the Gauss map condition $\sigma(x) = (a(x), b(x)) \in T_{k,l}$ is precisely equivalent to the condition that the surface $\mathcal{M}$ at x is tangent to a torus in the rotated family $\{\mathbb{T}_r\}$ for which $\mathbb{T}_0$ lies in the plane corresponding to (k,l) under the identification $G_{2,4} \simeq S^2 \times S^2$. By definition of Φ, we then have that (k,l) is hit by $T_{a(x),b(x)}$ a total of $\Phi(k,l)$ times, and hence that

$$V = \int_{S^2 \times S^2} \Phi = 16\pi^2 \fint_{G_{2,4}} \Phi. \tag{2}$$

Combining (1) and (2) we then have the desired result. ■

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