MA3B8 Complex Analysis

University of Warwick

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https://homepages.warwick.ac.uk/~maseq

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Contents

0 Preface

Almost all sections come with an accompanying video.

Click on the text in the green boxes to launch the videos.

Welcome to the Complex Analysis course that I prepared between 2021 and 2023 at the University of Warwick. Almost all sections have their own embedded video. My aspiration is that the lecture notes are correct and complete, while the videos may have the odd typo and minor slip. Generally I try to point these out next to the links to the videos.

These notes were supplemented by live lectures, which included an introduction to the subject and motivation. These notes currently have no introductory video or motivation section.

Current Warwick students should use the version of the notes on the module web page, which contains additional information about what is examinable, and links with previous Warwick modules. The starting point for this course is the basic grounding in complex analysis that is given by the final part of the Analysis 3 module at Warwick. However, we review most of this material, generally either without proofs or with different proofs.

I have used many sources for the material in this course. The classic book is that of Ahlfors, and this was useful to give perspective. I learned my basic complex analysis from Alan Beardon, and there may be elements of his lectures/exercises echoed in these notes. According to the book of Ahlfors, the proof we give of the general homology form of Cauchy's theorem is due to Beardon.

The most significant sources for these notes were the lecture notes of previous versions of this module at Warwick, most recently taught by Stefan Adams using notes that at the time of writing can be found here, and before that by Hendrik Weber. When constructing this module I generally followed the logical development of the subject used by previous lecturers, with major deviations in the treatment of winding numbers, isolated singularities, simply connected domains and homotopies of curves. In some subsections I followed very closely the earlier notes, particularly those of Hendrik Weber. For example, the first half of Section 6 and much of Section 7 would fit into this category, but quite a few other parts have their roots in Hendrik's presentation, and I am grateful to him for being happy for me to post these notes. In particular, I am not claiming any great originality. The notes are simply released in the hope that they might be useful to somebody.

Finally, many thanks are due to the many students who picked up errors and made suggestions, including Thomas Macdonagh, Liam O'Neill, Ricardo Antunes Ferreira, Sebastian Woodward, George Coote, Josh Bridges, Peter Job, Edward Masding, Andy Song, Ladislas Colonna Walewski, Adi Tangirala, Amiella Venturini, John Wang, Logan Aitchison etc.

Peter Topping, Warwick, Autumn 2023

1 Review of basic complex analysis I

Most of this material you will have seen, but we will write some of it in different language, and interpret some of it more geometrically.

1.1 Complex differentiability - definitions and intuition

Click on green bar for video

VIDEO: Complex differentiability

By now you should have a good working understanding of complex numbers, how to add, multiply and conjugate them. You will understand the topology of the complex plane, i.e. what it means for a sequence $\{z_n\}$ to converge to some $z \in \mathbb{C}$. You will understand what it means for a function $f \colon \mathbb{C} \to \mathbb{C}$ to be continuous.

The notion of a function being complex differentiable is so fundamental that we repeat it.

Definition 1.1. Suppose $\Omega \subset \mathbb{C}$ is open. Then a function $f : \Omega \to \mathbb{C}$ is complex differentiable at $z \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1.1.1}$$

exists in $\mathbb{C}$. We denote the limit by $f'(z)$ and call it the derivative of f at z.

In the discussion below, we sometimes view Ω as a subset of $\mathbb{R}^2$ rather than $\mathbb{C}$ in the obvious way without change of notation, i.e. we consider

$$\{(x, y) \in \mathbb{R}^2 \ : \ x + iy \in \Omega\},$$

and still write this as Ω. In this case we still use f to denote the resulting function. Sometimes (but less and less as time goes on) we decompose f into real and imaginary parts:

$$f(x + iy) = f(x, y) = u(x, y) + iv(x, y),$$

for real valued functions $u, v : \Omega \to \mathbb{R}$. That is, $u = \Re(f)$ and $v = \Im(f)$, the real and imaginary parts.

If the corresponding function $F : \Omega \to \mathbb{R}^2$ defined by

$$F(x, y) = (u(x, y), v(x, y))$$

is differentiable in the sense of multivariable calculus, then we say that f is *real* differentiable. As you know, f being complex differentiable is a lot stronger than f being *real* differentiable. This is because if we take limits $h \to 0$ in (1.1.1) by setting $h = \alpha \in \mathbb{R}$ and sending $\alpha \to 0$, or by setting $h = i\alpha$, $\alpha \in \mathbb{R}$ and sending $\alpha \to 0$, then we should obtain the same answer. This imposes some extra rigidity. In the first case, we see $f'(z) = \frac{\partial f}{\partial x}(z)$, and in the second case we find that $f'(z) = \frac{1}{i}\frac{\partial f}{\partial y}(z)$. The statement that these are equal, i.e.

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}, \tag{1.1.2}$$

is precisely equivalent to the Cauchy-Riemann equations. Indeed, f being complex differentiable is precisely equivalent to the pair of conditions that f is real differentiable AND the Cauchy-Riemann equations hold.

To write all this in the slickest way possible, we define

$$f_{\bar{z}} := \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \tag{1.1.3}$$

and

$$f_z := \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \tag{1.1.4}$$

which make sense at any point where f is real differentiable. Take great care with the signs here!

The Cauchy-Riemann equations can then be written simply as

$$f_{\bar{z}} = 0. \tag{1.1.5}$$

Although f_z makes sense for merely real differentiable f, if f is also complex differentiable at z (so $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$) then we find that $f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = f'(z)$, so we have

$$f_{\bar{z}} = 0 \qquad \text{and} \qquad f'(z) = f_z.$$

Remark 1.2. Let's pause to try to understand better the meaning of the Cauchy-Riemann equations. First, we can rephrase (1.1.5) or (1.1.2) as

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}. \tag{1.1.6}$$

Let's try to go from this algebraic statement to a geometric picture. By definition, the quantity $\frac{\partial f}{\partial x}$ is the velocity vector of the path $x \mapsto f(x+iy)$. Similarly, the quantity $\frac{\partial f}{\partial y}$ is the velocity vector of the path $y \mapsto f(x+iy)$. They are related by the multiplication by i. But geometrically this a rotation anticlockwise by 90 degrees, according to (1.1.6).

I'll draw some pictures in the lectures/videos.

Remark 1.3. We see that at a point z where f is complex differentiable, the derivative of F is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ that is a rotation and dilation (expansion or contraction). Indeed, in complex notation this linear map is given by $w \mapsto f'(z)w$. It preserves orthogonality and is invertible, provided $f'(z) \neq 0$. This will be handy to apply the Inverse Function Theorem later.

Now let's switch from considering complex differentiability at individual points to considering complex differentiability in open sets.

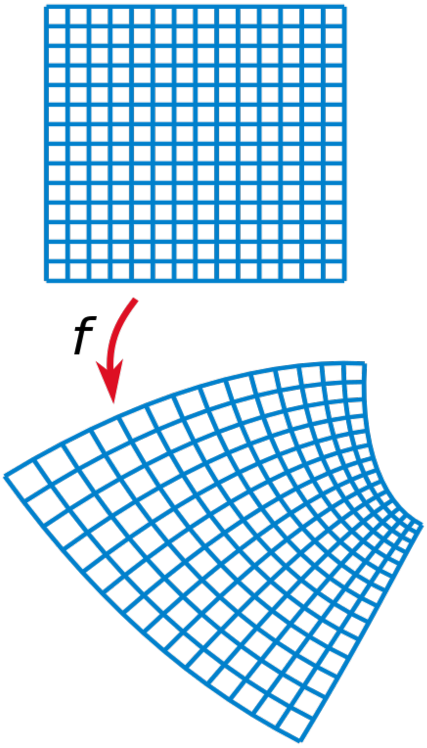


Figure 1: A conformal map. Picture created by Oleg Alexandrov (public domain).

Definition 1.4. Suppose $\Omega \subset \mathbb{C}$ is open. We say that a function $f : \Omega \to \mathbb{C}$ is **holomorphic** if it is complex differentiable at every point $z \in \Omega$. In the case that $\Omega = \mathbb{C}$, we sometimes say that f is **entire**.

Being holomorphic is a much stronger condition than being merely continuously differentiable from $\mathbb{R}^2$ to $\mathbb{R}^2$. As an example, there are many continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are 'real valued' in that they map into $\mathbb{R} \times \{0\}$, but the only ones of these that come from holomorphic functions are the *constant* functions.

By the discussion in Remark 1.2, a holomorphic function that has nonzero derivative everywhere has the property that it preserves angles/orthogonality. For example, it will map a grid of lines to deformed grid of curves that meet orthogonally as in Figure 1. *See the video for an explanation of this figure.*

Generally, a function or map preserving angles like this is called *conformal*. We will settle on a precise definition of 'conformal' that is adapted to this course in Section 2.9.

1.2 Product and chain rules

VIDEO: Product and chain rules

In this section we recast the familiar product and chain rules in the $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ notation.

By definition of $f_{\bar{z}}$ and f_z, given two functions f and g that are real differentiable at z, we have product rules

$$(fg)_{\bar{z}} = f.g_{\bar{z}} + f_{\bar{z}}g \qquad \text{and} \qquad (fg)_z = f.g_z + f_z g. \tag{1.2.1}$$

In particular, if f and g are both *complex* differentiable at z then so is fg, and

$$(fg)'(z) = f(z)g'(z) + f'(z)g(z). \tag{1.2.2}$$

We also need several variants of the chain rule in this notation.

Lemma 1.5. *Suppose $f : \Omega \to \mathbb{C}$, for $\Omega \subset \mathbb{C}$ open, and $\gamma : I \to \Omega$, for $I \subset \mathbb{R}$ some open interval. If γ is differentiable at $t \in I$ and f is real differentiable at $\gamma(t)$, then $f \circ \gamma : I \to \mathbb{C}$ is differentiable at t and*

$$(f \circ \gamma)'(t) = f_z(\gamma(t))\gamma'(t) + f_{\bar{z}}(\gamma(t))\overline{\gamma'(t)}. \tag{1.2.3}$$

In particular, if f is complex differentiable at $\gamma(t)$ then

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t). \tag{1.2.4}$$

Proof. If we write $\gamma(t) = u(t) + iv(t)$, then the usual chain rule tells us that

$$(f \circ \gamma)'(t) = f_x(\gamma(t))u'(t) + f_y(\gamma(t))v'(t).$$

On the other hand, if we expand out the right-hand side of (1.2.3), omitting the arguments, then we obtain

$$f_z\gamma' + f_{\bar{z}}\overline{\gamma'} = \frac{1}{2}(f_x - if_y)(u' + iv') + \frac{1}{2}(f_x + if_y)(u' - iv') = f_x u' + f_y v',$$

which is the same. □

Essentially the same calculation gives us the following chain rules, the proofs of which are left to the exercises.

Lemma 1.6. *Suppose that $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open sets. If $g : \Omega_1 \to \Omega_2$ is real differentiable at $z \in \Omega_1$, and $f : \Omega_2 \to \mathbb{C}$ is complex differentiable at $g(z)$, then $f \circ g$ is real differentiable at z and we have the two chain rules*

$$(f \circ g)_z(z) = f'(g(z))g_z(z), \tag{1.2.5}$$

and

$$(f \circ g)_{\bar{z}}(z) = f'(g(z))g_{\bar{z}}(z). \tag{1.2.6}$$

In particular, if g is also *complex* differentiable at z, then $f \circ g$ is *complex* differentiable at z and we have the chain rule

$$(f \circ g)'(z) = f'(g(z))g'(z). \tag{1.2.7}$$

1.3 Exercises

To start off, we try to get used to the notation f_z and $f_{\bar{z}}$, also written $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$. We will assume throughout that functions are sufficiently regular to admit the derivatives we consider!

1.1. Verify by computation that $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$.

Deduce (without repeating the computation) that $\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}$.

Remark: Another way of writing these identities would be $\overline{f_z} = \bar{f}_{\bar{z}}$ and $\overline{f_{\bar{z}}} = \bar{f}_z$. But if you use this notation you have to take care not to confuse $\overline{f_z}$ and $\bar{f}_z$.

1.2. Verify that $\frac{\partial z}{\partial z} = 1$, $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial z} = 0$ and $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$.

1.3. Verify that $f_{z\bar{z}} = \frac{1}{4}\Delta f$. Here, $f_{z\bar{z}}$ means $(f_z)_{\bar{z}}$.

1.4. Verify that the assertion $f_{\bar{z}} = 0$ is equivalent to the pair of Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$.

1.5. By expanding out definitions, verify the product rules (1.2.1) and (1.2.2).

1.6. Use appropriate product rules to verify that

(a) $(|z|^2)_z = \bar{z}$, and

(b) $(z^n)_z = nz^{n-1}$ for $n \in \mathbb{N}$ (by induction).

Remark: Do **not** do the computation using x and y. Please write everything in terms of z and $\bar{z}$.

1.7. Which of the following define holomorphic functions $f : \mathbb{C} \to \mathbb{C}$?

(a) $f(z) = \bar{z}$

(b) $f(z) = \Re(z)$

(c) $f(z) = |z|^2$

(d) $f(z) = \frac{1}{1+z^2}$

1.8. Prove that if $h : \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function (i.e. h is C^2 and $\Delta h \equiv 0$) then $h_z : \mathbb{C} \to \mathbb{C}$ is holomorphic.

1.9. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Prove that the function $z \mapsto \overline{f(\bar{z})}$ is also entire.

Remark: We'll use this principle later when we encounter so-called Schwarz reflection.

1.10. *This is a chance to revise some basic facts from topology that we will need a few times. It is not a complex analysis exercise.*

Recall that an open set $\Omega \subset \mathbb{C}$ is said to be *connected* if whenever we partition Ω into two open sets $A, B \subset \Omega$, then either A or B is the empty set (so the other is Ω). This is more abstract than the notion of *path connected* but easier to work with.

Suppose $\Omega \subset \mathbb{C}$ is a connected open set, and $f : \Omega \to \mathbb{C}$ is a locally constant function. That is, each point $z_0 \in \Omega$ has a neighbourhood around it on which f is constant. Prove that f is constant throughout.

1.11. Suppose $\Omega \subset \mathbb{C}$ is open and $f\colon \Omega \to \mathbb{C}$ is a holomorphic function with $f' \equiv 0$. Prove that f is constant on each connected component of Ω.

Remark: In other words, if f is a real differentiable function such that $f_z \equiv f_{\bar{z}} \equiv 0$, then f is constant on each connected component.

1.12. If you still have some energy, you could try to do the calculations to prove the following extension of Lemma 1.6. Otherwise, it's safe to move on! No answers will be given for this!

Lemma: Suppose that $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open sets. If $g : \Omega_1 \to \Omega_2$ is real differentiable at $z \in \Omega_1$, and $f : \Omega_2 \to \mathbb{C}$ is real differentiable at $g(z)$, then $f \circ g$ is real differentiable at z and we have the two chain rules

$$(f \circ g)_z(z) = f_z(g(z))g_z(z) + f_{\bar{z}}(g(z))\bar{g}_z(z), \tag{1.3.1}$$

and

$$(f \circ g)_{\bar{z}}(z) = f_z(g(z))g_{\bar{z}}(z) + f_{\bar{z}}(g(z))\bar{g}_{\bar{z}}(z). \tag{1.3.2}$$

Warning: when we write $\bar{g}_z$, we mean $\frac{\partial}{\partial z}(\bar{g})$, whereas $\overline{g_z}$ would be $\bar{g}_{\bar{z}}$, i.e. $\frac{\partial}{\partial \bar{z}}(\bar{g})$. See Q. 1.1.

2 Möbius transformations

In general, Complex Analysis gains a lot of its power and interest from being a combination of analysis and geometry; this section makes a start on the geometric side.

2.1 The Riemann sphere

VIDEO: The Riemann sphere

It will turn out to be a very good idea to extend the complex plane by adding an extra point at infinity; we write

$$\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$$

for this extended complex plane.

We delay a discussion of why this is such a good idea for a moment until we have equipped $\mathbb{C}_\infty$ with a bit more structure. At the moment, it is just a set. We can equip it with a topology by viewing it as the one point compactification of $\mathbb{C}$. It doesn't matter if you don't know what that means because we can see the topology extremely explicitly. In particular, a sequence $\{z_i\}$ in $\mathbb{C} \subset \mathbb{C}_\infty$ converges to $\infty \in \mathbb{C}_\infty$ if $z_i \to \infty$ in the usual sense.

But a topology is not enough geometric structure. We want to be able to see $\infty \in \mathbb{C}_\infty$ in just the same way as we would see any other point in $\mathbb{C}_\infty$. Eventually we want to be able to make sense of whether a function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ can be thought of as being holomorphic, including at ∞, and including at points that map to ∞. An efficient way of seeing all points in $\mathbb{C}_\infty$ as equal is to consider *stereographic projection*. This will turn $\mathbb{C}_\infty$ into the *Riemann sphere*.

2.2 Stereographic projection

VIDEO: Stereographic projection

Typo at around 06:30 in the video: the π in red needs to be moved to the other point. That is, swap (x_1, x_2, x_3) and $\pi(x_1, x_2, x_3)$.

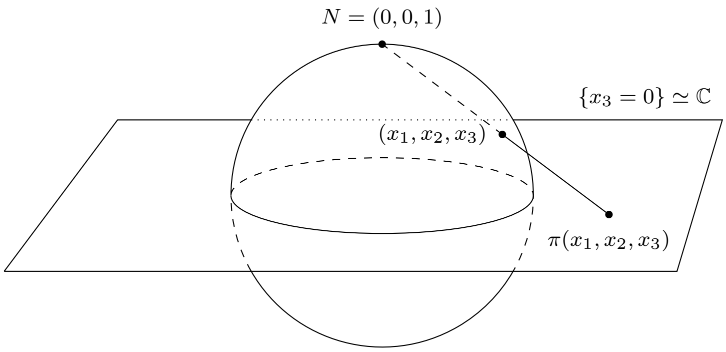
We would like to realise concretely the extended complex plane as a sphere.

To do this, we start by imagining the points $x + iy$ in the complex plane as points $(x, y, 0)$ in $\mathbb{R}^3$.

We will find a correspondence between this complex plane and $S^2 \setminus N$, where

$$S^2 := \{(x_1, x_2, x_3) \ : \ x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$

and $N := (0,0,1)$ is the 'north pole'. We do this by mapping each point (x_1, x_2, x_3) on the unit sphere, other than N, to the unique point on the plane that is on the line through N and (x_1, x_2, x_3).



Thus the south pole $S := (0,0,-1)$ is mapped to $0 \in \mathbb{C}$. The lower hemisphere is mapped to the unit disc in $\mathbb{C}$. Each point on the unit circle representing the equator is fixed under the correspondence. The upper hemisphere is mapped to the complement of the unit disc in $\mathbb{C}$.

This map is the so-called stereographic projection and by some basic trigonometry that is discussed in the exercises on page 28, one can find an explicit formula for it:

Definition 2.1. We define stereographic projection $\pi : S^2 \setminus N \to \mathbb{C}$ by

$$\pi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}. \tag{2.2.1}$$

It extends to a bijection $\pi : S^2 \mapsto \mathbb{C}_\infty$ by sending $N \in S^2$ to $\infty \in \mathbb{C}_\infty$.

The inverse of π can be computed to be the map $\mathbb{C} \mapsto S^2 \setminus N$ given by

$$z = x + iy \mapsto \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right), \tag{2.2.2}$$

Warning: Don't let the notation make you think that x, y, z are coordinates in $\mathbb{R}^3$! Of course, z is complex, x and y are real, $|z|^2 = x^2 + y^2$ etc.

The bijection $\pi : S^2 \mapsto \mathbb{C}_\infty$ can be used to transfer the standard topology on S^2 to a topology on $\mathbb{C}_\infty$; this coincides with the topology we alluded to in the previous section. With this topology in hand we can say, for example, that the function $z \mapsto 1/z$ is a homeomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$, without worrying about any singularity at 0 (now mapped to the point ∞ in the target $\mathbb{C}_\infty$) and without worrying about the function omitting 0 in the range (now ∞ in the domain is mapped to 0). In fact, the function $z \mapsto 1/z$ corresponds to a rotation of the sphere by 180° as we will see later in Section 2.5.

The bijection $\pi : S^2 \mapsto \mathbb{C}_\infty$ also allows us to endow $\mathbb{C}_\infty$ with some additional geometric structure. The essential idea is that just as every point in the sphere is much the same as ever other, for many purposes every point in $\mathbb{C}_\infty$ is much the same as every other, including ∞. This will allow us to make sense of a continuous function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ being complex differentiable at a point $z_0 \in \mathbb{C}$ for which $f(z_0) = \infty$. Geometrically we simply rotate the target under the transformation $z \mapsto \frac{1}{z}$ so the point ∞ becomes the point 0, and ask that $z \mapsto \frac{1}{f(z)}$ is complex differentiable at z_0. Alternatively, we could make sense of f being complex differentiable at ∞ by rotating the domain. If $f(\infty) \neq \infty$ this amounts to asking that $z \mapsto f(\frac{1}{z})$ is holomorphic at 0, while in the remaining case that $f(\infty) = \infty$ we can rotate both domain and target and ask that $z \mapsto \frac{1}{f(\frac{1}{z})}$ is holomorphic at 0.

In Section 2.3 we will find that the function $f(z) = 1/z$ is just one of a large class of homeomorphisms $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ known as Möbius transformations for which both f and f^{-1} are holomorphic (i.e. complex differentiable everywhere in the above sense).

Remark 2.2. Stereographic projection has the property that circles on S^2, i.e. intersections of S^2 with planes in $\mathbb{R}^2$ that consist of more than just one point, are mapped to either circles or lines in $\mathbb{C}$. More precisely, if the circle does not pass through N then π maps it to a circle in $\mathbb{C}$, while if the circle passes through N, then it is mapped to a line. You will check these facts in exercise sheet Q. 2.3 and Q. 2.4. This correspondence also works the other way round. That is, circles and lines in $\mathbb{C}$ are mapped to circles in S^2 by π^{-1}.

This fact motivates the following definition.

Definition 2.3. A circle in $\mathbb{C}_\infty$ is any subset of $\mathbb{C}_\infty$ that arises as the image under π of the intersection of S^2 with any plane that intersects the open unit ball in $\mathbb{R}^3$. In other words, it is either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with the point $\infty \in \mathbb{C}_\infty$.

The requirement that the plane intersects the open unit ball is there to ensure that the intersection is nonempty and does not consist of only one point.

To reiterate, a circle in $\mathbb{C}_\infty$, when restricted to $\mathbb{C}$, is precisely a line or a circle.

Historical remark: Stereographic projection was known at least as far back as Hipparchus, born about 2200 years ago.

Hipparchus of Nicaea, c.190BC - c.120BC. Greek astronomer, mathematician and geographer. Considered by some to be the founder of trigonometry.

2.3 Möbius transformations - definition and first properties

August Ferdinand Möbius (1790 - 1868).

VIDEO: Möbius transformations

These are going to be bijections, and even homeomorphisms, from the Riemann sphere to itself, with special properties.

Definition 2.4. The Möbius transformations (also known as Möbius maps) are the functions $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form
$$f(z) = \frac{az+b}{cz+d},$$
for coefficients $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. For $c = 0$, the point ∞ is sent to ∞. For $c \neq 0$, the point $z = -d/c$ is sent to $\infty \in \mathbb{C}_\infty$, and the point $z = \infty$ is sent to a/c.

All Möbius transformations are continuous functions on the whole of $\mathbb{C}_\infty$. You should verify that you agree with this; the claim is still valid at $z = \infty$, and for $c \neq 0$, it is still valid at $z = -d/c$ despite that point being mapped to ∞.

Why are we asking that $ad - bc \neq 0$? Because if not, then f will just map to one point because some cancellation is possible in the expression for f. For example, if $d \neq 0$ then we would have $az + b = \frac{bc}{d}z + b = \frac{b}{d}(cz + d)$, in which case $f(z) = \frac{b}{d}$, which is independent of z.

As it is, because of the assumption that $ad - bc \neq 0$, the Möbius transformations are invertible:

Lemma 2.5. *The Möbius transformation*
$$f(z) = \frac{az+b}{cz+d} \tag{2.3.1}$$
is invertible from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. The inverse f^{-1} is also a Möbius transformation and is given by
$$f^{-1}(z) = \frac{dz-b}{-cz+a}. \tag{2.3.2}$$

It would be a good idea to pause and give a proof by direct calculation. Brushing the issue of ∞ under the rug, you are essentially setting $w = \frac{az+b}{cz+d}$ and then rearranging in order to write z in terms of w. Or you could simply verify that
$$f\left(\frac{dz-b}{-cz+a}\right) = z.$$
We won't do either here because this lemma will follow from a more general picture in a moment.

The invertibility claimed in Lemma 2.5 above implies that Möbius transformations are not just continuous functions, they are even *homeomorphisms*.

Lemma 2.6. *Let f_1 and f_2 be two Möbius transformations given by*
$$f_i = \frac{a_i z + b_i}{c_i z + d_i} \qquad i = 1, 2.$$
Then $f_1 \circ f_2$ is again a Möbius transformation and is given by
$$f_1 \circ f_2(z) = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}. \tag{2.3.3}$$

The proof of formula (2.3.3) is a simple calculation that you must do yourself in order to get a feeling for what is going on. There is one slightly subtle point in the claim that this is a Möbius transformation, which is that Möbius transformations must have coefficients that satisfy the nondegeneracy condition $ad - bc \neq 0$, and it is initially a bit daunting to verify this directly for the transformation in (2.3.3). One could instead establish this indirectly by double-checking that failure to satisfy this nondegeneracy condition would imply that the composition was constant, as indicated above; but the composition of two homeomorphisms is another homeomorphism and so cannot be constant. However, there is a more illuminating approach that we describe in the next section, so we are happy to delay that part of the proof.

Already, equipped with Lemmata 2.5 and 2.6, we see that the set of Möbius transformations forms a group under composition, with the identity $f(z) = z$ being the identity of the group.

2.4 $PSL(2,\mathbb{C})$

VIDEO: $PSL(2,\mathbb{C})$

Consider the map from $GL(2,\mathbb{C})$, i.e. the group of invertible 2×2 matrices with complex entries, to the set of Möbius transformations, given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \longmapsto \qquad f_M(z) := \frac{az+b}{cz+d}. \tag{2.4.1}$$

The assumption $ad - bc \neq 0$ in the definition of Möbius transformation is precisely the condition that the matrix on the left-hand side of (2.4.1) is invertible. Indeed we have $\det(M) = ad - bc$. In particular, the map $M \mapsto f_M$ is well-defined and every Möbius transformation arises from some matrix $M \in GL(2,\mathbb{C})$.

Given two matrices $M_1, M_2 \in GL(2,\mathbb{C})$, written

$$M_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \qquad i = 1, 2,$$

consider the Möbius transformations $f_1 := f_{M_1}$ and $f_2 := f_{M_2}$ as in Lemma 2.6. Because the product $M_1 M_2$ has determinant $\det(M_1 M_2) = \det(M_1)\det(M_2) \neq 0$, it also lies in $GL(2,\mathbb{C})$, as we need for $GL(2,\mathbb{C})$ to be a group. Therefore its image $f_{M_1 M_2}$ under (2.4.1) is a Möbius transformation. A calculation reveals that

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix},$$

and so the Möbius transformation $f_{M_1 M_2}$ is precisely the map $f_1 \circ f_2$ in (2.3.3)! In particular, we now see that $f_1 \circ f_2$ is indeed a Möbius transformation, i.e. it satisfies the nondegeneracy condition, completing the proof of Lemma 2.6. In other notation, this can be written

$$f_{M_1} \circ f_{M_2} = f_{M_1 M_2}. \tag{2.4.2}$$

One consequence of (2.4.2) is that Lemma 2.5 concerning the inversion of Möbius transformations is now obviously true simply because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The identity (2.4.2) is also telling us that the map defined in (2.4.1) is a group homomorphism.

This group homomorphism is not a group isomorphism because although it is surjective, it is not injective. Indeed, if you take any matrix $M \in GL(2,\mathbb{C})$, and any nonzero $\lambda \in \mathbb{C}$, then the scaled matrix λM will also lie in $GL(2,\mathbb{C})$, and will give exactly the same Möbius transformation under (2.4.1). You can almost fix this by replacing $GL(2,\mathbb{C})$ with the smaller group $SL(2,\mathbb{C})$ in which the determinant is assumed to be not just nonzero, but to be equal to 1. (Note that $\det(\lambda M) = \lambda^2\det(M)$.) We still have a surjective group homomorphism, this time from $SL(2,\mathbb{C})$ to the group of Möbius transformations, but injectivity fails still because for any matrix $M \in SL(2,\mathbb{C})$, the negative matrix $-M$ will also lie in $SL(2,\mathbb{C})$, and will then give exactly the same Möbius transformation under (2.4.1) as before. A simple calculation shows that the kernel of this group homomorphism is precisely the subgroup $\{\pm I\}$. Indeed, if f_M from (2.4.1) is the identity $f_M(z) \equiv z$, then we must have $b = c = 0$ and $a/d = 1$. Together with the constraint that $\det(M) = 1$, this implies that $M = I$ or $M = -I$.

We can turn this homomorphism into an isomorphism by factoring out the kernel $\{\pm I\}$ and considering the quotient group

$$PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm I\},$$

where the P stands for *projective*. By the first isomorphism theorem for groups we obtain:

Lemma 2.7. *The map from $PSL(2,\mathbb{C})$ to the group of Möbius transformations induced by* (2.4.1) *is a group isomorphism.*

We will take a closer look at this in Section 2.10.

2.5 Decomposition of Möbius transformations

VIDEO: Decomposition of Möbius transformations

Minor mistake at 27:30 in the video. I got the 2 lines the wrong way round: the top half circle goes to the bottom line, and the bottom half circle to the top line.

There are several special classes of Möbius transformations from which all others can be derived, as we claim in Lemma 2.9.

Definition 2.8. We will call the following Möbius transformations *elementary*:

(i) *Translations*: For $b \in \mathbb{C}$, these are maps of the form $f(z) = z + b$.

(ii) *Rotations*: For $\theta \in \mathbb{R}$, these are maps of the form $f(z) = e^{i\theta}z$. Their effect is to rotate anti-clockwise about the origin by an angle θ.

(iii) *Dilations*: For $\lambda > 0$ these are maps of the form $f(z) = \lambda z$. They act as an expansion if $\lambda > 1$ or a contraction if $\lambda < 1$.

(iv) *Complex inversion*: This is the map $f(z) = \frac{1}{z}$. Its effect is most easily understood as a map from S^2 to S^2 using the stereographic projection. In that viewpoint it is a rotation by $180°$ about the x_1-axis in $\mathbb{R}^3$. In other words, the map $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$.

Beware: the use of term 'inversion' is potentially misleading. For many purposes, *inversion* would refer to the map $z \mapsto 1/\bar{z}$, which fixes the unit circle, sending $re^{i\theta}$ to $\frac{1}{r}e^{i\theta}$. The complex inversion is thus an inversion followed by a reflection given by complex conjugation.

The interpretations of translations, rotations and dilations that we have given are obvious. To verify the interpretation of the complex inversion as the rotation by $180°$, we can compute directly using the formula (2.2.1) for π in Section 2.2. To see this, first, note that the map $z \mapsto 1/z$, in coordinates, is $x + iy \mapsto \frac{x-iy}{x^2+y^2}$. Also note that because (x_1, x_2, x_3) lies in the unit sphere, we have $x_1^2 + x_2^2 = 1 - x_3^2 = (1 - x_3)(1 + x_3)$. Then as a map on S^2 we get

$$(x_1, x_2, x_3) \quad \overset{\pi}{\longmapsto} \quad \frac{x_1 + ix_2}{1 - x_3} \quad \overset{z \mapsto 1/z}{\longmapsto} \quad (1-x_3)\frac{x_1 - ix_2}{x_1^2 + x_2^2} = \frac{x_1 - ix_2}{1 + x_3} \quad \overset{\pi^{-1}}{\longmapsto} \quad (x_1, -x_2, -x_3),$$

Lemma 2.9. *Every Möbius transformation can be written as the composition of elementary Möbius transformations.*

Proof. First recall that given an arbitrary nonzero complex number, written as $re^{i\theta}$, with $r > 0, \theta \in \mathbb{R}$, the transformation $z \mapsto re^{i\theta}z$ is a composition of a rotation by θ and a dilation by a factor r, both of which are elementary.

For $c = 0$, the Möbius transformation would be $z \mapsto \frac{az+b}{d}$. This arises by composing the elementary transformations

$$z \longmapsto \frac{a}{d}z \longmapsto \frac{a}{d}z + \frac{b}{d}.$$

For $c \neq 0$ we can write

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}.$$

Then this Möbius transformation is obtained by composing the elementary transformations

$$z \longmapsto cz \longmapsto cz + d \longmapsto \frac{1}{cz+d} \longmapsto \frac{b - \frac{ad}{c}}{cz+d} \longmapsto \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}. \tag{2.5.1}$$

□

We can often use this decomposition into elementary Möbius transformations in order to prove that certain properties are preserved under general Möbius transformations: The proof is then reduced to checking that the property is preserved for elementary Möbius transformations, as in the following theorem. Keeping in mind our definition of 'circles' in $\mathbb{C}_\infty$ from Definition 2.3, we have

Theorem 2.10. *The image of every circle in $\mathbb{C}_\infty$ under any Möbius transformation is also a circle in $\mathbb{C}_\infty$.*

Proof. We only need check this property for each type of elementary Möbius transformation. The property is obvious for translations, rotations and dilations. The property for the complex inversion follows from its interpretation as a 180° rotation, together with the preservation of circles/lines by stereographic projection given in Remark 2.2. □

2.6 Three points determine a Möbius transformation

VIDEO: Three points determine a Möbius transformation

In this section we will see the fundamental (and very useful) property that if we ask for three specific distinct points in $\mathbb{C}_\infty$ to be mapped to another three specific distinct points in $\mathbb{C}_\infty$, then we determine a unique Möbius transformation. Precisely, we have:

Theorem 2.11. *Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and three distinct points $w_1, w_2, w_3 \in \mathbb{C}_\infty$, there exists a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.*

Thus the group of Möbius transformations has three complex degrees of freedom. This illustrates very clearly how it is a six real-parameter family.

The proof of Theorem 2.11 will be supported by a couple of sub-results. We start with an observation about the fixed points of Möbius transformations.

Lemma 2.12. *Every Möbius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ other than the identity $f(z) = z$ has at least one, but at most two, fixed points. In particular, if f is a Möbius transformation and $z_1, z_2, z_3 \in \mathbb{C}_\infty$ are distinct points such that $f(z_i) = z_i$, then f is the identity.*

Note that a Möbius transformation such as $f(z) = z + 1$ has no fixed points in $\mathbb{C}$. But ∞ is a fixed point in this case.

Proof. As usual, we write

$$f(z) = \frac{az + b}{cz + d}.$$

Observe that f being the identity corresponds to the case $a = d \neq 0$ and $b = c = 0$, so if we assume that this is not the case then we need to show that there can be at most two fixed points.

We split the proof into two cases: First, if $c = 0$, then f is just a linear transformation $f(z) = \frac{a}{d}z + \frac{b}{d}$, where $a, d \neq 0$ by nondegeneracy of Möbius transformations. Such a linear transformation has a fixed point at infinity, plus at most one further fixed point at $z = \frac{b}{d-a}$ if $a \neq d$.

In the remaining case that $c \neq 0$, we have

$$f(z) = \frac{az+b}{cz+d} = z \iff (az+b) = (cz+d)z$$
$$\iff 0 = cz^2 + (d-a)z - b\,.$$

The quadratic formula gives us two solutions, which might coincide. □

Next, we give a special case of the desired Theorem 2.11 in which the points w_i are explicit. In this special case we also give formulae for the Möbius transformation, which will be useful later.

Proposition 2.13. *Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$, there exists a Möbius transformation f that maps z_1, z_2, z_3 to $1, 0, \infty$ respectively. In the case that $z_i \neq \infty$ for $i = 1, 2, 3$, then it is*

$$f(z) := \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}. \tag{2.6.1}$$

In the case that $z_1 = \infty$ we set

$$f(z) = \frac{z-z_2}{z-z_3}, \tag{2.6.2}$$

in the case that $z_2 = \infty$ we set

$$f(z) = \frac{z_1-z_3}{z-z_3}, \tag{2.6.3}$$

and in the case that $z_3 = \infty$ we set

$$f(z) = \frac{z-z_2}{z_1-z_2}. \tag{2.6.4}$$

Proof. Clearly each of these functions f are Möbius transformations. By inspection, they send the points z_i to the required image points. □

In the final three cases (2.6.2) to (2.6.4) in which one of the z_i is ∞, the formulae for f arise from (2.6.1) by dropping the factor on the numerator and the factor on the denominator containing that z_i. These two factors would cancel in the limit $z_i \to \infty$, so this makes sense.

We are finally in a position to prove the main result of this section.

Proof of Theorem 2.11.
Existence: Let f_1 be the function from Proposition 2.13 that sends z_1, z_2, z_3 to $1, 0, \infty$ respectively. Let f_2 be the function from Proposition 2.13 that sends w_1, w_2, w_3 to $1, 0, \infty$ respectively. The Möbius transformation f we seek is simply $f_2^{-1} \circ f_1$.

Uniqueness: Suppose that we have two Möbius transformations f and g, both of which send the points z_i to w_i respectively. Then $g^{-1} \circ f$ is a Möbius transformation that has all three distinct points z_i as fixed points. Lemma 2.12 then tells us that $g^{-1} \circ f$ is the identity, i.e. $f \equiv g$. □

2.7 The cross ratio

Definition 2.14. The cross-ratio of pairwise distinct $z_0, z_1, z_2, z_3 \in \mathbb{C}_\infty$, is the point in $\mathbb{C}$ that is the image of z_0 under the unique Möbius transformation that sends z_1, z_2, z_3 to $1, 0, \infty$ respectively.

Remark 2.15. Formulae for the Möbius transformation f, and hence for $f(z_0)$, are given by Proposition 2.13.

Theorem 2.16. *The cross-ratio is invariant under Möbius transformations.*

Proof. If f is any Möbius transformation, we have to show that the cross-ratio of (z_0, z_1, z_2, z_3) is the same as that of $(f(z_0), f(z_1), f(z_2), f(z_3))$. If we denote by g the unique Möbius transformation that sends z_1, z_2, z_3 to $1, 0, \infty$ respectively, then this cross ratio is $g(z_0)$ by definition. But then the unique Möbius transformation that sends $f(z_1), f(z_2), f(z_3)$ to $1, 0, \infty$ respectively is $g \circ f^{-1}$, and so the cross-ratio of $(f(z_0), f(z_1), f(z_2), f(z_3))$ is $g \circ f^{-1}(f(z_0)) = g(z_0)$, which is the same. □

Theorem 2.17. *The cross ratio of (z_0, z_1, z_2, z_3) is real-valued if and only if z_0, z_1, z_2, z_3 all lie on a common circle in $\mathbb{C}_\infty$.*

Recall from Definition 2.3 that circles in $\mathbb{C}_\infty$ restrict to circles and lines in $\mathbb{C}$.

Proof. Since both the cross-ratio and the property of lying on a circle are invariant under Möbius transformations, we may assume that z_1, z_2, z_3 equal $1, 0, \infty$ respectively. The cross ratio of $(z_0, 1, 0, \infty)$ is z_0 by definition. But the circle passing through 1, 0 and ∞ is the real line. □

Let's digest a consequence of what we have proved. If $z_0, z_1, z_2, z_3 \in \mathbb{C}$, then they all lie on a common line or circle in the plane if and only if

$$\frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)} \in \mathbb{R}.$$

Amazing! Try it out on a few examples.

Remark 2.18. If you can remember the formula for the cross-ratio then you can use it to compute the Möbius transformation from Theorem 2.11 (let's call it h) sending three distinct points z_1, z_2, z_3 to distinct points w_1, w_2, w_3. This is because the invariance of cross-ratio under Möbius transformations from Theorem 2.16 tells us that

$$(z, z_1, z_2, z_3) = (h(z), w_1, w_2, w_3),$$

so we can expand out both sides using the formula for cross-ratio from Proposition 2.13 (recall Remark 2.15) and solve for $h(z)$.

The alternative is to consider a general Möbius transformation $h(z) = \frac{az+b}{cz+d}$, plug in $h(z_i) = w_i$ for $i = 1, 2, 3$, and solve for a, b, c, d (up to a factor).

2.8 Examples and special classes of Möbius transformations

VIDEO: Examples of Möbius transformations

Around 23:10 when I said 'reflection', I meant 'rotation'. I'll explain more in the live lecture.

There are numerous distinguished examples and subclasses of Möbius transformations, of which we give a few prominent examples. Many of them will map a specific region bijectively to the unit disc

$$D := \{z \in \mathbb{C} \,:\, |z| < 1\}.$$

Example 2.19 (The Cayley transform: A Möbius transformation that gives a bijection from the upper half-space to the disc)**.** Consider the Möbius transformation

$$f(z) = \frac{z-i}{z+i}.$$

Then

$$f(z) \in D \iff |f(z)| < 1 \iff |z-i| < |z+i| \iff \Im(z) > 0,$$

because such $z \in \mathbb{C}$ satisfying $|z-i| < |z+i|$ would be closer to i than $-i$. Thus f is a bijection from the upper half-space $H_{\Im>0} := \{z \in \mathbb{C} \,:\, \Im(z) > 0\}$ to the unit disc D.

Example 2.20 (Möbius transformation that gives a bijection from a disc to a half-space, constructed by picking 3 points)**.** It's easy enough to find a Möbius transformation that is a bijection from D to $H_{\Im>0}$ by inverting the Cayley transform of Example 2.19. Here is a direct way of constructing such functions f. Since Möbius transformations are homeomorphisms from $\mathbb{C}_\infty$ to itself, f will map the boundary of D, i.e. the unit circle, to the boundary of $H_{\Im>0}$, i.e. the real axis plus ∞. There are many maps with this property. By Theorem 2.11 we can pick any three distinct points on the unit circle and map them to 1, 0 and ∞ respectively. For example, we could pick 1, $-i$ and i on the unit circle, in which case the map is given by (2.6.1) as

$$f(z) = \frac{(z-(-i))(1-i)}{(z-i)(1-(-i))} = \frac{z+i}{iz+1}.$$

By Theorem 2.10, this Möbius transformation will map the entire unit circle ∂D to the entire real axis plus ∞. Because f is a homeomorphism, this map must send the disc D either to the upper half-plane $H_{\Im>0}$, or the lower half-plane. But since it sends 0 to i, it must be the former case, as required.

Example 2.21 (Möbius transformations that give bijections from the disc to itself)**.** Consider the Möbius transformations of the form

$$f(z) = \frac{z-w}{\bar{w}z-1} \qquad \text{for } w \in \mathbb{C}, \text{ with } |w| < 1. \tag{2.8.1}$$

We claim that Möbius transformations of this form map D onto itself, and map the boundary of D to itself. To see this, we use the easily-checked identity

$$|z-w|^2 = |\bar{w}z-1|^2 - (1-|z|^2)(1-|w|^2)$$

to compute

$$|f(z)|^2 = \frac{|z-w|^2}{|\bar{w}z-1|^2} = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|\bar{w}z-1|^2}.$$

Because we are assuming that $1-|w|^2 > 0$, we see that $|f(z)| < 1$ if and only if $|z| < 1$, and $|f(z)| = 1$ if and only if $|z| = 1$.

This argument also implies that f maps *onto* D, since f is a bijection from $\mathbb{C}_\infty$ to itself. An alternative argument to obtain the surjectivity would be to observe that the inverse of f is f itself!

Remark 2.22. We can slightly generalise the class of Möbius transformations from Example 2.21 that map the disc D to itself, by composing with a rotation about the origin, giving maps of the form

$$f(z) = e^{i\theta}\left(\frac{z-w}{\bar{w}z-1}\right), \tag{2.8.2}$$

still with $w \in D$, and now with $\theta \in (-\pi, \pi]$. A stunning fact that will follow from the so-called Schwarz lemma, Theorem 7.15, is that every holomorphic map $D \mapsto D$ that is a bijection is of the form (2.8.2). We are not starting with the assumption that the map is a Möbius transformation here. This is true in far greater generality! Think how different this is to what you have seen before. Just imagine if there were only a finite-dimensional family of real differentiable bijective functions from $(-1,1)$ to itself!

Example 2.23 (Möbius transformations that give bijections from $H_{\Im>0}$ to itself)**.** A Möbius transformation $g(z)$ from D to itself can be converted into a Möbius transformation $h := f^{-1} \circ g \circ f$ from $H_{\Im>0}$ to itself, where $f : H_{\Im>0} \to D$ is the Cayley transform from Example 2.19. One can also conjugate in the other direction to give $g = f \circ h \circ f^{-1}$. Because of this, it may seem pointless to consider Möbius transformations from $H_{\Im>0}$ to itself after already considering Möbius transformations from D to itself in Example 2.21.

However, working in the $H_{\Im>0}$ viewpoint has some significant advantages in certain situations. To see one, we return to the isomorphism between the group of Möbius transformations and the group $PSL(2,\mathbb{C})$ from Section 2.4. The key observation is that the subgroup

$$PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\},$$

essentially restricting from complex matrices in $SL(2,\mathbb{C})$ to real matrices in $SL(2,\mathbb{R})$, but as before identifying each pair A and $-A$, corresponds to an interesting subgroup of Möbius transformations. Indeed, we claim that they send the upper half plane $H_{\Im>0}$ to itself. To see this, we rewrite

$$f(z) = \frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)} = \frac{ac|z|^2 + ad\,z + bc\,\bar{z} + bd}{|cz+d|^2}.$$

Therefore, keeping in mind that $\Im(\bar{z}) = -\Im(z)$ and $ad-bc = 1$, we have

$$\Im(f(z)) = \frac{\Im(ad\,z + bc\,\bar{z})}{|cz+d|^2} = \frac{(ad-bc)\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2}$$

In particular, if $z \in H_{\Im>0}$, equivalently $\Im(z) > 0$, if and only if $\Im(f(z)) > 0$, equivalently $f(z) \in H_{\Im>0}$. Thus f maps $H_{\Im>0}$ to itself.

Similarly to before, we see that f also maps $H_{\Im>0}$ *onto* itself. For example, one can observe that the inverse of f is another Möbius transformation of the same form, and thus maps $H_{\Im>0}$ to itself.

One can check that every Möbius transformation that maps $H_{\Im>0}$ bijectively to itself arises in this way. To see this, consider the circle $\mathcal{C}$ in $\mathbb{C}_\infty$ consisting of the real line plus ∞. This is the boundary of $H_{\Im>0}$ in $\mathbb{C}_\infty$. Any Möbius transformation mapping $H_{\Im>0}$ bijectively to itself must then map $\mathcal{C}$ bijectively to itself (since f is a homeomorphism). If we now take the points $x_1, x_2, x_3 \in \mathcal{C}$ that map to the points $1, 0, \infty$ respectively, then we can construct a Möbius transformation in $PSL(2,\mathbb{R})$ using Proposition 2.13 that has the same effect on these three points. By Theorem 2.11 itself, the two Möbius transformations must then coincide.

Example 2.24. The rotations of the unit sphere $S^2 \hookrightarrow \mathbb{R}^3$ are Möbius transformations when we identify S^2 and $\mathbb{C}_\infty$ as above. By rotations, we mean elements of $SO(3)$. Since the group of Möbius transformations is isomorphic to $PSL(2,\mathbb{C})$, by Lemma 2.7, it is natural to ask to which subgroup of $PSL(2,\mathbb{C})$ these rotations correspond. It turns out that it is the subgroup $PSU(2)$. The corresponding Möbius transformations can be written

$$z \mapsto \frac{az - \bar{c}}{cz + \bar{a}},$$

for $a, c \in \mathbb{C}$ with $|a|^2 + |c|^2 = 1$.

2.9 Conformal maps

VIDEO: Conformal maps

Around 37:50 for a couple of minutes I keep saying omega when I mean gamma. It should be clear because I am writing γ and occasionally I correct myself!

A general principle in mathematics is that one defines an object with some structure, for example a vector space with its linear structure, and then one considers bijective maps between different objects that preserve this structure, for example a bijection between vector spaces that preserves the linear structure. Such bijections are typically called isomorphisms, and intuitively we view two objects that are isomorphic as being the same. You will have seen many other examples of this viewpoint, for example isometries between metric spaces, isomorphisms between groups, homeomorphisms between topological spaces or maybe diffeomorphisms between manifolds (if you have studied manifolds).

What is the right notion of equivalence for domains in $\mathbb{C}$?

(By *domain* here we mean a **nonempty, open and connected subset**. Beware that sometimes domain means this, and sometimes it simply means the space that a function maps from. You have got to work out which from context.)

In order to work towards a sensible answer to this question, we need to define what it means for a map/function f to be *conformal* and what it means to be *biholomorphic*.

Definition 2.25. Given an open set $\Omega \subset \mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to be a *conformal map*, if f is holomorphic and $f'(z) \neq 0$ for all $z \in \Omega$.

The point to keep in mind here is that such a function preserves angles in the sense discussed in Section 1.1. For example, an anticlockwise rotation of $\frac{\partial f}{\partial x}$ by 90 degrees gives $\frac{\partial f}{\partial y}$. The word conformal is used in several slightly different ways, even in this precise subject, but the idea of preserving angles is always present.

A conformal map as we have defined it is not necessarily injective. Consider the function $f(z) = z^2$ defined on the domain $\mathbb{C} \setminus \{0\}$ for example.

Definition 2.26. A function $f : \Omega_1 \to \Omega_2$ between open sets $\Omega_1, \Omega_2 \subset \mathbb{C}$ is said to be *biholomorphic* if it is a bijection such that both f and f^{-1} are conformal maps.

We will see later the remarkable fact that every bijective holomorphic function f is automatically biholomorphic. Thus the facts that the inverse f^{-1} is holomorphic and the derivatives of both f and f^{-1} don't vanish will come for free. Let's not worry about why right now, but later you will be able to refer to Theorem 7.14.

We can now give an answer to our earlier question.

Definition 2.27. Two domains Ω_1 and Ω_2 in $\mathbb{C}$ are said to be *conformally equivalent* if there exists a biholomorphic function $\varphi : \Omega_1 \to \Omega_2$.

This notion of equivalence gives us an equivalence relation. In particular, if Ω_1 and Ω_2 are conformally equivalent via the biholomorphic map $\varphi : \Omega_1 \to \Omega_2$, and Ω_2 and Ω_3 are conformally equivalent via the biholomorphic map $\psi : \Omega_2 \to \Omega_3$, then Ω_1 and Ω_3 are conformally equivalent via the biholomorphic map $\psi \circ \varphi : \Omega_1 \to \Omega_3$. We are implicitly using the chain rule to be sure that the composition of these biholomorphic maps is biholomorphic.

The key property that is preserved by this notion of equivalence is the concept of a function being holomorphic. More precisely, by the chain rule, a function $f : \Omega_2 \to \mathbb{C}$ is holomorphic if and only if the composition $f \circ \varphi : \Omega_1 \to \mathbb{C}$ is holomorphic.

For the rest of this section we try to get a feeling for which domains are conformally equivalent to which other domains. Our knowledge of Möbius transformations will help. Throughout the discussion we continue to write the unit disc as $D := \{z \in \mathbb{C} \ : \ |z| < 1\}$.

Before we begin, let's recall that by writing down the Cayley transform in Example 2.19 we already showed that the upper half-space is conformally equivalent to the disc D. Let's find some more such domains.

Come to the lectures (or watch the video) for pictures!

Example 2.28. We claim that the upper right quarter of the complex plane

$$Q := \{z \in \mathbb{C} \mid \Re(z) > 0 \text{ and } \Im(z) > 0\}$$

is conformally equivalent to the disc D.

To see this, we first observe that Q is conformally equivalent to the upper half plane $H_{\Im>0}$ by virtue of the map $z \mapsto z^2$ which is biholomorphic from Q to the upper half plane.

The upper half plane is then conformally equivalent to D via the Cayley transform of Example 2.19.

Example 2.29. We claim that the upper half disc

$$D_{\Im>0} := \{z \in \mathbb{C} \; : \; |z| < 1 \text{ and } \Im(z) > 0\}$$

is conformally equivalent to the whole disc D.

Danger: It is very tempting to try to use the conformal map $z \mapsto z^2$ to do this job, but this actually shows the quite different fact that $D_{\Im>0}$ is conformally equivalent to the disc D with the real interval $[0, 1)$ removed!

Instead, we notice that $D_{\Im>0}$ is conformally equivalent to Q via the unique Möbius transformation that sends -1, 0 and 1 to 0, 1 and ∞ respectively. One could compute this explicitly, but we can argue more geometrically as follows: First, we know from Theorem 2.11 that this Möbius transformation exists. Second, by the preservation of circles in $\mathbb{C}_\infty$ from Theorem 2.10 it sends the interval $[-1, 1]$ to the interval $[0, \infty]$. Third, by Theorem 2.10 again it must send the semicircle $\{z \in \mathbb{C} \; : \; |z| = 1 \text{ and } \Im(z) \geq 0\}$ to a half line starting at 0 and going in some direction off to infinity. Finally, the right-angle in the boundary of $D_{\Im>0}$ at -1 must induce a right angle in the boundary of the image of $D_{\Im>0}$ under this Möbius transformation, because the Möbius transformation preserves angles, and so the half line must be the positive imaginary axis.

Now we have shown that $D_{\Im>0}$ is conformally equivalent to Q, the claim follows by Example 2.28.

By this point you may be getting the false impression that *every* domain is conformally equivalent to D. But this is not true. One type of counterexample would be to take the domain that is the whole of $\mathbb{C}$. If we could find a biholomorphic function from $\mathbb{C}$ to D then this would be a bounded holomorphic function on $\mathbb{C}$, and therefore constant by Liouville's theorem that we will see later in Corollary 6.7 once we have rigorously proved Cauchy's theorem. In this case it could not be surjective onto D.

Another sort of domain that would certainly fail to be conformal to the disc D would be a domain 'with holes' such as an annulus $\{z \in \mathbb{C} \; : \; a < |z| < b\}$, where $0 < a < b < \infty$. This domain is not even homeomorphic to D so it is a bit much to ask for it to be homeomorphic via a homeomorphism that is additionally a conformal map! To see that they are not homeomorphic, it suffices to notice that one is simply connected while the other is not, as will be easy to prove rigorously later – see Q. 4.5. The notion of being simply connected has been briefly mentioned in Analysis 3. Informally it means that every loop in the space can be deformed to a point. In order to make this precise we need to define the notion of homotopy.

Definition 2.30 (Homotopic)**.** Let $\Omega \subset \mathbb{C}$ be open and let $\gamma_1, \gamma_2 : [a, b] \to \Omega$ be two continuous paths, i.e. continuous maps from an interval $[a, b]$, with the same endpoints $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$. Then γ_1 and γ_2 are said to be *homotopic* if there exists a continuous map $h : [0, 1] \times [a, b] \to \Omega$ such that for all $s \in [0, 1]$ we have

$$h(s, a) = \gamma_1(a), \quad \text{and} \quad h(s, b) = \gamma_1(b),$$

i.e. the paths $t \mapsto h(s, t)$ have the same endpoints also, and for all $t \in [a, b]$ we have

$$h(0, t) = \gamma_1(t), \quad \text{and} \quad h(1, t) = \gamma_2(t),$$

i.e. the paths $t \mapsto h(s,t)$ interpolate between γ_1 and γ_2 as s increases from 0 to 1. Such a map h is called a *homotopy* from γ_1 to γ_2.

The notion of Ω being simply connected is intuitively that it does not contain any holes. To make this precise, we recall:

Definition 2.31. A continuous path $\gamma : [a,b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$, i.e. the path closes up.

Definition 2.32 (Simply connected)**.** An open set $\Omega \subset \mathbb{C}$ is said to be *simply connected* if it is connected and every closed continuous path $\gamma : [a,b] \to \Omega$ is homotopic to the constant path $\tilde{\gamma} : [a,b] \to \Omega$ defined by $\tilde{\gamma}(t) = \gamma(a) = \gamma(b)$.

In more general situations the definition of simply connected would ask for *path* connectedness, but path connectedness is equivalent to connectedness when considering an open set. In this very special situation of open sets Ω in $\mathbb{C}$, the notion of being simply connected is equivalent to **both** Ω being connected, **and** its complement in the Riemann sphere $\mathbb{C}_\infty$ being connected. We will not explicitly use this formulation, so will not prove the equivalence.

At this point we can look forward to the end of the course when we prove one of the greatest theorems in the subject, namely the Riemann mapping theorem. That theorem will tell us that every simply connected domain $\Omega \subset \mathbb{C}$ other than $\Omega = \mathbb{C}$ is conformally equivalent to the disc D. See Theorem 11.1.

2.10 Complex projective space $\mathbb{C}P^1$

Whenever you find some magic correspondence as in the link between $PSL(2,\mathbb{C})$ and the group of Möbius transformations in Section 2.4, you can be sure there is an underlying picture that can explain it. Here this arises by considering complex projective space

$$\mathbb{C}P^1 := \mathbb{C}^2 \setminus \{(0,0)\}/\sim,$$

where the equivalence relation $\sim$ is defined by $(z_1, w_1) \sim (z_2, w_2)$ if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $z_2 = \lambda z_1$ and $w_2 = \lambda w_1$.

It turns out that $\mathbb{C}P^1$ is another way of viewing the Riemann sphere.

With the exception of the one point $[(1,0)]$ when $w = 0$, we can represent every point in $\mathbb{C}P^1$ uniquely by $(z,1)$ for some $z \in \mathbb{C}$. Thus once we've removed this one point $[(1,0)]$, we are left with something that can be viewed as a complex plane (parametrised by z). This is directly analogous to removing the north pole N of S^2 and using stereographic projection to view the remainder as a complex plane.

Similarly, with the exception of the one point $[(0,1)]$ when $z = 0$, we can represent every point in $\mathbb{C}P^1$ uniquely by $(1,w)$ for some $w \in \mathbb{C}$. Now we have a plane parametrised by w. Away from the two points $[(1,0)]$ and $[(0,1)]$, a general point can be represented by either $(1,w)$ or $(z,1) \sim (1, 1/z)$. That is, z and w are related by $w = 1/z$. Recall from Section 2.5 that the transformation $z \mapsto 1/z$

corresponds to a rotation of the Riemann sphere S^2 by 180° about the x_1 axis. Thus to go from the z coordinate to the w coordinate, we can use (inverse) stereographic projection to map to S^2, then do this rotation by 180°, and then map back to the complex plane using stereographic projection. Note how w gives a nice complex coordinate around the point $z = \infty$ in the Riemann sphere. This means we can make sense of being holomorphic to and/or from the whole Riemann sphere: We just work with respect to z or w as is convenient.

Don't fret if this is a bit quick or vague. I am just trying to smooth your transition to understanding the basics of *Riemann surfaces*.

Meanwhile, the group $GL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ as follows: If we represent points $[(z,w)]$ in $\mathbb{C}P^1$ as column vectors $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$ then we can simply left-multiply by the matrix (checking that this is well-defined). In particular, a general matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$$

maps a general point $[(z,1)]$ in $\mathbb{C}P^1 \setminus [(1,0)]$ to the point represented by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix},$$

which can be represented by $(az+b, cz+d) \sim (\frac{az+b}{cz+d}, 1)$.

To conclude, M maps $[(z,1)]$ to $[(\frac{az+b}{cz+d}, 1)]$, and we realise where the formula for Möbius transformations came from.

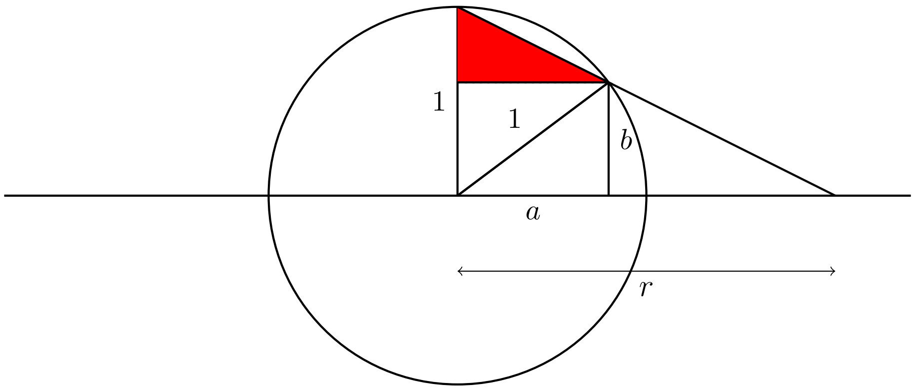


Figure 2: Stereographic projection, cross-section

2.11 Exercises

2.1. By considering a triangle similar to the red triangle in Figure 2, prove that $r = \frac{a}{1-b}$. Deduce the formula (2.2.1) for stereographic projection.

Think of Figure 2 as a cross-section in the picture describing stereographic projection.

2.2. Invert formula (2.2.1) to give (2.2.2).

Hint: Solve first for x_3. Then get x_1 and x_2.

2.3. We claimed in Remark 2.2 that straight lines in $\mathbb{C}$ are in one-to-one correspondence with circles in S^2 that pass through the north pole N, with the correspondence being given by stereographic projection π. By considering appropriate planes in $\mathbb{R}^3$ that pass through N, give a geometric justification of this fact.

You could also check this correspondence by solving equations as in the next question.

2.4. We claimed in Remark 2.2 that circles in $\mathbb{C}$ are in one-to-one correspondence with circles in S^2 that don't pass through the north pole N, with the correspondence being given by stereographic projection π. Verify this by writing down the equations of the appropriate circles.

You might also try to give a geometric proof of this, but it's trickier than the previous question!

Hint: Let's remember some basic geometric facts from school mathematics. If $n := (a, b, c)$ is a unit vector, then the plane through the origin with normal vector n is given by $(x_1, x_2, x_3).n = 0$. More generally if we shift that plane in the direction n by a distance $d \in (-1, 1)$, then the equation is $(x_1, x_2, x_3).n = d$, i.e.

$$ax_1 + bx_2 + cx_3 = d. \tag{2.11.1}$$

The case that $N = (0, 0, 1)$ lies within this plane is then precisely that $(0, 0, 1).n = d$, i.e. $c = d$. The intersection of any such plane with S^2 is a circle in S^2. All such circles arise in this way.

Now plug in the values (x_1, x_2, x_3) given by the formula (2.2.2) into the formula (2.11.1), to give a formula for the image of this circle under π.

In the case $c = d$ you should end up with an equation of a line in the plane. This is the case of the last question.

In the case $c \neq d$, you should get the equation of a circle. By completing the square, you should find that the centre is $(\frac{a}{d-c}, \frac{b}{d-c})$. The radius should be $\frac{\sqrt{1-d^2}}{|d-c|}$.

One then needs to check the other direction, i.e., that a circle in $\mathbb{C}$ is mapped to a circle in S^2. You can try to give a geometric argument for this (exploiting what we have already proved) or show it by direct computation.

2.5. The points 0 and ∞ in $\mathbb{C}_\infty$, when viewed as points in S^2 by applying π^{-1}, are the south and north poles respectively. They are therefore antipodal points. Given any other point $z \in \mathbb{C}_\infty$, i.e. $z \in \mathbb{C}$ that is nonzero, show that $-1/\bar{z} \in \mathbb{C}$ corresponds to the antipodal point. More precisely, show that $\pi^{-1}(z)$ and $\pi^{-1}(-1/\bar{z})$ are antipodal.

Hint: This is equivalent to showing that $\pi(x_1, x_2, x_3)\overline{\pi(-x_1, -x_2, -x_3)} = -1$.

2.6. In Example 2.20 we mapped the unit disc to the upper half plane with a Möbius transformation. Viewed as a transformation of the Riemann sphere, seen as a 2-sphere in $\mathbb{R}^3$, what is this transformation?

What is the square of this transformation? That is, what do we get if we apply the transformation twice? Give the formula for the resulting Möbius transformation, and describe what it looks like as a transformation of the Riemann sphere seen as a 2-sphere in $\mathbb{R}^3$?

2.7. Where does the map

$$f(z) = \frac{1+z}{1-z}$$

send the unit disc? What is the inverse of this map? What is it as a transformation of the Riemann sphere S^2?

2.8. Show that the quarter disc

$$\hat{Q} := \{z \in \mathbb{C} \ : \ |z| < 1 \text{ and } \Im(z) > 0 \text{ and } \Re(z) > 0\}$$

is conformally equivalent to the unit disc D.

2.9. Show that the slit plane

$$S := \mathbb{C} \setminus [0, \infty)$$

is conformally equivalent to the unit disc D.

2.10. Construct explicitly the Möbius transformation mapping the upper half disc to the upper right quarter Q that was determined geometrically during Example 2.29.

2.11. Construct explicitly the unique Möbius transformation f that sends D to the half space $H_{\Re>0}$, while sending 0 to $\frac{1}{2}$ and sending -1 to 0. By composing f with the map $z \mapsto z^2$ and then $z \mapsto z - \frac{1}{4}$, show that the map

$$K(z) := \frac{z}{(1-z)^2}$$

is a conformal map from D to the slit plane $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

This function is known as the Koebe function, and we will revisit it once we know a little more theory. Amongst all the conformal maps from D to $\mathbb{C}\backslash(-\infty, -\frac{1}{4}]$, it is the unique one satisfying the normalisation $K(0) = 0$ and $K'(0) = 1$.

3 Review of basic complex analysis II

3.1 Power series

VIDEO: Power Series

We will consider power series, i.e. expressions of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where a_n is a complex-valued sequence. By convention, $z^0 = 1$ above, even if $z = 0$.

Theorem 3.1. *Given a complex-valued sequence (a_n), define the so-called* ***radius of convergence*** *by*

$$R := \frac{1}{\limsup |a_n|^{1/n}} \in [0, \infty].$$

Then the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for all $|z| < R$ and diverges for all $|z| > R$.

Nothing is claimed in this theorem about z for which $|z| = R$. That question can be somewhat delicate. The cases considered in the theorem follow easily from the root test.

Power series can be differentiated term-by-term within their radius of convergence:

Theorem 3.2. *If the radius of convergence R of the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is positive (or ∞), then within the disc $B_R := \{z \in \mathbb{C} \ : \ |z| < R\}$, the function f is holomorphic, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

with this new power series having the same radius of convergence R.

A function that can be written as a power series over some ball about each point in its domain is called *analytic*. We see now that analytic implies holomorphic. Later we will see that holomorphic implies analytic. Indeed, Taylor's theorem will tell us that every holomorphic function can be written as a power series locally. For this reason, many people use the terms analytic and holomorphic interchangeably. But not us.

By repeatedly applying Theorem 3.2, we obtain

Corollary 3.3. *If the radius of convergence R of the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is positive (or ∞), then within the disc $B_R := \{z \in \mathbb{C} : |z| < R\}$, the function f is infinitely differentiable, and the n-th derivative of f at $0 \in \mathbb{C}$ is given by

$$f^{(n)}(0) = a_n n!. \tag{3.1.1}$$

Although the convergence of a power series is a bit delicate near the circle $\{z \in \mathbb{C} : |z| = R\}$, if we restrict to compact subsets of B_R then we have uniform convergence:

Theorem 3.4. *If the radius of convergence R of the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is positive (or ∞), then for all $r \in (0, R)$, the convergence

$$\sum_{n=0}^{k} a_n z^n \to \sum_{n=0}^{\infty} a_n z^n$$

is uniform within the disc B_r as $k \to \infty$.

3.2 Definitions of e^z, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$

VIDEO: Exponentials, sine, cosine etc.

You will already be familiar with the idea of writing a complex number as $re^{i\theta}$; we used that earlier. At school you probably took Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3.2.1}$$

as a definition of $e^{i\theta}$ and then defined $e^{\alpha+i\theta} = e^{\alpha}e^{i\theta}$ in order to make sense of a general complex exponential e^z, and gave ad hoc proofs that this gave something with the expected behaviour, with a promise that you would see a better motivated definition later on.

The theory of power series described in the previous section allows us to give this better motivated definition of $\exp : \mathbb{C} \to \mathbb{C}$, and also define other familiar functions such as $\sin$ and $\cos$ etc. as functions from $\mathbb{C}$ rather than $\mathbb{R}$. It does this by using the familiar Taylor series expansions of these functions on $\mathbb{R}$ and using them instead on $\mathbb{C}$.

Definition 3.5. We define $\exp : \mathbb{C} \to \mathbb{C}$, also written as $z \mapsto e^z$ by

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

From the theory we have described, for this definition to make sense we have to verify that the radius of convergence is $R = \infty$. From the definition of R, it suffices to establish that $(n!)^{1/n} \to \infty$, which is a mini analysis exercise. (You can also use the ratio test to establish this without doing any exercise, but we did not mention that.)

By differentiating term by term using Theorem 3.2, we find that the derivative of e^z is e^z itself:

$$(e^z)' = e^z. \tag{3.2.2}$$

The familiar property of the exponential function that $e^{x+y} = e^x e^y$ extends to the complex case:

Lemma 3.6. *For all* $a, b \in \mathbb{C}$ *we have*

$$e^{a+b} = e^a e^b. \tag{3.2.3}$$

Proof. Consider the function $f(z) := e^{a+b-z} e^z$. By the product and chain rules, f is entire and

$$\begin{aligned} f'(z) &= e^{a+b-z}(e^z)' + (e^{a+b-z})' e^z \\ &= e^{a+b-z} e^z - e^{a+b-z} e^z = 0. \end{aligned} \tag{3.2.4}$$

Therefore f is constant, as we saw in Q. 1.11, giving $f(z) = f(0) = e^{a+b}$ for all $z \in \mathbb{C}$, and in particular for $z = b$. But $f(b) = e^a e^b$, yielding (3.2.3). □

The definitions of $\sinh$ and $\cosh$ extend immediately to entire functions from $\mathbb{C}$ rather than just $\mathbb{R}$:

$$\sinh(z) := \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) := \frac{e^z + e^{-z}}{2},$$

and by (3.2.2) we have $\sinh'(z) = \cosh(z)$ and $\cosh'(z) = \sinh(z)$.

Moreover, if we define entire functions

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) := \frac{e^{iz} + e^{-iz}}{2}, \tag{3.2.5}$$

then these are the familiar $\sin$ and $\cos$ functions when restricted to $\mathbb{R}$; one can see this by comparing the power series, for example.

By adding the formulae (3.2.5) for $\sin$ and $\cos$, and setting $z = \theta \in \mathbb{R}$, we obtain Euler's formula (3.2.1) and we see that our new definitions are consistent with what you were told at the beginning about exponentiating complex numbers.

3.3 Argument and logarithm

VIDEO: Argument and logarithm

Euler's formula (3.2.1) shows that $e^{i\theta}$ represents a point on the unit circle, an angle θ anticlockwise from the positive real axis (i.e. the x-axis) and we confirm the familiar fact that $z \in \mathbb{C}$ can be written in the form $|z|e^{i\theta}$, where θ is determined up to the addition of 2π if $z \neq 0$ (and θ is arbitrary if $z = 0$).

This is essentially viewing z in polar coordinates (r, θ) with $r = |z|$.

For $z \neq 0$, this value θ, determined up to the addition of 2π, is known as the *argument* $\arg(z)$. Strictly speaking it should be viewed as a function taking values in $\mathbb{R}/(2\pi\mathbb{Z})$.

If you understand the picture here, then you can read off properties of $\arg(z)$ at will. For example, if z and w are nonzero complex numbers then

$$|zw|e^{i\arg(zw)} = zw = (|z|e^{i\arg(z)})(|w|e^{i\arg(w)}) = |zw|e^{i(\arg(z)+\arg(w))},$$

and so

$$\arg(zw) = \arg(z) + \arg(w) \qquad \text{modulo } 2\pi. \tag{3.3.1}$$

The multi-valued nature of arg can be annoying. There are several ways we can avoid it, and the best strategy depends on the context. A simple idea would be to ask that it takes values in $(-\pi, \pi]$, say, yielding the *principal value*. This has a major disadvantage that it makes arg discontinuous along the negative real axis. It does, however, give a nice continuous[1] function on $\mathbb{C} - \{x \in \mathbb{R} \ : \ x < 0\}$.

This idea can be generalised. If we remove any ray $\{re^{i\theta} \ : \ r > 0\}$ from $\mathbb{C}$, for some $\theta \in \mathbb{R}$, then we can define a single valued choice of $\arg(z)$ on the slit plane that remains, although different people may make a choice of $\arg(z)$ that differs by a fixed constant multiple of 2π. The ray here is known as a *branch cut*.

A particularly useful resolution of this 2π ambiguity in one specific situation will be given in Lemma 4.1.

The (complex) logarithm can be defined for $z \neq 0$ by

$$\log(z) := \log|z| + i\arg(z).$$

Here the logarithm on the right-hand side is taking real values, so it is uniquely defined. But the argument is only defined up to the addition of an integer multiple of 2π, so $\log(z)$ is only defined up to the addition of an integer multiple of $2\pi i$. As for the argument $\arg(z)$, we are sometimes content with this state of affairs, and sometimes we ask $\arg(z)$ to take its principal value in $(-\pi, \pi]$, in which case $\log(z)$ is a well defined function that extends the usual logarithm. Unfortunately it is discontinuous across the negative real axis $\{x < 0\} \subset \mathbb{C}$, and we sometimes make a *branch cut* by removing the half-line $\{x \leq 0\} \subset \mathbb{C}$. As soon as we can view the logarithm as a function, it can

[1] even smooth, i.e. infinity real differentiable

be seen to be holomorphic. You could check that with bare hands now, e.g. by verifying that the Cauchy-Riemann equations hold, but we will put the issue aside because it will come for free later on.

The complex logarithm inherits many of the useful properties of the real logarithm. For example, it is the inverse of the exponential function in the sense that

$$e^{\log z} = e^{\log|z| + i\arg(z)} = |z|e^{i\arg(z)} = z, \tag{3.3.2}$$

and

$$\begin{aligned} \log(e^z) &= \log|e^z| + i\arg(e^z) \\ &= \log e^x + i\arg(e^{iy}) \\ &= x + iy \\ &= z, \end{aligned} \tag{3.3.3}$$

where the latter computation is carried out modulo $2\pi i$.

By (3.3.1) we have the familiar identity

$$\begin{aligned} \log(zw) &= \log|zw| + i\arg(zw) = \log|z| + \log|w| + i(\arg(z) + \arg(w)) \\ &= \log z + \log w, \end{aligned} \tag{3.3.4}$$

again modulo $2\pi i$. Beware that identities that are claimed modulo $2\pi i$ should not be expected to work if we take the principal values of the functions $\log$ or $\arg$.

The function $\log(z)$ allows us to define what it means to raise a complex number to a complex power, albeit with complications arising from $\log(z)$ only being defined up to a multiple of $2\pi i$. We will revisit this later.

3.4 Complex integration

VIDEO: Complex integration

You have probably seen that to integrate a suitable (e.g. continuous) function $f : [a, b] \to \mathbb{C}$, we define

$$\int_a^b f(t)dt := \int_a^b \Re[f(t)]dt + i\int_a^b \Im[f(t)]dt. \tag{3.4.1}$$

A basic property is that

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt, \tag{3.4.2}$$

which follows by setting

$$\int_a^b f(t)dt = Re^{i\theta},$$

and computing

$$\left|\int_a^b f(t)dt\right| = R = \int_a^b \Re[e^{-i\theta}f(t)]dt \leq \int_a^b |f(t)|dt.$$

The definition in (3.4.1) allows us to define what it means to integrate a suitable function $f:\Omega\to\mathbb{C}$, where $\Omega\subset\mathbb{C}$ is open, along some 'curve' within Ω.

Definition 3.7. We say that $\gamma:[a,b]\to\mathbb{C}$ is a $\mathcal{C}^1$ curve if not only is it continuous on $[a,b]$ but the derivative γ' exists on (a,b) and extends to a continuous function $\gamma':[a,b]\to\mathbb{C}$.

Note that our convention is generally to use the word path when $\gamma:[a,b]\to\mathbb{C}$ is merely continuous, and use the word curve when γ is more regular.

This definition allows us to make sense of a one-sided derivative of γ at both endpoints a and b.

Definition 3.8. Given a continuous function $f:\Omega\to\mathbb{C}$, and a $\mathcal{C}^1$ curve $\gamma:[a,b]\to\Omega$, we define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Definition 3.8 does not require the full strength of f being continuous, but this is a simple way of stopping us from encountering integrals of nonintegrable functions, and will be fine for our purposes.

One can check that if you take a different parametrisation of γ, e.g. you take a new $\mathcal{C}^1$ curve $\tilde{\gamma}:[\tilde{a},\tilde{b}]\to\Omega$ such that $\tilde{\gamma}(t)=\gamma(\phi(t))$ for some $\mathcal{C}^1$ bijection $\phi:[\tilde{a},\tilde{b}]\to[a,b]$ with[2] $\phi':[\tilde{a},\tilde{b}]\to\mathbb{R}$ positive then

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

If we reverse the parametrisation by asking for $\phi'<0$, i.e. for the bijection ϕ to be *decreasing*, then we have

$$\int_\gamma f(z)dz = -\int_{\tilde{\gamma}} f(z)dz.$$

Later we will often need that if $|f(z)|\leq M$ then from Definition 3.8 and (3.4.2) we have

$$\left|\int_\gamma f(z)dz\right| \leq M\int_a^b |\gamma'(t)|dt = ML(\gamma), \tag{3.4.3}$$

where

$$L(\gamma) := \int_a^b |\gamma'(t)|dt \tag{3.4.4}$$

is the length of the image of γ, which is also invariant under reparametrisations of γ.

The definitions and observations above extend in an obvious way to curves γ that can have some corners in the following sense:

[2]as before, ϕ' is defined initially on $(\tilde{a},\tilde{b})$ and is assumed to extend continuously to $[\tilde{a},\tilde{b}]$

Definition 3.9. We say that $\gamma : [a, b] \to \mathbb{C}$ is a piecewise $\mathcal{C}^1$ curve if it is continuous on $[a, b]$ and there exist finitely many intermediate points $a = c_0 < c_1 < c_2 < \cdots c_n = b$ such that the restriction of γ to each interval $[c_i, c_{i+1}]$ is a $\mathcal{C}^1$ curve.

An example would be a curve tracing out a rectangle or a triangle. For example, if T is a closed triangle in $\mathbb{C}$ with vertices $z_1, z_2, z_3 \in \mathbb{C}$ (distinct points that are not colinear) then we can define a closed piecewise $\mathcal{C}^1$ curve $\gamma : [0, 3] \to \mathbb{C}$ connecting the vertices, e.g. for $t \in [0, 1]$ we can take $\gamma(t) = z_1 + t(z_2 - z_1)$ connecting z_1 to z_2. For $t \in [1, 2]$ we can take $\gamma(t) = z_2 + (t-1)(z_3 - z_2)$ connecting z_2 to z_3. For $t \in [2, 3]$ we can take $\gamma(t) = z_3 + (t-2)(z_1 - z_3)$ connecting z_3 back to z_1.

We can arrange that this path γ moves round the boundary in an anticlockwise direction by switching two of the points z_i to achieve this if necessary. The essential intuitive point is that this ensures that as we move around the boundary, we have the triangle always on the left-hand side rather than the right-hand side. Then we can write

$$\int_{\partial T} f(z)dz := \int_\gamma f(z)dz.$$

As we have mentioned above, the exact parametrisation of ∂T is not important, but the direction we travel determines the sign of the integral.

We can use similar notation for integration around the boundaries of regions other than triangles, but we don't try to make any general definitions. We only ever consider explicit situations in which the boundary is sufficiently nice to be unambiguously parametrised by a piecewise $\mathcal{C}^1$ curve γ with $\gamma'(t) \neq 0$ away from the corners, i.e. for each t in any of the intervals (c_i, c_{i+1}), and then we integrate around the boundary curve keeping the region itself on the left-hand side, as above. For example, for $a \in \mathbb{C}$ and $r > 0$ we could consider the ball

$$B_r(a) := \{z \in \mathbb{C} \, : \, |z - a| < r\} \tag{3.4.5}$$

and write

$$\int_{\partial B_r(a)} f(z)dz := \int_\gamma f(z)dz,$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$ is defined by $\gamma(\theta) = a + re^{i\theta}$. Alternatively, if

$$A := \{z \in \mathbb{C}\colon R_1 < |z| < R_2\}$$

is an annulus, then we have two boundary components that are parametrised by curves $\gamma_1, \gamma_2 : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma_2(\theta) = R_2 e^{i\theta}$ and $\gamma_1(\theta) = R_1 e^{-i\theta}$, and we analogously define

$$\int_{\partial A} f(z)dz := \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

We also need notation for the integral along straight line intervals in the plane. If $z_1, z_2 \in \mathbb{C}$, then we write $[z_1, z_2]$ for the straight line connecting z_1 to z_2. If we parametrise this line by $\gamma : [0, 1] \to \mathbb{C}$ defined by $\gamma(t) = tz_2 + (1-t)z_1$, then we write

$$\int_{[z_1, z_2]} f(z)dz := \int_\gamma f(z)dz.$$

Returning to consider the triangle T with vertices $z_1, z_2, z_3 \in \mathbb{C}$ as above, we could then alternatively write

$$\int_{\partial T} f(z)dz = \int_{[z_1,z_2]} f(z)dz + \int_{[z_2,z_3]} f(z)dz + \int_{[z_3,z_1]} f(z)dz.$$

3.5 Anti-derivatives, and a baby version of Cauchy's theorem

VIDEO: Anti-derivatives; a baby version of Cauchy's theorem

*Around 2:30 I missed out the hypothesis that γ should be **closed**. See the lecture notes below.*

Later, in Section 5, we will turn our attention to a theorem that is at the heart of complex analysis, namely Cauchy's theorem. Loosely speaking it will tell us that in certain situations when we integrate holomorphic functions around closed curves we obtain zero. In this section we see a baby version of this theory in which our holomorphic function f is the derivative of some other holomorphic function F. In fact, in this result we only need assume that $f\colon \Omega \to \mathbb{C}$ is continuous rather than holomorphic, provided it is the derivative of a holomorphic function $F\colon \Omega \to \mathbb{C}$, although this will turn out to imply that f is necessarily holomorphic anyway.

Lemma 3.10. *Suppose $\Omega \subset \mathbb{C}$ is open. Suppose further that $f : \Omega \to \mathbb{C}$ is continuous and $F : \Omega \to \mathbb{C}$ is holomorphic with $F'(z) = f(z)$. If γ is a piecewise $\mathcal{C}^1$ **closed** curve in Ω. Then*

$$\int_\gamma f(z)dz = 0.$$

This lemma follows immediately from the following type of fundamental theorem of calculus.

Lemma 3.11. *Suppose that $F : \Omega \to \mathbb{C}$ is holomorphic, with F' continuous. Suppose further that $\gamma : [a, b] \to \Omega$ is a piecewise $\mathcal{C}^1$ curve. Then*

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)). \tag{3.5.1}$$

In particular, if γ is a closed curve (i.e. $\gamma(b) = \gamma(a)$) then we have $\int_\gamma F'(z)\,dz = 0$.

Proof. By the definition of contour integration and the chain rule of Lemma 1.5, we have

$$\int_\gamma F'(z)\,dz = \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)).$$

Note that to be able to apply the usual fundamental theorem of calculus in the last equality, we need some regularity on the integrand such as continuity, which is why we are assuming that F' is continuous. $\square$

The result assumes that the derivative F' of the holomorphic function F is continuous. Later we will see that this is always true, but we can't assume that now or our arguments will be circular.

Corollary 3.12. *Suppose $n \in \mathbb{Z}$ does not equal -1. Then for $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ any piecewise $\mathcal{C}^1$ closed curve, we have*

$$\int_\gamma z^n dz = 0.$$

Proof. If we define $F(z) := z^{n+1}/(n+1)$, then $F'(z) = z^n$, so the result follows from Lemma 3.10. □

This corollary fails in a very important way if $n = -1$! In Q. 3.8 you will compute:

Example 3.13. For $r > 0$ and $k \in \mathbb{Z}$ let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed $\mathcal{C}^1$ curve $\gamma(\theta) = re^{ik\theta}$ that travels anticlockwise k times around the circle of radius r. Then

$$\int_\gamma \frac{dz}{z} = 2\pi i\, k.$$

As we will learn in Section 4, what is important about γ here is not that it traces out a circle k times but that it winds around the origin k times. Our first task in that section will be to make the idea of winding around precise.

3.6 Exercises

3.1. Suppose we have a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R_1 > 0$, and a second power series $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence $R_2 \geq R_1$. Suppose we are told that these power series give the same function on the ball $B_{R_1}(0)$ where they are both converging. Prove that $a_n = b_n$ for every $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, i.e., the power series and their radii of convergence agree.

3.2. Write the function $z \mapsto \frac{1}{1-z}$ as a power series $\sum_{k=0}^{\infty} a_k z^k$ and give its radius of convergence.

3.3. By using the previous question and a theorem from Section 3.1, write down the power series of the function $z \mapsto \frac{1}{(1-z)^2}$ and its radius of convergence.

We will use this exercise when we take a closer look at the Koebe function.

3.4. Suppose $w \in \mathbb{C} \setminus \{0\}$. Write the function $z \mapsto \frac{1}{w-z}$ as a power series $\sum_{k=0}^{\infty} a_k z^k$ and give its radius of convergence.

We will use this fact when we review the proof of Taylor's theorem.

3.5. Suppose that $n \in \mathbb{N}$, and that the power series $\sum_{k=n}^{\infty} a_k z^k$, which omits the first n terms of a general power series, has radius of convergence $R > 0$ and thus defines a holomorphic function $f : B_R(0) \to \mathbb{C}$. Prove that there exists a holomorphic function $g : B_R(0) \to \mathbb{C}$ such that

$$f(z) = z^n g(z) \text{ for all } z \in B_R(0),$$

and that g can be written as a power series with radius of convergence R.

We'll use this fact when we study the zeros of holomorphic functions.

3.6. Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{\sin z}{z}.$$

Prove that we can extend f to a function on the whole of $\mathbb{C}$ (by defining $f(0)$ to be a suitable value in $\mathbb{C}$) that is entire, i.e. holomorphic on the whole of $\mathbb{C}$.

The 'singularity' of f at 0 in this example will be known as a 'removable singularity'.

3.7. (a) For $R > 0$, Compute

$$\frac{1}{2i} \int_{\partial B_R(0)} \bar{z}\, dz,$$

and show that it agrees with the area of the ball $B_R(0)$.

(b) For $\mathcal{R} := \{z \in \mathbb{C} : \Re(z) \in [0, a], \Im(z) \in [0, b]\}$, a rectangle of side-lengths $a, b > 0$, compute

$$\frac{1}{2i} \int_{\partial \mathcal{R}} \bar{z}\, dz,$$

and show that it agrees with the area of $\mathcal{R}$.

This is not a fluke. We are seeing a couple of instances of a general fact that could be derived from an appropriate form of Stokes' theorem.

3.8. For $r > 0$ and $k \in \mathbb{Z}$ let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed $\mathcal{C}^1$ curve $\gamma(\theta) = re^{ik\theta}$ that travels anticlockwise k times around the circle of radius r. Prove that

$$\int_\gamma \frac{dz}{z} = 2\pi i\, k.$$

4 Winding numbers

4.1 Winding numbers of continuous closed paths

VIDEO: Winding numbers of continuous closed paths

See the lectures/video for an instant explanation-by-pictures of what the winding number is!

In Section 3.3 we have discussed the function $\arg(z)$, and the issue that it is only defined modulo an integer multiple of 2π. The following lemma will tell us that if we decide on a choice of $\arg(z)$ at one point z, and then move along a continuous path/curve that stays away from 0, then this determines a *unique* continuously varying choice of the argument along this path.

Lemma 4.1 (Lifting lemma)**.** *Suppose* $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ *is continuous, and fix* $\theta_0 \in \mathbb{R}$ *such that* $\gamma(a) = |\gamma(a)|e^{i\theta_0}$*. Then there exists a unique continuous function* $\theta : [a, b] \to \mathbb{R}$ *such that* $\theta(a) = \theta_0$ *and* $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ *for all* $t \in [a, b]$*.*

For example, for a curve $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$, if we choose $\theta_0 = 0$ rather than any other value in $2\pi\mathbb{Z}$ then $\theta(t) = t$. In particular, even though the start and end points are the same, the argument differs by 2π.

Those of you who are studying topology may learn one way to prove this lemma. There is an obvious 'covering map' $\mathbb{R} \mapsto \mathbb{R}/(2\pi\mathbb{Z})$ given by $\theta \mapsto \theta + 2\pi\mathbb{Z}$, and we are taking a *lift* of the function $\arg \circ \gamma : [a, b] \to \mathbb{R}/(2\pi\mathbb{Z})$.

For those who are not studying topology, here is a self-contained proof.

The lectures/video could be useful in order to understand this proof!

Proof of Lemma 4.1. First observe that if γ avoids a slit $\{-re^{i\theta_0} \ : \ r \geq 0\}$ on the opposite side of the starting point, then the existence of $\theta(t)$ is clear: In this case we can make a global continuous choice of $\arg$ on the slit plane by asking that it takes values within the interval $(\theta_0 - \pi, \theta_0 + \pi)$, and can then define $\theta(t)$ to be $\arg(\gamma(t))$.

In the general case, we are free to replace $\gamma(t)$ by the curve $\tilde{\gamma}(t) := \gamma(t)/|\gamma(t)|$ without changing the argument. Since $\tilde{\gamma}$ is a continuous function from a closed interval, it is uniformly continuous, and by dividing up $[a, b]$ into a large enough number of equal intervals, we can be sure that $\tilde{\gamma}$ does not move too far when restricted to each of these sub-intervals. More precisely, by taking $n \in \mathbb{N}$ large enough, we can be sure that for each $k \in \{0, 1, \ldots, n-1\}$ and each $t \in [c_k, c_{k+1}]$, where $c_k := a + \frac{k}{n}(b - a)$, we have $|\tilde{\gamma}(t) - \tilde{\gamma}(c_k)| < 1$.

The idea then is to do the lifting on each of these sub-intervals in turn. Indeed, the restriction of $\tilde{\gamma}$ to $[c_0, c_1]$ must avoid a slit $\{-re^{i\theta_0} \ : \ r \geq 0\}$ on the opposite side of the starting point, so by the comment at the start of the proof we can find our function $\theta(t)$ at least for $t \in [c_0, c_1]$, with $\theta(c_0) = \theta_0$. At this point we can use $\theta(c_1)$ as a new starting argument analogous to θ_0, and do our lifting on the next interval $[c_1, c_2]$, where $\tilde{\gamma}(t)$ avoids the new opposite slit $\{-re^{i\theta(c_1)} : r \geq 0\}$. This extends $\theta(t)$ to the interval $[c_0, c_2]$. By repeating this process a total of n times, we obtain a lift to the whole interval $[c_0, c_{n-1}] = [a, b]$.

To establish uniqueness of $\theta(t)$, suppose we have a second continuous lift $\hat{\theta}(t)$ also with $\hat{\theta}(a) = \theta_0$. Then $t \mapsto \theta(t) - \hat{\theta}(t)$ is a continuous function that vanishes at $t = a$, and takes values in $2\pi\mathbb{Z}$ because both $\theta(t)$ and $\hat{\theta}(t)$ represent the argument, modulo 2π. Thus $\theta(t) - \hat{\theta}(t) = 0$ for all $t \in [a, b]$, as required. □

Lemma 4.1 allows us to unambiguously define the total change in argument as we move all the way along a continuous path.

Definition 4.2. Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is continuous, and let $\theta : [a, b] \to \mathbb{R}$ be a function arising in Lemma 4.1. We define

$$\measuredangle(\gamma) := \theta(b) - \theta(a).$$

The function θ was only defined up to a constant multiple of 2π that was determined by θ_0. However, when we subtract $\theta(a)$ from $\theta(b)$ this unknown multiple of 2π will disappear, making $\measuredangle(\gamma)$ well-defined.

Definition 4.3. Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is a **closed** continuous path. Then we define the **index** or **winding number** of γ around 0 to be

$$I(\gamma, 0) := \frac{1}{2\pi}\measuredangle(\gamma) \in \mathbb{Z}.$$

More generally, if $w \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ is a closed continuous path then we define the index or winding number of γ around w to be

$$I(\gamma, w) := \frac{1}{2\pi}\measuredangle(\gamma_w),$$

where $\gamma_w : [a, b] \to \mathbb{C} \setminus \{0\}$ is the path γ translated to send w to the origin, i.e. $\gamma_w(t) := \gamma(t) - w$.

Example 4.4. For $n \in \mathbb{Z}$, consider the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = re^{in\theta}$, for some $r > 0$, which winds around the origin n times in an anticlockwise direction. Then

$$I(\gamma, 0) = n.$$

Remark 4.5. Suppose that $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ is a closed continuous path taking values within a region on which we can make a global continuous choice of $\arg(z)$. For example, for some $\alpha \in \mathbb{R}$, γ might map into the slit plane $\mathbb{C} \setminus \{-re^{i\alpha} : r \geq 0\}$, in which case we could decide to insist that $\arg(z) \in (\alpha - \pi, \alpha + \pi)$. Then one possibility for the function $\theta(t)$ of Lemma 4.1 would be $\arg(\gamma(t))$, and hence $\theta(a) = \arg(\gamma(a)) = \arg(\gamma(b)) = \theta(b)$ and we deduce that $I(\gamma, 0) = 0$. The branch cut $\{-re^{i\alpha} : r \geq 0\}$ prevents γ from winding around the origin. By translation of this picture we see that if $\gamma : [a, b] \to \mathbb{C}$ is a closed continuous path that avoids a radial line from some point $w \in \mathbb{C}$ out to infinity then $I(\gamma, w) = 0$.

4.2 Nearby closed paths have the same winding number

VIDEO: Nearby closed paths have the same winding number

In this section we prove that if we have a closed path $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$, then a small-enough perturbation of γ will wind round 0 the same number of times as γ itself.

In the lectures/video I will attempt to make this completely self evident!

However, a little care is required. If γ goes very close to 0 then we must ensure that we perturb very little so the path does not jump to the other side of 0.

Lemma 4.6 (Dog walking lemma)**.** *Suppose $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ and $\tilde{\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$ are continuous closed paths, with $|\gamma(t) - \tilde{\gamma}(t)| < |\gamma(t)|$ for every $t \in [a, b]$. Then*

$$I(\gamma, 0) = I(\tilde{\gamma}, 0).$$

The dog-walking analogy is as follows. Picture γ as your path, and $\tilde{\gamma}$ as your dog's path. There is a tree at the origin, but we are assuming from the outset that your paths lie in $\mathbb{C} \setminus \{0\}$ so neither of you hit the tree. The hypothesis says that the length of your flexi-lead[3], $|\gamma(t) - \tilde{\gamma}(t)|$, is always kept below the distance $|\gamma(t)|$ from you to the tree. Then both you and the dog go round the tree the same number of times. It's pretty obvious at an intuitive level, but the slickest proof may not be instantly clear.

Proof. Let $\theta(t)$ and $\tilde{\theta}(t)$ be lifts of the arguments of $\gamma(t)$ and $\tilde{\gamma}(t)$, respectively, as given by Lemma 4.1. Define a continuous function $\alpha : [a, b] \to \mathbb{R}$ by $\alpha(t) := \tilde{\theta}(t) - \theta(t)$. If we consider a new continuous closed path $\sigma : [a, b] \to \mathbb{C} \setminus \{0\}$ defined by

$$\sigma(t) := \frac{\tilde{\gamma}(t)}{\gamma(t)}$$

then $\sigma(t) = |\sigma(t)|e^{i\alpha(t)}$, so $\alpha(t)$ is a lift of the argument of $\sigma(t)$. By definition of winding number, we have

$$\begin{aligned} I(\tilde{\gamma}, 0) - I(\gamma, 0) &= \frac{1}{2\pi}[\tilde{\theta}(b) - \tilde{\theta}(a)] - \frac{1}{2\pi}[\theta(b) - \theta(a)] \\ &= \frac{1}{2\pi}(\alpha(b) - \alpha(a)) \\ &= I(\sigma, 0), \end{aligned} \tag{4.2.1}$$

so we are reduced to proving that $I(\sigma, 0) = 0$. But

$$|1 - \sigma(t)| = \left| \frac{\gamma(t) - \tilde{\gamma}(t)}{\gamma(t)} \right| < 1$$

by hypothesis, and so $\sigma(t)$ remains in a ball $B_1(1)$ of radius 1 centred at the point $1 \in \mathbb{C}$. Therefore $I(\sigma, 0) = 0$ by Remark 4.5. □

[3] i.e. your variable-length dog leash

Given our definition of the winding number $I(\gamma, w)$ as the number of times a curve γ winds around a point w, it is intuitively obvious that it will be constant as we vary w continuously without touching the image of γ. A precise version of this statement arises as a special case of Lemma 4.6.

Lemma 4.7. *Suppose $\gamma : [a,b] \to \mathbb{C}$ is a continuous closed path. Then on each connected component of $\mathbb{C} \setminus \gamma([a,b])$, the function $w \mapsto I(\gamma, w)$ is constant.*

Remark 4.8. Note that as the continuous image of a compact set, we know that $\gamma([a,b])$ is compact, and therefore (being a subset of $\mathbb{C}$) it is closed. We deduce that $\mathbb{C} \setminus \gamma([a,b])$ is open.

Proof. By Q. 1.10, it suffices to prove that each point in $\mathbb{C} \setminus \gamma([a,b])$ has a neighbourhood in which $I(\gamma, \cdot)$ is constant. By translation, it suffices to assume that 0 is not in the image of γ, and hence also some nonempty open ball $B_\varepsilon(0)$ does not intersect the image of γ, and to show that $I(\gamma, w)$ is constant as we vary w within $B_\varepsilon(0)$. For a given such w we can define a new path $\tilde{\gamma} : [a,b] \to \mathbb{C} \setminus \{0\}$ by $\tilde{\gamma}(t) = \gamma(t) - w$, and by definition we have $I(\gamma, w) = I(\tilde{\gamma}, 0)$. By construction we have $|\gamma(t) - \tilde{\gamma}(t)| = |w| < \varepsilon \leq |\gamma(t)|$ for every $t \in [a,b]$, and so Lemma 4.6 applies giving

$$I(\gamma, 0) = I(\tilde{\gamma}, 0) = I(\gamma, w),$$

as required. □

Remark 4.9. The previous proof used a claim that because 0 is not in the image of γ, a whole ball $B_\varepsilon(0)$ is also not in the image of γ. A more general fact is that given a continuous map $h : X \to \mathbb{C} \setminus \{0\}$ from a compact topological space X, there exists $\varepsilon > 0$ such that the image of h omits not just 0 but also the entire ball $B_\varepsilon(0)$. To see this, note that $|h|$ is a continuous function on a compact space. It therefore achieves its infimum $\varepsilon \geq 0$, which must then be positive, rather than zero, since the image omits 0.

4.3 Winding number under homotopies

VIDEO: Winding number under homotopies

During the recording of the video I realised that my notation γ_1, γ_2 would be better as γ_0, γ_1. I changed it below. It is now more logical in these notes, but slightly different to the video.

We saw in the previous section that if we move a path a little then its winding number will not change. If now we consider a homotopy[4] of closed paths then this can be broken up into a possibly large number of small adjustments to the path, none of which will change the winding number by the previous section. Homotopic closed paths will then, under reasonable conditions, have the same winding number.

Theorem 4.10. *Let $w \in \mathbb{C}$. If $\gamma_0, \gamma_1 : [a,b] \to \mathbb{C} \setminus \{w\}$ are homotopic continuous closed paths, then $I(\gamma_0, w) = I(\gamma_1, w)$. In particular, if $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ is a continuous closed path that is homotopic to a constant path, then $I(\gamma, w) = 0$.*

[4] recall Definition 2.30

It may be worth stressing that the convention is that the homotopy must remain within the image, which in this case is $\mathbb{C} \setminus \{w\}$.

Before proving this theorem, let's notice the following consequence that is immediate from the definition of simply connected, i.e. Definition 2.32.

Corollary 4.11. *If an open set $\Omega \subset \mathbb{C}$ is simply connected then for every $w \in \mathbb{C} \setminus \Omega$ and every continuous closed path $\gamma : [a, b] \to \Omega$, we have $I(\gamma, w) = 0$.*

In the lectures/video we will draw some pictures.

Proof of Theorem 4.10. By translation of w and the paths γ_0, γ_1, we may assume that $w = 0$.

The fact that γ_0 and γ_1 are homotopic continuous closed paths means that there exists a continuous map $h : [0, 1] \times [a, b] \to \mathbb{C} \setminus \{0\}$ such that

$$h(0, t) = \gamma_0(t) \quad \text{and} \quad h(1, t) = \gamma_1(t) \qquad \text{for all } t \in [a, b], \tag{4.3.1}$$

i.e. the homotopy starts at γ_0 and ends at γ_1, and such that

$$h(s, a) = z_0, \quad \text{and} \quad h(s, b) = z_0 \qquad \text{for all } s \in [0, 1], \tag{4.3.2}$$

where $z_0 := \gamma_0(a) = \gamma_1(a) = \gamma_0(b) = \gamma_1(b)$ is the fixed end point. In particular, for each $s \in [0, 1]$, we have a continuous closed curve $\gamma_s : [a, b] \to \mathbb{C} \setminus \{0\}$ defined by $\gamma_s(t) := h(s, t)$. The theorem will be proved if we can show that the winding number $I(\gamma_s, 0)$ is the same for each $s \in [0, 1]$.

By Remark 4.9, and the compactness of $[0, 1] \times [a, b]$, not only does h omit 0, it also omits an entire open ball $B_\varepsilon(0)$ for some $\varepsilon > 0$.

Because h is continuous on its compact domain, it is also uniformly continuous. In particular, we can pick $\delta > 0$ so that whenever $t \in [a, b]$ and $s_1, s_2 \in [0, 1]$ with $|s_1 - s_2| < \delta$, we must have

$$|h(s_1, t) - h(s_2, t)| < \varepsilon,$$

and therefore

$$|\gamma_{s_1}(t) - \gamma_{s_2}(t)| = |h(s_1, t) - h(s_2, t)| < \varepsilon \leq |\gamma_{s_1}(t)|.$$

By Lemma 4.6, we then have $I(\gamma_{s_1}, 0) = I(\gamma_{s_2}, 0)$ for our arbitrary $s_1, s_2 \in [0, 1]$ with $|s_1 - s_2| < \delta$. This implies that $s \mapsto I(\gamma_s, 0)$ is locally constant, and then constant for all $s \in [0, 1]$. □

4.4 The winding number as an integral

VIDEO: The winding number as an integral

If we have a closed path that is not just continuous but also piecewise $\mathcal{C}^1$, then we can characterise the winding number as an integral:

Lemma 4.12. *If $w \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ is a closed piecewise $\mathcal{C}^1$ curve, then*

$$I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w}.$$

This expression for the winding number is often used as the definition. It is often easier to use in rigorous proofs, but the definition we gave is more immediately visual, more general, and is better adapted to considering homotopies.

In Example 3.13 and Q. 3.8 we showed that for $r > 0$, $n \in \mathbb{Z}$, and $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = re^{in\theta}$, which winds around the origin n times in an anticlockwise direction, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} = n.$$

Meanwhile, in Example 4.4 we found that $I(\gamma, 0) = n$ in this case. Thus the formula claimed in Lemma 4.12 for $I(\gamma, 0)$ at least works in this case.

Proof of Lemma 4.12. By translation, we may assume that $w = 0$. We give the proof assuming that γ is $\mathcal{C}^1$. The modifications to handle piecewise $\mathcal{C}^1$ curves are straightforward. We must control

$$\int_\gamma \frac{dz}{z} = \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

Taking any function $\theta(t)$ from the Lifting lemma 4.1, so $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$, we notice that because γ is $\mathcal{C}^1$ and keeps away from 0, the function $\theta(t)$ is also $\mathcal{C}^1$ and we can compute

$$\gamma'(t) = e^{i\theta(t)} \frac{d}{dt}|\gamma(t)| + |\gamma(t)| i\theta'(t) e^{i\theta(t)},$$

and so

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{d}{dt} \log|\gamma(t)| + i\theta'(t).$$

Integrating gives

$$\int_\gamma \frac{dz}{z} = \int_a^b \left[\frac{d}{dt} \log|\gamma(t)| + i\theta'(t) \right] dt = 0 + i[\theta(b) - \theta(a)] = i\measuredangle(\gamma) = 2\pi i I(\gamma, 0)$$

because γ is closed. □

Remark 4.13. Now that we are considering piecewise $\mathcal{C}^1$ curves γ rather than just continuous paths, one could give an alternative proof of Lemma 4.7, which states that the function $w \mapsto I(\gamma, w)$ is constant on each connected component of $\mathbb{C} \setminus \gamma([a,b])$. Once one has verified real differentiability of this function, and justified differentiating under the integral sign, one can compute

$$\frac{\partial}{\partial \bar{w}} I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial \bar{w}} \left(\frac{1}{z-w} \right) dz = 0,$$

to establish that $I(\gamma, \cdot)$ is holomorphic, and then

$$\frac{\partial}{\partial w} I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial w} \left(\frac{1}{z - w} \right) dz = \frac{1}{2\pi i} \int_\gamma \frac{1}{(z-w)^2} \, dz = 0$$

by Corollary 3.12, to deduce that the derivative of $I(\gamma, \cdot)$ is zero. By Q. 1.11 we deduce that $I(\gamma, \cdot)$ is constant on each connected component of $\mathbb{C} \setminus \gamma([a, b])$. Alternatively, we could appeal to the fact that $I(\gamma, w)$ takes discrete values, in which case we only need to use Lemma 4.12 to show that $w \mapsto I(\gamma, w)$ is continuous.

4.5 Exercises

4.1. Given a continuous path $\gamma\colon [a,b] \to \mathbb{C}\setminus\{0\}$, write $-\gamma$ for the continuous path $[a,b] \mapsto \mathbb{C}\setminus\{0\}$ defined by $t \mapsto \gamma(a+b-t)$, which reverses the direction of the parametrisation (and not for the reflection of γ through the origin, i.e., not for $t \mapsto -\gamma(t)$).

(a) Verify that $\measuredangle(-\gamma) = -\measuredangle(\gamma)$.

(b) Now suppose additionally that γ is closed. What is $I(-\gamma, w)$ in terms of $I(\gamma, w)$?

4.2. Suppose that $\Omega \subset \mathbb{C}$ contains the closure of the ball $B_r(a)$ of radius $r > 0$ centred at $a \in \Omega$. Prove that for all $z_0 \in B_r(a)$ we have

$$I(\partial B_r(a), z_0) = 1$$

by showing that we may as well take $z_0 = a$ and computing.

Recall that $\partial B_r(a)$ refers to a curve γ passing once around $\partial B_r(a)$ in an anticlockwise direction.

4.3. Prove that if $\gamma : [a,b] \to \mathbb{C}$ is a closed continuous path then the set of points $w \in \mathbb{C} \setminus \gamma([a,b])$ for which $I(\gamma, w) \neq 0$ is bounded.

Remark: You might like to think of an alternative proof in the special case that γ is a piecewise C^1 curve, by using the integral formulation of winding number.

4.4. In Q. 3.4, hopefully you wrote (for $w \in \mathbb{C}\setminus\{0\}$) the function $z \mapsto \frac{1}{w-z}$ as a power series $\sum_{k=0}^{\infty} w^{-k-1} z^k$, valid for $|z| < |w|$. By replacing z by $1/z$ and setting $w = 1/z_0$, where $z_0 \in B_r(0)$ for some $r > 0$, obtain an expansion for $\frac{1}{z-z_0}$ in terms of negative powers of z, that is valid for $|z| > |z_0|$ and converges uniformly for z in ∂B_r. By integrating around $\partial B_r(0)$, prove that

$$I(\partial B_r(0), z_0) = 1$$

thus reproving Q. 4.2.

The expansion considered here is an example of a Laurent series, as discussed in Section 8.

4.5. Use winding numbers to prove the intuitively obvious statement that an annulus $A := \{z \in \mathbb{C} : a < |z| < b\}$, for $0 \leq a < b$, is not simply connected.

5 Cauchy's Theorem

Baron Augustin-Louis Cauchy (1789 - 1857).

5.1 Preamble

VIDEO: Cauchy's theorem: Analysis 3 reminder

This section is about the iconic theorem(s) of Cauchy from which a spectacular amount of wonderful theory gushes forth. The following version needs the notion of *simply connected* from Definition 2.32.

Theorem 5.1 (Cauchy's theorem on simply connected domains)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and **simply connected**. Suppose further that $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a piecewise $\mathcal{C}^1$ closed curve in Ω. Then*

$$\int_\gamma f(z)dz = 0.$$

Cauchy's theorem has a huge number of applications. For example, it eventually implies that a function $f : \Omega \to \mathbb{C}$ that is holomorphic is necessarily infinitely differentiable. Amazing. This is nothing like what happens for real differentiable functions. Just think of the function $f : \mathbb{R} \to \mathbb{R}$ that is zero for $x < 0$ and equal to x^2 for $x \geq 0$.

If you make additional hypotheses on γ, and add an additional hypothesis on f that its derivative is continuous, then there is a relatively simple proof of Theorem 5.1 using Green's theorem. This is how Cauchy originally viewed the result. It will turn out that the derivative of f is automatically continuous since a holomorphic f will be infinitely differentiable, but we will need Cauchy's theorem on star-shaped domains along the way to proving this!

In contrast, we will adopt a fully rigorous approach. In Theorem 5.7 we will prove the result above in the special case of so-called *star-shaped domains*. In due course (Theorem 9.3) we will also see a much more general form of Cauchy's theorem that includes Theorem 5.1 as a special case.

Before doing that, we recall the following special case of Example 3.13 and Q. 3.8, which shows that Cauchy's theorem fails on $\mathbb{C} \setminus \{0\}$, so the requirement that Ω is simply connected cannot simply be dropped.

Example 5.2. Consider the holomorphic function $f(z) = \frac{1}{z}$ on $\Omega := \mathbb{C} \backslash \{0\}$, and for $r > 0$ let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed $\mathcal{C}^1$ curve $\gamma(\theta) = re^{i\theta}$ that travels anticlockwise around the circle of radius r. Then

$$\int_\gamma f(z)dz = 2\pi i.$$

Although you have already done a more complicated computation in Q. 3.8, let's redo it in this special case. Note that $f(\gamma(\theta)) = f(re^{i\theta}) = r^{-1}e^{-i\theta}$, and $\gamma'(\theta) = ire^{i\theta}$, and so

$$\int_\gamma f(z)dz = \int_0^{2\pi} r^{-1}e^{-i\theta}ire^{i\theta}d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

5.2 Goursat's theorem - Cauchy's theorem on triangles

VIDEO: Goursat's theorem: Cauchy's theorem on triangles

Édouard Jean-Baptiste Goursat (1858 - 1936).

A first situation in which one rigorously proves Cauchy's theorem for general holomorphic f is when one heavily restricts the curves γ one allows and integrates around the boundary of a triangle. This will then later be used as a tool in order to prove more general results such as Theorem 5.1. This special case is named after Goursat, who came long after Cauchy.

Theorem 5.3 (Goursat's theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and contains a closed triangle T. Suppose further that $f : \Omega \to \mathbb{C}$ is holomorphic. Then*

$$\int_{\partial T} f(z)dz = 0.$$

Recall that the notation $\int_{\partial T}$ was introduced in Section 3.4.

The proof will be given in the lecture/video with pictures, which will make it far easier to understand!

Proof. The triangle T can be divided into four congruent triangles T_0, T_1, T_2 and T_3. Each vertex of these smaller triangles is either a vertex of T, or a midpoint of one of its sides. Keeping in mind cancellation along the inner edges, we can expand

$$\int_{\partial T} f(z)dz = \sum_{i=0}^{3} \int_{\partial T_i} f(z)dz,$$

and the triangle inequality then tells us that

$$\left|\int_{\partial T} f(z)dz\right| \leq \sum_{i=0}^{3} \left|\int_{\partial T_i} f(z)dz\right|.$$

Thus we can pick one of the four smaller triangles T_0, T_1, T_2 and T_3, denoted T^1 (now with a superscript), such that

$$\left|\int_{\partial T} f(z)dz\right| \leq 4\left|\int_{\partial T^1} f(z)dz\right|.$$

Now we can repeat this whole procedure starting with T^1 instead of T. We obtain an even smaller triangle $T^2 \subset T^1$ with the property that

$$\left|\int_{\partial T^1} f(z)dz\right| \leq 4\left|\int_{\partial T^2} f(z)dz\right|.$$

Iterating gives a nested sequence T^n of triangles whose diameters and boundary lengths decay geometrically in that $\operatorname{diam}(T^n) = 2^{-n}\operatorname{diam}(T)$ and $L(\partial T^n) = 2^{-n}L(\partial T)$, while

$$\left|\int_{\partial T} f(z)dz\right| \leq 4^n\left|\int_{\partial T^n} f(z)dz\right|. \tag{5.2.1}$$

Now pick, for each $n \in \mathbb{N}$, a point $z_n \in T^n$. Because the triangles are nested, with diameter converging to zero, z_n is a Cauchy sequence and thus has a limit $z_\infty \in T \subset \Omega$. Indeed, $z_n \in T^n$ for every n. By definition of the complex differentiability of f at z_∞, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in B_\delta(z_\infty)$ we have

$$f(z) = f(z_\infty) + f'(z_\infty)(z - z_\infty) + R(z),$$

where the remainder is controlled by $|R(z)| \leq \varepsilon|z - z_\infty|$.

For sufficiently large n, we have $T^n \subset B_\delta(z_\infty)$ because the diameter of T^n is converging to zero, and therefore

$$\begin{aligned}\int_{\partial T^n} f(z)dz &= \int_{\partial T^n} \left[f(z_\infty) + f'(z_\infty)(z - z_\infty) + R(z)\right] dz \\ &= \big(f(z_\infty) - f'(z_\infty)z_\infty\big)\int_{\partial T^n} dz + f'(z_\infty)\int_{\partial T^n} z\, dz + \int_{\partial T^n} R(z)dz \\ &= \int_{\partial T^n} R(z)dz,\end{aligned} \tag{5.2.2}$$

where we have used that that

$$\int_{\partial T^n} dz = 0 \qquad \text{and} \qquad \int_{\partial T^n} z\, dz = 0$$

by Corollary 3.12. Therefore, by (3.4.3), we have

$$\left|\int_{\partial T^n} f(z)dz\right| \leq L(\partial T^n)\varepsilon \sup_{\partial T^n}|z - z_\infty| \leq 2^{-n}L(\partial T)\varepsilon\operatorname{diam}(T^n) \leq 4^{-n}\varepsilon L(\partial T)\operatorname{diam}(T).$$

Combining with (5.2.1) gives

$$\left|\int_{\partial T} f(z)dz\right| \leq \varepsilon L(\partial T)\operatorname{diam}(T),$$

and because $\varepsilon > 0$ was arbitrary, this completes the proof. □

5.3 Goursat's conclusion gives us an anti-derivative

VIDEO: Goursat's conclusion gives us an anti-derivative

Goursat's theorem may seem a rather feeble special case of Cauchy's theorem as stated in Theorem 5.1, and even more so of the generalised version of Cauchy's theorem that we'll see later. However, it is the engine that makes the general theory work. The conclusion of Goursat's theorem allows us to construct anti-derivatives for continuous functions on sufficiently nice domains, as described in the following definition.

Definition 5.4. An open set $\Omega \subset \mathbb{C}$ is called a *star-shaped domain* if there exists $z_0 \in \Omega$ such that for all $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z also lies in Ω. We call such a point z_0 a *central point*.

We'll draw some pictures of star-shaped domains in the lectures/video!

If Ω is convex then it is certainly a star-shaped domain, but we will need this more general class of sets in practice in order to rigorously prove the so-called Cauchy integral formula.

Theorem 5.5 (The output of Goursat's theorem implies the existence of an anti-derivative)**.** *Suppose that Ω is a star-shaped domain, and $f : \Omega \to \mathbb{C}$ is a continuous function. Suppose that for every closed triangle $T \subset \Omega$ we have*

$$\int_{\partial T} f(z)dz = 0.$$

Then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$. Indeed, if z_0 is a central point of the star-shaped domain then we can take F defined by

$$F(z) = \int_{[z_0,z]} f(w)dw. \tag{5.3.1}$$

Recall that we defined the integral on the right-hand side of (5.3.1) to be the integral of f over the $\mathcal{C}^1$ curve $\gamma : [0,1] \to \Omega$ given by $\gamma(t) = z_0 + t(z - z_0)$.

Proof. Although the theorem defines $F(z)$ at a general point $z \in \Omega$, for the remainder of the proof we fix z to be an arbitrary point in Ω at which we want to prove that F is complex differentiable, with $F'(z) = f(z)$. Let $r > 0$ be sufficiently small so that $B_r(z) \subset \Omega$. For each $h \in B_r(0)$, the point $z + h$, and indeed the whole segment $[z, z + h]$, lies in Ω. Because Ω is star-shaped with respect to z_0, the entire closed triangle T with vertices z_0, z and $z + h$ must lie in Ω. By hypothesis,

$$\int_{\partial T} f(w)dw = 0,$$

and hence

$$F(z+h) - \int_{[z,z+h]} f(w)dw - F(z) = 0.$$

Keeping in mind that

$$\int_{[z,z+h]} dw = \int_0^1 \gamma'(t)dt = \gamma(1) - \gamma(0) = (z+h) - z = h,$$

where $\gamma(t) = z + t((z+h) - z) = z + th$, we can use (3.4.3) to compute

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{[z,z+h]} (f(w) - f(z))dw\right| \leq \max_{w\in[z,z+h]} |f(w) - f(z)| \to 0$$

as $h \to 0$ since f is continuous at z. Thus F is complex differentiable at z and $F'(z) = f(z)$ □

5.4 Cauchy's theorem on star-shaped domains

VIDEO: Cauchy's theorem on star-shaped domains

Theorem 5.5 will be useful later in order to prove the so-called Morera theorem 6.9. However, for now we are most interested in combining it with Goursat's theorem 5.3, immediately giving the following.

Corollary 5.6. *Suppose that Ω is a star-shaped domain, and $f : \Omega \to \mathbb{C}$ is a holomorphic function. Then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$.*

If z_0 is a central point of the star-shaped domain then we can take F defined by

$$F(z) = \int_{[z_0,z]} f(w)dw.$$

This corollary will be useful later in order to construct so-called conjugate harmonic functions, but for now we are most interested in combining it with Lemma 3.10, immediately yielding an accurate proof of Cauchy's theorem 5.1 in the special case that Ω is star-shaped.

Theorem 5.7 (Cauchy's theorem on a star-shaped domain)**.** *Suppose that Ω is a star-shaped domain, $f : \Omega \to \mathbb{C}$ is holomorphic and γ is a piecewise C^1 closed curve in Ω. Then*

$$\int_\gamma f(z)dz = 0.$$

At the risk of repetition, we emphasise that although this theorem will be valid for somewhat more general Ω, e.g. simply connected, it will fail on completely general Ω as we know by considering $\Omega = \mathbb{C} \setminus \{0\}$ in Example 5.2.

At this point we're itching to use Cauchy's theorem on star-shaped domains to give a rigorous proof of *Cauchy's integral formula*, but let's first record how it implies Cauchy's theorem on annuli.

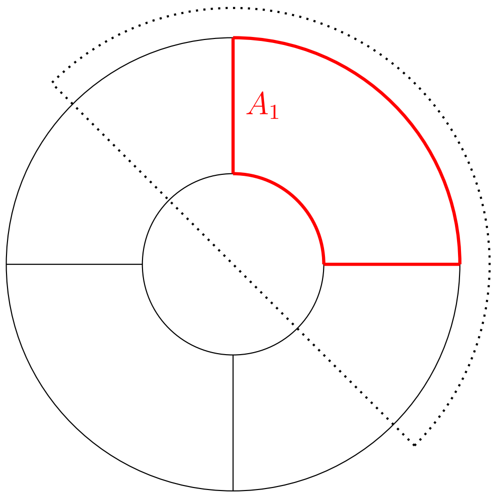


Figure 3: Annulus A_{R_1,R_2} divided into quarters, and a star-shaped domain containing A_1

5.5 Cauchy's theorem on annuli

VIDEO: Cauchy's theorem on annuli

The following form of Cauchy's theorem will be used repeatedly while building the theory, although it will eventually be subsumed into a far more general Theorem 9.3. Recall that integrals around boundaries such as ∂A were discussed in Section 3.4.

Corollary 5.8. *Suppose that* $0 \leq r_1 < R_1 < R_2 < r_2$*, and that* f *is a holomorphic function on the annulus* $A_{r_1,r_2} := \{z \in \mathbb{C} \ : \ r_1 < |z| < r_2\}$*. Then writing* $A := \{z \in \mathbb{C} \ : \ R_1 < |z| < R_2\}$*, we have*

$$\int_{\partial A} f(z)dz = 0, \tag{5.5.1}$$

or equivalently that

$$\int_{\partial B_{R_2}(0)} f(z)dz = \int_{\partial B_{R_1}(0)} f(z)dz. \tag{5.5.2}$$

See the lecture/video for a proper explanation of the following proof, with pictures!

Proof. Let's work out the idea in the case that $r_1 = 0$. We can divide the annulus A_{R_1,R_2} up into four quarters. We call their interiors A_1, A_2, A_3, A_4. See Figure 3. We can write

$$\int_{\partial A} f(z)dz = \sum_{j=1}^{4} \int_{\partial A_j} f(z)dz, \tag{5.5.3}$$

since the four 'spokes' (i.e. interior lines) are integrated over once in each direction on the right-hand side, leaving only the integrals around the circles of radius R_1 and R_2 (in appropriate directions).

Each of the quarters A_j is contained in a half disc (rotated at 45 degrees) where f is holomorphic, as in Figure 3. Since each half-disc is star-shaped, we can apply Cauchy's theorem 5.7 to deduce that each of the four terms on the right-hand side of (5.5.3) are zero, as required to complete the proof in the $r_1 = 0$ case.

Note that A itself does not lie within a star-shaped domain on which f is holomorphic. That is why we chopped it up into pieces A_j. Each of those ***does*** *lie within a star-shaped domain (even a convex domain) on which f is holomorphic.*

If instead $r_1 > 0$, essentially the same proof works, but we have to be more careful in working out which star-shaped domains to take. You should convince yourself that for every annulus A_{r_1,r_2}, when we intersect with a narrow enough sector $\{z \in \mathbb{C} : \arg(z) \in (0, \delta)\}$, we obtain a star-shaped domain. The closer $r_2/r_1 > 1$ is to 1 (we're assuming $r_1 \neq 0$) the smaller we will have to take δ. We then just have to choose spokes separated by an angle less than δ.

I will explain this general case in the lectures/video. □

5.6 Cauchy's integral formula

VIDEO: Cauchy's integral formula

We can use Cauchy's theorem on star-shaped domains (Theorem 5.7) to prove the so-called Cauchy integral formula. Later this will be used to prove Taylor's theorem.

Theorem 5.9 (Cauchy integral formula on a disc)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Suppose that the closed disc/ball $\overline{B_r(a)}$ of radius $r > 0$, centred at $a \in \Omega$, lies within Ω. Then for every $z_0 \in B_r(a)$ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(z)}{z - z_0} \, dz. \tag{5.6.1}$$

Proof. For the given $z_0 \in B_r(a)$, suppose $\delta > 0$ is small enough to ensure that $B_\delta(z_0) \subset B_r(a)$. The function $\frac{f(z)-f(z_0)}{z-z_0}$ is defined and holomorphic on $\Omega \setminus \{z_0\}$, and by Cauchy's theorem for star-shaped domains, Theorem 5.7, we have

$$\int_{\gamma_1} \frac{f(z) - f(z_0)}{z - z_0} dz = 0,$$

where γ_1 is as in Figure 4.

I will explain what star-shaped domain to take (and why it's star-shaped) in the lectures/video if it's not clear to you. The point z_0 is not allowed in this domain because the function being integrated is not defined there! And removing z_0 does break the star-shapedness of many of the domains you might have wanted to consider.[5]

[5] Although we used the notation z_0 for the central point of a star-shaped domain, the z_0 here is something different!

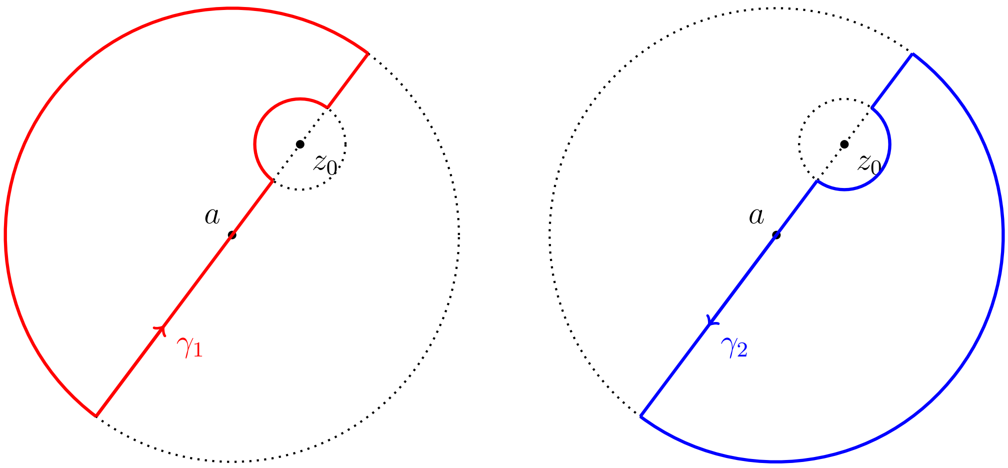


Figure 4: Curves γ_1 and γ_2 in two copies of $B_r(a)$

We can repeat this for γ_2, and add to give

$$\int_{\partial B_r(a)} \frac{f(z)-f(z_0)}{z-z_0} dz = \int_{\partial B_\delta(z_0)} \frac{f(z)-f(z_0)}{z-z_0} dz. \tag{5.6.2}$$

Note the cancellation of the integrals along the straight line portions.

Because $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$, and the length of $\partial B_\delta(z_0)$ is $2\pi\delta$, we see that the right-hand side of (5.6.2) converges to zero as $\delta \downarrow 0$, by (3.4.3). Therefore

$$\int_{\partial B_r(a)} \frac{f(z)}{z-z_0} dz = \int_{\partial B_r(a)} \frac{f(z_0)}{z-z_0} dz = f(z_0) 2\pi i I(\partial B_r(a), z_0) = f(z_0) 2\pi i,$$

by the integral characterisation of winding number given in Lemma 4.12, and the formula for the winding number derived in Q. 4.2. □

5.7 Exercises

5.1. For given $\theta \in \mathbb{R}$, consider the slit domain

$$\Omega := \mathbb{C} \setminus \{re^{i\theta} \ : \ r \geq 0\}.$$

Is Ω convex? Is it star-shaped?

5.2. Suppose $\Omega \subset \mathbb{C}$ is a star-shaped open set and z_0 is a central point for Ω. Prove that every closed continuous path $\gamma : [a, b] \to \Omega$ with $\gamma(a) = \gamma(b) = z_0$ is homotopic to the constant path $\tilde{\gamma} : [a, b] \to \Omega$ defined by $\tilde{\gamma}(t) \equiv z_0$.

Show that a star-shaped domain is necessarily simply connected. In particular, convex subsets of $\mathbb{C}$ are simply connected.

You need to homotop a general closed path in Ω to a constant path. In particular, this closed path now needn't start/end at z_0. Although the statement of this question is important, the proof could be left to those studying 'Introduction to Topology.' The idea is simple but it is a bit tedious to write down.

5.3. In Theorem 5.9 you saw Cauchy's integral formula on a disc. In this question you are asked to adapt the proof to prove an analogous integral formula on a square. More precisely, prove the following theorem:

Theorem 5.10. *Suppose $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Suppose that $Q \subset \Omega$ is a closed square, i.e. $Q = \{x + iy \ : \ x \in [x_0, x_0 + d], y \in [y_0, y_0 + d]\}$ for some $x_0, y_0 \in \mathbb{R}$ and $d > 0$. Then for every z_0 lying in the interior of Q we have*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial Q} \frac{f(z)}{z - z_0} \, dz. \tag{5.7.1}$$

This exercise gives you a chance to work through the proof of Cauchy's integral formula, including subtleties like how the star-shaped condition affects our choice of curves. But it will also be useful when we prove the so-called homology version of Cauchy's theorem. If you have already fully understood the proof of the usual Cauchy integral formula then this exercise will be an easy adaptation. In particular, no answer will be provided!

5.4. By integrating the function $f(z) = e^{-z^2}$ around a large piece of cake, i.e. the contour in Figure 5, prove that

$$\int_0^\infty \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

You may assume the standard integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Compute also

$$\int_0^\infty \cos(x^2) dx.$$

Hint: The integral over the circular arc part of the contour will converge to zero as $R \to \infty$. I am happy if you simply assume that for the purposes of this question. If you are feeling

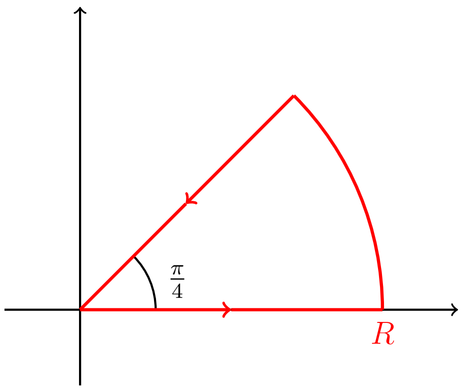


Figure 5: A useful contour for Fresnel integrals

ambitious and want to prove that part also then you may find the following simple inequality useful: If $t \in [0, \pi/2]$ then $\sin t \geq 2t/\pi$. And similarly for cosine. Just draw the graphs to see what is going on...

Integrals of $\sin(x^2)$ and $\cos(x^2)$ are called Fresnel integrals, and arise in the study of optics and elsewhere. Later, the so-called Residue theorem 9.10 will give a much more sophisticated technique for computing integrals.

5.5. Suppose $u : B \to \mathbb{R}$ is a harmonic function on some ball $B = B_r(a)$ in $\mathbb{C} \simeq \mathbb{R}^2$. That is, u is C^2 and $\Delta u \equiv 0$. Prove that there exists a holomorphic function $F : B \to \mathbb{C}$ such that $u = \Re(F)$.

The imaginary part $\Im(F)$ is often called a conjugate harmonic function, and is unique up to the addition of a constant.

Hint: To get an idea of where to go, suppose we manage to find holomorphic $F = u + iv$ for some real function v. Then $0 = F_{\bar{z}} = u_{\bar{z}} + iv_{\bar{z}}$, and conjugating gives $iv_z = u_z$. On the other hand we have $F_z = u_z + iv_z = 2u_z$. This suggests we should consider the function $2u_z$ and try to use Corollary 5.6.

6 Taylor series and applications

6.1 Taylor series - main result

VIDEO: Taylor series - main result

Brook Taylor (1685 - 1731).

Cauchy's integral formula allows one to write any holomorphic function as a power series.

Theorem 6.1 (Taylor's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Suppose $z_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$. Then for all $z \in B_r(z_0)$ we have*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{6.1.1}$$

where

$$a_k = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw. \tag{6.1.2}$$

We will give the (short) proof in the next section.

In contrast to this theorem, to be able to write a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ as a power series is very rare! Such functions are called real analytic. Most infinitely differentiable real functions fail to have this property. For example,

$$f(x) = \begin{cases} \exp(-1/x) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

has all its derivatives at $x = 0$ equal to zero, and this is enough to force every term in any Taylor series about $x = 0$ to vanish, as we will review in a moment. Yet the function is not zero in any neighbourhood of $x = 0$.

Remark 6.2. By Cauchy's theorem on annuli, Corollary 5.8, we could also reduce the radius of the circle over which we are integrating in (6.1.2) to give

$$a_k = \frac{1}{2\pi i} \int_{\partial B_s(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$$

for any $s \in (0, r]$. In particular, as we already know from Q. 3.1, the Taylor coefficients a_k do not depend on r.

Remark 6.3 (Holomorphic/analytic terminology)**.** Taylor's theorem tells us that if $f : \Omega \to \mathbb{C}$ is holomorphic (i.e. complex differentiable at each point in Ω) then it is analytic (i.e. we can expand it as a power series in some ball about each point $z_0 \in \Omega$). Conversely, if a function $f : \Omega \to \mathbb{C}$ is analytic, then Theorem 3.2 from Section 3.1 tells us that f is holomorphic. For this reason, many people use the terms holomorphic and analytic interchangeably. But for us, their equivalence is a triumph of the theory.

A holomorphic function is only assumed to be differentiable at each point, and that derivative is not assumed to be continuous. That is, the function is not even assumed to be $\mathcal{C}^1$. But now that we know our holomorphic function is analytic, we can appeal to the theory of power series, in particular Corollary 3.3, to deduce the following dramatic consequence.

Corollary 6.4. *If $f : \Omega \to \mathbb{C}$ is a holomorphic function on an open set $\Omega \subset \mathbb{C}$ then it is infinitely complex differentiable.*

In fact, Corollary 3.3 also gives us a formula $f^{(n)}(z_0) = a_n n!$ for the nth derivative of a power series, and combining with the formula for the Taylor coefficients in Theorem 6.1, we deduce (cf. Q. 6.5):

Corollary 6.5 (cf. Cauchy's integral formula)**.** *If $f : \Omega \to \mathbb{C}$ is a holomorphic function on an open set $\Omega \subset \mathbb{C}$, and $\overline{B_r(z_0)} \subset \Omega$ for some $z_0 \in \Omega$ and $r > 0$, then for each $n \in \mathbb{N}$ we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

This formula will be useful later to prove smooth local convergence of any sequence of holomorphic functions that is known to converge locally uniformly (see Theorem 10.1).

6.2 Taylor's theorem - proof

VIDEO: Taylor's theorem - proof

In the lectures/video we will reduce to the case $z_0 = 0$ first to make everything cleaner

Proof. Cauchy's integral formula tells us that for all $z \in B_r(z_0)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z)}\, dw. \tag{6.2.1}$$

We can rewrite

$$\frac{1}{w - z} = \frac{1}{w - z_0} \left[\frac{1}{1 - \frac{z - z_0}{w - z_0}} \right] \tag{6.2.2}$$

and using the assumption $|z - z_0| < r = |w - z_0|$, the part in square brackets in (6.2.2) can be written as a geometric series

$$\frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k. \tag{6.2.3}$$

Combining (6.2.1), (6.2.2) and (6.2.3) gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k dw \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\Big(\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\,dw\Big)\,(z-z_0)^k \\ &= \sum_{k=0}^{\infty} a_k(z-z_0)^k. \end{aligned} \tag{6.2.4}$$

The interchange of the summation and the integration is justified because the sum converges uniformly in the integration variable w. □

6.3 Basic consequences of Taylor's theorem. Liouville's theorem.

VIDEO: Basic consequences of Taylor's theorem

We've already started describing applications of Taylor's theorem, e.g. Corollary 6.4 tells us that a holomorphic function is infinitely differentiable. It turns out that much of the subject is based, one way or another, on Taylor's theorem. In the remainder of Section 6 we focus on some of the other basic applications.

Corollary 6.6. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $B_R(0)$, for some $R > 0$, and that for all $z \in B_R(0)$ we have $|f(z)| \le M < \infty$. Then*

$$|a_k| \le \frac{M}{R^k} \tag{6.3.1}$$

for all k.

Proof. Although we are given $f(z)$ already as a power series, by appealing to Taylor's theorem 6.1 we obtain a formula for the coefficients a_k. More precisely, for each $r \in (0, R)$, Taylor's theorem 6.1 applied on $\overline{B_r(0)}$ gives $f(z)$ as a power series, and by Q. 3.1 the coefficients a_k in the statement of the corollary must agree with the Taylor coefficients a_k in Taylor's theorem. According to (6.1.2) we have

$$|a_k| \le \frac{1}{2\pi}\left|\int_{\partial B_r(0)} \frac{f(w)}{w^{k+1}}\,dw\right| \le \frac{1}{2\pi}2\pi r\frac{M}{r^{k+1}} = \frac{M}{r^k}, \tag{6.3.2}$$

by (3.4.3). Then let $r \uparrow R$. □

Corollary 6.7 (Liouville's theorem)**.** *Any bounded entire function is constant.*

Proof. By applying Taylor's theorem 6.1 with $z_0 = 0$ and arbitrarily large R (keeping in mind that changing R does not change the Taylor coefficients) we can write our entire function $f : \mathbb{C} \to \mathbb{C}$ as a

Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{6.3.3}$$

We are assuming that f is bounded, i.e. $|f(z)| \le M$ for all $z \in \mathbb{C}$, so by Corollary 6.6 we have

$$|a_k| \le \frac{M}{R^k}, \tag{6.3.4}$$

for every $k \in \mathbb{N}$ and every $R > 0$. By taking $R \to \infty$ we deduce that $a_k = 0$ for each $k \ge 1$. □

Liouville's theorem implies the following fundamental fact.

Corollary 6.8 (Fundamental theorem of Algebra)**.** *Every non-constant polynomial has at least one zero in* $\mathbb{C}$*.*

Essentially the idea is that if a polynomial $p(z)$ does not have a zero then one can show that $1/p$ is a bounded entire function, and must therefore be constant by Liouville's theorem, Corollary 6.7.

Proof. For some $n \in \mathbb{N}$ we can write our polynomial as $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ with $a_n \neq 0$. Suppose that p does *not* have a zero. Then $1/p(z)$ is an entire function. By rewriting

$$\left| \frac{p(z)}{z^n} - a_n \right| = \left| a_{n-1} z^{-1} + \cdots a_1 z^{1-n} + a_0 z^{-n} \right| \le \sum_{k=0}^{n-1} |a_k|.|z|^{k-n},$$

and observing that the right-hand side converges to zero as $z \to \infty$, we can pick a large $R > 0$ so that for all $z \in \mathbb{C}$ with $|z| \ge R$ the right-hand side is less than $|a_n|/2$, and thus

$$\left| \frac{p(z)}{z^n} \right| \ge |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2},$$

and so

$$\left| \frac{1}{p(z)} \right| \le \frac{2}{|a_n|} |z|^{-n} \le \frac{2}{|a_n|} R^{-n}.$$

We deduce that $1/p(z)$ is bounded over $\mathbb{C} \setminus \overline{B_R(0)}$. But being a continuous function, it is also bounded over the compact set $\overline{B_R(0)}$. By Liouville's theorem, the boundedness of $1/p(z)$ implies that it is constant, giving a contradiction. □

6.4 Morera's theorem

VIDEO: Morera's theorem

Giacinto Morera (1856 - 1909).

The following is an inverse to Goursat's theorem.

Theorem 6.9 (Morera's theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and $f\colon \Omega \to \mathbb{C}$ is a continuous function. Suppose that for all closed triangles $T \subset \Omega$ we have*

$$\int_{\partial T} f(z)\, dz = 0. \tag{6.4.1}$$

Then f is holomorphic on Ω.

Note that initially we don't know that f is differentiable. Simply the condition (6.4.1) implies that it is infinitely differentiable! Amazing!

Remark 6.10. If you have studied some PDE theory then you might have seen some results that are related to this. For example, a real-valued function u on some open set $U \subset \mathbb{R}^n$ such that for all $x \in U$, $u(x)$ agrees with the average of u over every ball $B_r(x) \subset U$, must be harmonic. In both cases we are assuming something not involving any derivatives, and deducing infinite differentiability.

Proof. We need to show that f is complex differentiable at an arbitrary point $a \in \Omega$. Pick $r > 0$ sufficiently small so that $B_r(a) \subset \Omega$. By Theorem 5.5, we can construct a holomorphic function $F : B_r(a) \to \mathbb{C}$ with $F'(z) = f(z)$ for all $z \in B_r(a)$. Because F is holomorphic, Corollary 6.4 tells us that it is infinitely complex differentiable. In particular, $f = F'$ is complex differentiable at a. □

6.5 Local invertibility of holomorphic functions

VIDEO: Local invertibility of holomorphic functions

Now we have the extra regularity for holomorphic functions given by Corollary 6.4, we can locally invert them where their derivative is nonzero.

Lemma 6.11. *Suppose $\Omega \subset \mathbb{C}$ is open and $f\colon \Omega \to \mathbb{C}$ is holomorphic with $f'(z_0) \neq 0$ at some $z_0 \in \Omega$. Then there exists a neighbourhood $V_0 \subset \Omega$ of z_0 and a neighbourhood $V_1 \subset \mathbb{C}$ of $f(z_0)$ such that the restriction of f to V_0 is a biholomorphic map from V_0 to V_1.*

Without the hypothesis that $f'(z_0) \neq 0$ we could have some ridiculous example like $f \equiv 0$, or a subtler example like $f(z) = z^2$ with $z_0 = 0$. The latter example here maps $B_r(0)$ *twice* onto $B_{r^2}(0)$, away from 0, so is certainly not invertible.

Proof. The essential point is that by virtue of Corollary 6.4 we now know that f is a C^1 function when viewed as a real-differentiable function. That is, we now know that the derivative at z varies continuously in z. Moreover, the hypothesis $f'(z_0) \neq 0$, together with the fact that f is holomorphic, implies that the real derivative at z_0 is invertible. (As discussed in Remark 1.3, it is just a rotation and scaling by a positive factor.) Consequently we may now apply the Inverse Function Theorem to deduce the existence of V_0 and V_1, and the invertibility of the restriction of f to V_0. We also learn that

$f^{-1} : V_1 \to V_0$ is $\mathcal{C}^1$. By shrinking the neighbourhoods V_0 and V_1 if necessary, we may assume that $f'(z) \neq 0$ for all $z \in V_0$, because $f'(z_0) \neq 0$ and f' is continuous.

We need to show that the inverse of f is holomorphic, i.e. that the Cauchy-Riemann equation(s) $(f^{-1})_{\bar{z}} = 0$ are satisfied, and also that the derivative of the inverse does not vanish. The chain rule (1.2.6) of Lemma 1.6 tells us that, because $f : V_0 \to V_1$ is holomorphic, for any $\mathcal{C}^1$ function $g : V_1 \to V_0$ we have

$$(f \circ g)_{\bar{z}}(z) = f'(g(z))g_{\bar{z}}(z)$$

for all $z \in V_1$. Applying this with $g = f^{-1}$, we deduce that

$$0 = z_{\bar{z}} = f'(f^{-1}(z))(f^{-1})_{\bar{z}}(z),$$

and hence $(f^{-1})_{\bar{z}} \equiv 0$ because $f' \neq 0$. Meanwhile, to prove that $(f^{-1})'(z) \neq 0$ at any point $z \in V_1$, we apply now the chain rule (1.2.7), i.e. $(f \circ g)'(z) = f'(g(z))g'(z)$, again with $g = f^{-1}$, but this time knowing that g is holomorphic, to deduce that

$$1 = (z)'(z) = f'(f^{-1}(z))(f^{-1})'(z).$$

In particular, because $f' \neq 0$ throughout V_1 we have

$$(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))} \neq 0,$$

for all $z \in V_1$. □

Remark 6.12. The eagle-eyed may spot that it was not really necessary to shrink the sets V_0 and V_1 in the proof above. For each $z \in V_0$, the real derivative of f^{-1} at $f(z)$ can be seen to be the inverse of the real derivative of f at z. In particular, the real derivative of f at each z is invertible. If we had any point $z \in V_0$ with $f'(z) = 0$ then the real derivative of f at z would be the linear map that sent all vectors to zero, and in particular it would not be invertible.

6.6 Exercises

We could have done the first few questions below formally a long time ago, e.g. in the exercises from Section 1.3. But only now, thanks to Corollary 6.4, can we be sure that a holomorphic function f admits enough derivatives to justify the calculations.

6.1. Suppose that $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Prove that the real and imaginary parts of f are harmonic functions.

6.2. Suppose that $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is holomorphic. Prove that $\Delta|f(z)|^2 = 4|f'(z)|^2$.

6.3. Suppose $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C} \setminus \{0\}$ is holomorphic. Prove that $\log|f(z)|$ is harmonic.

6.4. (Important exercise.) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$. We can write this as a Taylor series

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k},$$

and despite f being infinitely differentiable on the whole of $\mathbb{R}$, the power series only converges for $|x| < 1$.

By considering f as the restriction to $\mathbb{R}$ of the function $f : \mathbb{C}\backslash\{\pm i\} \to \mathbb{C}$, given by $f(z) = \frac{1}{1+z^2}$, explain geometrically why the radius of convergence is precisely $R = 1$.

What would the radius of convergence be if we wrote $f(x)$ as a power series $\sum_{k=0}^{\infty} a_k(x-1)^k$?

6.5. Verify the following slight modification of Theorem 6.1 in which we allow the ball $B_R(z_0)$ to go right up to the boundary of Ω.

Theorem 6.13 (Variant of Taylor's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose $z_0 \in \Omega$ and $R > 0$ such that $B_R(z_0) \subset \Omega$. Then for all $z \in B_R(z_0)$ we can write*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{6.6.1}$$

where

$$a_k = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} \, dw \tag{6.6.2}$$

for all $r \in (0, R)$.

7 Zeros of holomorphic functions

Taylor's theorem is going to give us a detailed, and very useful, theory describing the structure of a holomorphic function near points where it vanishes.

7.1 Zeros of holomorphic functions - basic structure

VIDEO: Zeros of holomorphic functions - basic structure

Consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^n$, for some $n \in \mathbb{N}$. This function is zero precisely at $z = 0$. The *order* of the zero, as will be defined in a moment in general, will be n.

Definition 7.1. Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be a holomorphic function with $f(z_0) = 0$ for some $z_0 \in \Omega$. We define the *order* of the zero of f at z_0 to be

$$\operatorname{ord}(f, z_0) := \begin{cases} \infty & \text{if } f^{(k)}(z_0) = 0 \text{ for all } k \in \mathbb{N}, \\ \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\} & \text{otherwise.} \end{cases} \tag{7.1.1}$$

Example 7.2. If $g : \Omega \to \mathbb{C}$ is a holomorphic function for which $g(z_0) \neq 0$, and we define the holomorphic function $f(z) := (z - z_0)^n g(z)$, then the order of the zero of f at z_0 is n. This is because as we differentiate $k < n$ times, using the product rule, each of the resulting terms will have at least a factor $(z - z_0)^{n-k}$ within it, so will vanish at z_0. But if we differentiate n times, and evaluate at z_0, then there will be one nonzero term $n!g(z_0)$.

As it turns out, Example 7.2 gives a full description of zeros of finite order:

Theorem 7.3. *Suppose that $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$ is a holomorphic function that has a zero of **finite** order $n \in \mathbb{N}$ at $z_0 \in \Omega$. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ such that*

$$f(z) = (z - z_0)^n g(z),$$

*and g is nonzero in a neighbourhood of z_0. In particular, each zero of finite order is an **isolated** point of the set of zeros.*

To clarify, if z_0 is a zero of finite order, then saying it is isolated in the set of zeros is saying that there is no other zero (of finite or infinite order) in some small neighbourhood of z_0.

Proof. Starting with any $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$, we can use Taylor's theorem 6.1 to write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Now we have f as a power series, Corollary 3.3 tells us that $a_k = \frac{f^{(k)}(z_0)}{k!}$, and because f has a zero of order n at z_0, we must then have $a_k = 0$ for $k < n$, and $a_n \neq 0$, and we can write, for all $z \in B_r(z_0)$,

$$f(z) = \sum_{k=n}^{\infty} a_k(z-z_0)^k = (z-z_0)^n \sum_{k=0}^{\infty} a_{k+n}(z-z_0)^k = (z-z_0)^n g(z), \tag{7.1.2}$$

where $g(z) := \sum_{k=0}^{\infty} a_{k+n}(z-z_0)^k$ is defined on $B_r(z_0)$, and $g(z_0) \neq 0$. Note that the power series defining g must converge pointwise for each $z \in B_r(z_0)$, by construction, and so its radius of convergence must be at least r by Theorem 3.1. By Theorem 3.2, g is holomorphic on $B_r(z_0)$. You may recognise this argument from doing Q. 3.5.

We can extend g to the rest of Ω by setting $g(z) = f(z)(z-z_0)^{-n}$, which agrees with what we already have for g on $B_r(z_0) \setminus \{z_0\}$, and is holomorphic elsewhere by the product rule. Because g is continuous and $g(z_0) \neq 0$, it is nonzero in some neighbourhood of z_0. □

Theorem 7.4. *Suppose that $\Omega \subset \mathbb{C}$ is open **and connected**, and $f : \Omega \to \mathbb{C}$ is a holomorphic function that has a zero of **infinite** order at some point $z_0 \in \Omega$. Then $f \equiv 0$.*

Proof. Consider the set $\Omega_0 := \{z \in \Omega : f \text{ has a zero of order infinity at } z\}$. Our aim is to prove that $\Omega_0 = \Omega$. First, we know that Ω_0 is nonempty, since $z_0 \in \Omega_0$. Because Ω is connected, it therefore suffices to prove that Ω_0 is both open and closed.

Suppose we pick an arbitrary point $w_0 \in \Omega_0$. If we write out the Taylor series of f at $z = w_0$ using Theorem 6.1 and the fact, from Corollary 3.3, that $a_k = \frac{f^{(k)}(w_0)}{k!} = 0$, we find that f is identically zero in any ball $B_r(w_0) \subset \Omega$. Therefore $B_r(w_0) \subset \Omega_0$ and we find that Ω_0 must be open.

On the other hand, if we take a sequence $z_i \in \Omega_0$ that converges to some $z_\infty \in \Omega$, then $f(z_\infty) = 0$ by continuity of f. But z_∞ cannot be a zero of finite order since we have seen that such zeros are isolated within the set of all zeros. Therefore $z_\infty \in \Omega_0$, and we can deduce that Ω_0 is (relatively) closed. □

7.2 The identity theorem

VIDEO: The identity theorem

A simple consequence of the previous section is:

Theorem 7.5 (Identity theorem). *Let $\Omega \subset \mathbb{C}$ be open **and connected** and let f_1 and f_2 be holomorphic functions $\Omega \mapsto \mathbb{C}$. If the set $\Sigma := \{z \in \Omega : f_1(z) = f_2(z)\}$ has at least one accumulation point in Ω then $f_1 \equiv f_2$ throughout Ω.*

To clarify, $z_\infty \in \Omega$ is an accumulation point of Σ if there exists a sequence $z_i \in \Sigma \setminus \{z_\infty\}$ such that $z_i \to z_\infty$. We are not assuming that z_∞ lies in Σ in the definition of accumulation point, but by continuity of the functions f_1 and f_2, it will have to in this case. Thus an equivalent formulation of the

Identity theorem is that two holomorphic functions on an open and connected set are either identical or agree only at isolated points.

It's important that the limit point z_∞ is asked to lie within Ω, and not on the boundary $\partial\Omega$. See Q. 7.3.

Proof. By hypothesis, the function $g := f_1 - f_2$ is holomorphic and has a non-isolated zero. By Theorem 7.3, this zero must be of infinite order, and then by Theorem 7.4 we must have $g \equiv 0$ throughout the connected open set Ω, i.e. $f_1 \equiv f_2$. $\square$

Example 7.6. An explicit example to which we can apply the Identity theorem 7.5: Suppose f is a holomorphic function on the ball $B_2(0) \subset \mathbb{C}$, and suppose we know that $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Then we can deduce that f is identically zero on $B_2(0)$.

Example 7.7. Related to the example above, consider the function $z \mapsto z^2 \sin(\frac{\pi}{z})$ on $\mathbb{C}$. You might mistake this for a holomorphic function. But certainly it has zeros at all points $z = \frac{1}{n}$, for $n \in \mathbb{N}$, so by the previous example it would have to be identically zero, which it isn't. Indeed, on more careful analysis, this function behaves horribly at iy, for real y converging to zero. Put another way, the function $\sin$ behaves very badly at the point $\infty \in \mathbb{C}_\infty$. In fact, it could be viewed as an *essential singularity*, which we will define later.

7.3 Zeros of holomorphic functions - refined structure

VIDEO: Zeros of holomorphic functions - refined structure

We now want to improve our description of a holomorphic function near a zero.

Theorem 7.8. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. If f has a zero of finite order $k \geq 1$ at $z_0 \in \Omega$ then there exist a neighbourhood $V_0 \subset \Omega$ of z_0, a radius $r > 0$ and a biholomorphic function $h : V_0 \to B_r(0)$ such that for every $z \in V_0$ we have*

$$f(z) = \big(h(z)\big)^k. \tag{7.3.1}$$

In particular, f is locally k-to-one near z_0. More precisely, f takes every value in $B_{r^k}(0) \setminus \{0\}$ exactly k times in V_0.

A simple case to keep in mind is when $f(z) = (z - z_0)^k$, in which case $h(z) = z - z_0$ would work. Other cases can essentially be reduced to perturbations of this basic picture.

The map h in Theorem 7.8 is not unique if $k \geq 1$ because we could also have taken $\xi^j h(z)$, $j \in \{1, \dots, k-1\}$, where $\xi = e^{\frac{2\pi i}{k}}$.

The proof of Theorem 7.8 will essentially involve taking a kth root of a suitable function. Intuitively, we can take a kth root of a function $g(z)$ by considering $e^{\frac{1}{k}\log g(z)}$, but to make this work we need to make unambiguous sense of the logarithm in this context. One way of doing this is to work 'bare hands' and define $\log$ locally to some $g(z_0) \neq 0$ by taking a suitable branch cut. This is fine, although we did not spell out yet that $\log$ is holomorphic. Alternatively one could construct a local holomorphic $\log$ by inverting the function $z \mapsto e^z$ locally using Lemma 6.11. Instead we use a more elaborate method that will be reusable later on.

Lemma 7.9. *Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $g\colon \Omega \to \mathbb{C} \setminus \{0\}$ is a holomorphic function such that the 'logarithmic derivative' $\frac{g'(z)}{g(z)}$ admits an anti-derivative. That is, we assume that there exists a holomorphic $F : \Omega \to \mathbb{C}$ such that $F'(z) = \frac{g'(z)}{g(z)}$. Then there exists $w_0 \in \mathbb{C}$ so that when we define a holomorphic function $\ell\colon \Omega \to \mathbb{C}$ by*

$$\ell(z) := F(z) + w_0, \tag{7.3.2}$$

we have

$$g(z) = e^{\ell(z)} \qquad \text{for all } z \in \Omega. \tag{7.3.3}$$

The function ℓ is unique up to an additive constant $2\pi i n$ for $n \in \mathbb{Z}$.

You can think of $\ell(z)$ as a choice of $\log g(z)$.

Later on, in Lemma 11.4, we will be able to apply this lemma for simply connected domains Ω. For now, we exploit the existence of an anti-derivative of the function $f(z) = \frac{g'(z)}{g(z)}$ given by Corollary 5.6 to immediately give:

Corollary 7.10. *Suppose $\Omega \subset \mathbb{C}$ is **star-shaped** and $g\colon \Omega \to \mathbb{C} \setminus \{0\}$ is holomorphic. Then there exists a holomorphic function $\ell : \Omega \to \mathbb{C}$, unique up to an integer multiple of $2\pi i$, such that*

$$g(z) = e^{\ell(z)}.$$

In particular, for $k \in \mathbb{N}$, the function $z \mapsto e^{\frac{1}{k}\ell(z)}$ gives a holomorphic function on Ω whose kth power is $g(z)$.

Proof of Lemma 7.9. Fix an arbitrary $z_0 \in \Omega$. As $g(z_0) \neq 0$ by assumption, we can pick $w_0 \in \mathbb{C}$ such that $e^{w_0} = g(z_0)e^{-F(z_0)}$. This w_0 is uniquely determined up to a multiple of $2\pi i$, cf. Section 3.3. The corresponding function $\ell(z)$ defined by (7.3.2) induces a holomorphic function $g(z)e^{-\ell(z)}$ that has derivative

$$\Big(g(z)e^{-\ell(z)}\Big)' = g'(z)e^{-\ell(z)} - g(z)e^{-\ell(z)}\ell'(z) = e^{-\ell(z)}\Big(g'(z) - g(z)F'(z)\Big) = 0, \tag{7.3.4}$$

and is thus equal to some constant $c \in \mathbb{C}$ throughout the connected open set Ω by Q. 1.11. Evaluating at z_0 gives

$$c = g(z_0)e^{-\ell(z_0)} = g(z_0)e^{-F(z_0)}e^{-w_0} = 1,$$

and so $g(z) = e^{\ell(z)}$ throughout Ω as required. □

Proof of Theorem 7.8. We have seen in Theorem 7.3 that we can write

$$f(z) = (z - z_0)^k g(z), \tag{7.3.5}$$

where g is holomorphic on Ω and g does not attain the value 0 on a whole neighbourhood $B_s(z_0) \subset \Omega$ of z_0, for some $s > 0$. Intuitively we want to define $h(z) = (z - z_0)[g(z)]^{\frac{1}{k}}$. We do this rigorously by applying Corollary 7.10 with Ω there equal to $B_s(z_0)$ here, to obtain a function $\ell : B_s(z_0) \to \mathbb{C}$ such that $g(z) = e^{\ell(z)}$, and then defining

$$h(z) = (z - z_0)e^{\frac{1}{k}\ell(z)}.$$

We have now found a holomorphic function h on $B_s(z_0)$ such that $h(z)^k = f(z)$, and we notice that $h'(z_0) = e^{\frac{1}{k}\ell(z_0)} \neq 0$. By the local invertibility lemma 6.11, we can find neighbourhoods $V_0 \subset B_s(z_0)$ of z_0 and V_1 of $h(z_0) = 0$, so that the restriction $h\colon V_0 \to V_1$ is biholomorphic. By shrinking these neighbourhoods, we may assume that $V_1 = B_r(0)$ for some small $r > 0$. More precisely, we take $r > 0$ small enough so that $B_r(0) \subset V_1$, and then redefine $V_1 = B_r(0)$ and $V_0 = h^{-1}(B_r(0))$.

This completes the construction of h. To see the k-to-one property, pick an arbitrary point $w \in B_{r^k}(0) \setminus \{0\}$. Then there are precisely k points $\xi_1, \ldots, \xi_k$, all lying in $B_r(0)$, such that $\xi_j^k = w$ for each $j \in \{1, \ldots, k\}$. Thus we see that within V_0, precisely the k points $h^{-1}(\xi_j)$ are mapped to w by f. □

7.4 Open mapping theorem; Maximum modulus principle; Mean value property

VIDEO: Open mapping theorem; Maximum modulus principle; Mean value property

Theorem 7.11 (Open mapping theorem)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $f\colon \Omega \to \mathbb{C}$ is holomorphic but not constant. Then the image $f(\Omega)$ of Ω under f is also open and connected.*

By definition, the pre-image of every continuous function on any topological space is open. But the statement for forward images is not true for continuous, or even $\mathcal{C}^1$, functions in general. For example, the function $z \mapsto \Re(z)$ on $\mathbb{C}$ has the real line as image, which is not open in $\mathbb{C}$.

Proof. It is a general fact from topology that the image of every connected set under a continuous map is connected. In order to see that the image is open, pick an arbitrary point $w_0 = f(z_0) \in f(\Omega)$. We need to show that $f(\Omega)$ contains a whole neighbourhood of w_0.

The function $g(z) := f(z) - w_0$ has a zero at z_0. This zero must be of finite order, say of order $k \in \mathbb{N}$, because otherwise Theorem 7.4 would tell us that g would be identically zero throughout the (connected) domain Ω and then f would be a constant function $f \equiv w_0$ contrary to our assumptions. By Theorem 7.8, locally we have that $f(z) = w_0 + (h(z))^k$, where h is a biholomorphic map from some neighbourhood V_0 of z_0 to a ball $B_r(0)$. Therefore the image of f contains the ball $B_{r^k}(w_0)$. □

Corollary 7.12 (Maximum modulus principle)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $f\colon \Omega \to \mathbb{C}$ is holomorphic but not constant. Then $|f|$ does not have any local maxima.*

Proof. Suppose that $|f|$ attains a local maximum at $z_0 \in \Omega$. By the Open mapping theorem 7.11, the image of any neighbourhood of z_0 is a neighbourhood of $f(z_0)$, and therefore must contain points with larger absolute value, contradicting our assumption. □

In Q. 7.5 you will give an alternative proof of this maximum modulus principle using the following mean value property.

Lemma 7.13 (Mean value property). *Suppose $\Omega \subset \mathbb{C}$ is open with $\overline{B_r(z_0)} \subset \Omega$, for some $r > 0$ and $z_0 \in \Omega$. Suppose that $f\colon \Omega \to \mathbb{C}$ is holomorphic. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \tag{7.4.1}$$

that is, $f(z_0)$ equals the average of f over the circle of radius r centred at z_0.

Proof. By Cauchy's integral formula (5.6.1), we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) ire^{i\theta}}{re^{i\theta}}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned} \tag{7.4.2}$$

□

If you have studied harmonic functions, then you may be familiar with the mean value property already, and we know that the real and imaginary parts of f are harmonic functions.

7.5 Injective holomorphic functions are biholomorphic onto their image

VIDEO: Injective holomorphic functions are biholomorphic onto their image

As discussed before, in this section we are using the word domain to refer to a nonempty open and connected set within $\mathbb{C}$.

Back in Section 2.9 we defined a function $f : \Omega_1 \to \Omega_2$ between domains Ω_1 and Ω_2 to be biholomorphic if it is a holomorphic bijection whose inverse is also holomorphic, and so that the derivatives of both f and f^{-1} are nowhere vanishing.

Now we have learned more about the zeros of holomorphic functions, we find that very mild hypotheses lead to a function being biholomorphic.

Theorem 7.14. *Suppose $\Omega \subset \mathbb{C}$ is a domain and $f\colon \Omega \to \mathbb{C}$ is both injective and holomorphic. Then $f(\Omega)$ is a domain and $f\colon \Omega \to f(\Omega)$ is a biholomorphic map.*

Proof. Since Ω is connected and f is continuous, the image $f(\Omega)$ is connected. By the open mapping theorem 7.11, $f(\Omega)$ is an open subset of $\mathbb{C}$. (Note that f cannot be constant since it is injective.) Therefore $f(\Omega)$ is a domain in $\mathbb{C}$.

Suppose there exists $z_0 \in \Omega$ such that $f'(z_0) = 0$. Then the function $F(z) = f(z) - f(z_0)$ satisfies $F(z_0) = F'(z_0) = 0$. The zero of F at z_0 cannot be of infinite order because Theorem 7.4 would then force F, and hence also f, to be constant, which would contradict the injectivity of f. Therefore F has a zero of order $k \geq 2$ at z_0, and Theorem 7.8 then implies that the function F, and hence also f, is not injective, giving a contradiction.

Now we have established that $f'(z) \neq 0$ for all $z \in \Omega$, Lemma 6.11 tells us that f is locally biholomorphic. Because f is bijective, we see that it must then be (globally) biholomorphic. $\square$

7.6 The Schwarz lemma

VIDEO: The Schwarz lemma

Karl Hermann Amandus Schwarz (1843 - 1921).

The Schwarz lemma restricts how extreme a holomorphic function from the disc to itself can be. In a moment it will be useful for classifying biholomorphic functions from the disc to itself. We will also need it in the proof of the Riemann mapping theorem.

Theorem 7.15 (Schwarz lemma)**.** *Let $f\colon D \to D$ be holomorphic on D with $f(0) = 0$. Then*

(i) $|f'(0)| \leq 1$, *and*
(ii) $|f(z)| \leq |z|$ *for all* $z \in D$.

If equality holds in (i), or in (ii) even for a single value $z \in D \setminus \{0\}$, then f is a rotation in the sense that $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Proof. The zero of f at $z = 0$ can be assumed to be of finite order because otherwise Theorem 7.4 would imply $f \equiv 0$, in which case the theorem is trivial. By Theorem 7.3 there exists a holomorphic function $g\colon D \to \mathbb{C}$ such that for all $z \in D$ we have $f(z) = zg(z)$.

Suppose $r \in (0, 1)$. By assumption, for all z with $|z| = r$ we have

$$1 > |f(z)| = |z|\,|g(z)| = r\,|g(z)|, \tag{7.6.1}$$

and hence $|g(z)| < \frac{1}{r}$. By the Maximum modulus principle, Corollary 7.12, $|g|$ must attain its maximum over the ball $\overline{B_r(0)}$ on the boundary $\{|z| = r\}$, and thus we have $|g(z)| < \frac{1}{r}$ for all $|z| \leq r$. By taking the limit $r \uparrow 1$, we obtain $|g(z)| \leq 1$ throughout D. This implies part (i) because $f'(0) = g(0)$, and implies (ii) because $|f(z)| = |z|\,|g(z)|$.

Now we need to consider the equality cases in (i) and (ii). But equality in (i) is equivalent to $|g(0)| = 1$, while equality in (ii) at a point $z \in D \setminus \{0\}$ is equivalent to $|g(z)| = 1$. Either way, we need to consider the case that $|g(z_0)| = 1$ for some $z_0 \in D$. In this case, $|g|$ attains a local maximum at z_0. Appealing to the Maximum modulus principle again, we find that g must be constant, and of magnitude 1, i.e. we can write $g(z) = e^{i\theta}$ for some fixed $\theta \in \mathbb{R}$, and so $f(z) = ze^{i\theta}$. $\square$

In Remark 2.22 we saw that every Möbius transformation of the form

$$f(z) = e^{i\theta}\left(\frac{z-a}{\bar{a}z-1}\right) \qquad \text{for } |a| < 1 \quad \text{and} \quad \theta \in [-\pi, \pi), \tag{7.6.2}$$

is a biholomorphic function from the unit disc to itself. At the time we did not emphasise that they were the *only* Möbius transformations that were biholomorphic from D to itself because we knew that at this precise point of the course we would have the technology to prove a spectacularly more general result: These maps are the only biholomorphic functions from D to itself whether or not we restrict to considering Möbius transformations.

Corollary 7.16 (Classification of biholomorphic maps of the disc)**.** *Every biholomorphic function $f\colon D \to D$ is a Möbius transformation of the form* (7.6.2).

Proof. Suppose for the moment that our biholomorphic function $f\colon D \to D$ satisfies $f(0) = 0$. We can then apply the Schwarz lemma, Theorem 7.15, to f to deduce that $|f(z)| \leq |z|$ for all $z \in D$. But we can also apply it to f^{-1} to deduce that $|z| = |f^{-1}(f(z))| \leq |f(z)|$. Thus $|f(z)| = |z|$ throughout. The final part of the Schwarz lemma implies that merely knowing that $|f(z)| = |z|$ at a single point in $D \setminus \{0\}$, let alone at all points, is enough to deduce that f is a rotation.

In the general case, i.e. not assuming $f(0) = 0$, we set $a = f^{-1}(0)$ and define $\varphi(z) = \frac{z-a}{\bar{a}z-1}$. As in Example 2.21, φ is a biholomorphic map from D to itself that maps 0 to a. Therefore $f \circ \varphi$ is a biholomorphic map from D to itself that maps 0 to 0, and by the first part of the proof we know that $f \circ \varphi$ is a rotation $z \mapsto e^{i\theta}z$. As remarked after Example 2.21, φ is its own inverse, i.e. $\varphi \circ \varphi(z) = z$. Therefore

$$f(z) = f \circ \varphi \circ \varphi(z) = e^{i\theta}\frac{z-a}{\bar{a}z-1},$$

i.e. a Möbius transformation of the form (7.6.2). $\square$

Remark 7.17. The natural home of the Schwarz lemma is, in fact, that of hyperbolic geometry. When you equip the unit disc not with the Euclidean distance but with the so-called hyperbolic distance, then the Möbius transformations that map the disc bijectively to itself (as discussed in Example 2.21) turn out to be isometries. The Schwarz lemma, in a marginally more general form known as the Schwarz-Pick lemma, says that if you take any holomorphic function f from the disc to itself (not necessarily surjective) then the hyperbolic distance between any two points can only decrease after applying f.

7.7 Exercises

7.1. For each of the following parts, write down the complete list of holomorphic functions from D to $\mathbb{C}$ with the given infinitely many points prescribed. (There may be no such functions.)

(a) $f(\frac{1}{n}) = \frac{1}{n^3}$ for all $n \in \{2, 3, \ldots\}$;

(b) $f(\frac{1}{n^3}) = \frac{1}{n}$ for all $n \in \{2, 3, \ldots\}$;

(c) $f(\frac{1}{n}) = \frac{(-1)^n}{n^2}$ for all $n \in \{2, 3, \ldots\}$;

(d) $f^{(n)}(\frac{1}{n}) = 3^n n!$, where $f^{(n)}$ is the nth derivative, as before.

7.2. Given a function $f : \mathbb{R} \to \mathbb{C}$, prove that there can be at most one extension to a holomorphic function $\mathbb{C} \mapsto \mathbb{C}$.

Follow-up: can you think of any other proofs?

Remark: The function f would have to be extremely regular (real analytic) in order to have even one extension.

7.3. Suppose $\Omega \subset \mathbb{C}$ is an open and connected set, $f \colon \Omega \to \mathbb{C}$ is holomorphic and there exists a sequence of points z_n in Ω with $f(z_n) = 0$ for each n and such that $z_n \to z_\infty \in \mathbb{C}$, but with $z_n \neq z_\infty$ for each $n \in \mathbb{N}$. Is it necessarily true that $f \equiv 0$?

This is a little bit subtler than it may seem at first glance!

7.4. Suppose that $\Omega \subset \mathbb{C}$ is open and connected, and $f \colon \Omega \to \mathbb{C}$ is holomorphic.

(a) Prove, using the Open mapping theorem 7.11, that if any of the functions $u = \Re(f)$, $v = \Im(f)$, or $|f|$ are constant on Ω, then f itself is constant.

You could have proved this by bare-hands sometime ago, using the Cauchy-Riemann equations, but now that you have the open mapping theorem 7.11, it is much easier!

(b) Give an alternative proof of the previous statement that $|f|$ being constant implies f is constant, by differentiating $|f|^2$ with respect to z.

7.5. Use the mean value property of Lemma 7.13 to give a proof of the maximum modulus principle that doesn't use the open mapping theorem.

You may find Q. 7.4b useful.

7.6. *Preamble: Given an injective holomorphic function from D to $\mathbb{C}$, we can always add a constant so that 0 is mapped to itself. By the theory in Section 7.5, we know that the derivative at zero cannot be zero (or the function would fail to be injective) so we can further normalise by multiplying by a nonzero constant so that the derivative at zero is 1.*

Suppose that $f : D \to \mathbb{C}$ is an injective holomorphic map such that $f(0) = 0$ and $f'(0) = 1$. De Branges's theorem, previously known as the Bieberbach conjecture, says that the Taylor coefficients of f (expanding about 0) must satisfy $|a_n| \leq n$. By considering the Koebe function $K : D \to \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$, mentioned in Q. 2.11 and Q. 3.3, show that this bound is sharp (i.e. can't be improved).

Optional extra reading: Look up the Koebe quarter theorem.

7.7. For $r, s > 0$, suppose we have a holomorphic function $f : B_r \to B_s$ from the ball of radius r to the ball of radius s, such that $f(0) = 0$. Prove that

(a) $|f'(0)| \leq \frac{s}{r}$, and
(b) $|f(z)| \leq \frac{s}{r}|z|$ for all $z \in B_r$.

Hint: If your proof is more than a few lines long then you are probably working too hard, and might want to reduce the problem to something you already know.

7.8. Prove Liouville's theorem, Corollary 6.7, from Q. 7.7.

8 Isolated singularities

8.1 Riemann's removable singularity theorem

VIDEO: Riemann's removable singularity theorem

The function $x \mapsto |x|$ illustrates that you can have a continuous function $\mathbb{R} \mapsto \mathbb{R}$ that fails to be differentiable at a single point. In complex analysis this is impossible!

For example, a consequence of the removable singularity theorem that we are about to meet is that if a continuous function $f\colon D \to \mathbb{C}$ is holomorphic on $D \setminus \{0\}$, then it is in fact holomorphic throughout D!

More generally, the theorem will not even require f to be defined at the centre. The standard terminology for this is:

Definition 8.1. A function f that is holomorphic on $B_r(a) \setminus \{a\} \subset \mathbb{C}$, for some $r > 0$ and $a \in \mathbb{C}$, is said to have an *isolated singularity* at a.

Theorem 8.2 (Riemann's removable singularity theorem)**.** *Let $f : B_r(a)\setminus\{a\} \to \mathbb{C}$ be a holomorphic function from a ball of radius $r > 0$ centred at $a \in \mathbb{C}$. Suppose that*

$$|f(z)| \leq M \text{ for some } M < \infty \text{ and every } z \in B_r(a) \setminus \{a\}, \tag{8.1.1}$$

or more generally that

$$\lim_{z\to a}(z-a)f(z) = 0. \tag{8.1.2}$$

Then we can extend f to a holomorphic function $f : B_r(a) \to \mathbb{C}$.

In other words, we can assign a suitable value for $f(a)$ to make f not only continuous but even holomorphic throughout the ball!

Proof. Define a function $g : B_r(a) \to \mathbb{C}$ by $g(z) = (z-a)^2 f(z)$ for all $z \in B_r(a) \setminus \{a\}$, and $g(a) = 0$. By the product rule, g is holomorphic when restricted to $B_r(a) \setminus \{a\}$. For $z \in B_r(a) \setminus \{a\}$, by computing

$$\frac{g(z) - g(a)}{z - a} = (z-a)f(z) \to 0$$

as $z \to a$, using (8.1.2), we find that g is complex differentiable also at $z = a$, with $g'(a) = 0$, and hence holomorphic throughout $B_r(a)$ with a zero of order at least 2 at $z = a$.

If the zero of g at $z = a$ is of infinite order then both g and then f must be identically zero by Theorem 7.4, and the result is obvious by setting $f(a) = 0$.

If, instead, the zero of g at $z = a$ is of finite order $n \geq 2$ then we can apply Theorem 7.3 to g to deduce that we can write

$$g(z) = (z-a)^n h(z),$$

for holomorphic $h : B_r(a) \to \mathbb{C}$ with $h(a) \neq 0$. But then $(z-a)^{n-2}h(z)$ is a holomorphic function on the whole of $B_r(a)$ that agrees with f on $B_r(a) \setminus \{a\}$, and so we can use it as an extension of f. □

Magic! If you read the proof in slow-motion you may spot that I had Taylor's theorem up my sleeve.

8.2 Classification of isolated singularities; description of poles

VIDEO: Classification of isolated singularities; description of poles

Isolated singularities of a holomorphic function $f\colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ can be classified into three types.

Of course, one possible situation is that we can just make a definition for f at z_0 and end up with a holomorphic function on the whole of $B_r(z_0)$. According to Riemann's removable singularity theorem 8.2, this corresponds to the first case of the following trichotomy, and justifies its name.

Definition 8.3. The function f is said to have a

(1) *removable singularity* at z_0 if $f(z)$ has a limit in $\mathbb{C}$ as $z \to z_0$;

(2) *pole* at z_0 if $f(z) \to \infty$ as $z \to z_0$;

(3) *essential singularity* at z_0 if neither of the two cases above hold.

The function $f(z) = 1/z$ on $D \setminus \{0\}$ has a pole at 0.

The function $f(z) = \exp(1/z)$ has an essential singularity at 0. In fact, if g is any entire function that is not just a polynomial (i.e. its Taylor series has infinitely many terms) then the function $g(1/z)$ has an essential singularity at 0.

Essential singularities are very wild, as we will see in Section 8.3. Poles are actually rather nice. Indeed, if we are happy with ∞ being just another point in the extended complex plane $\mathbb{C}_\infty$, then poles are rather similar to removable singularities. Our analysis stems from the following simple observation:

If $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and has a pole at z_0, as defined above, then after reducing $r > 0$ if necessary so that $|f(z)| \geq 1$ for all $z \in B_r(z_0) \setminus \{z_0\}$, we find that $1/f(z)$ is bounded and holomorphic on $B_r(z_0) \setminus \{z_0\}$, and so by Riemann's removable singularity theorem 8.2, it is the restriction of some holomorphic function $F : B_r(z_0) \to \mathbb{C}$ with $F(z_0) = 0$.

To clarify, F is zero at z_0 otherwise $f(z) = 1/F(z)$ would be bounded near z_0, and z_0 would not have been a pole.

The zero of F at z_0 cannot be of infinite order, or Theorem 7.4 would tell us that F would have to be identically zero in $B_r(z_0)$, but $F(z) = 1/f(z)$ so that F is not zero anywhere in $B_r(z_0) \setminus \{z_0\}$. Therefore F has a zero of some finite order $n \in \mathbb{N}$

At this point we can apply Theorem 7.3 to argue that we can write $F(z) = (z - z_0)^n G(z)$, where G is holomorphic and nonzero on $B_r(z_0)$. Defining $g(z) = 1/G(z)$ gives another holomorphic and nonzero function on $B_r(z_0)$, and we see that we have proved the following analogue of Theorem 7.3:

Theorem 8.4. *Suppose that a holomorphic function $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ has a pole at z_0. Then there exist $n \in \mathbb{N}$ and a holomorphic function $g\colon B_r(z_0) \to \mathbb{C}$ that is nonzero in a neighbourhood of z_0, such that*

$$f(z) = (z - z_0)^{-n} g(z).$$

The pole in this theorem is said to be of order n. If $n = 1$ then we also refer to it as a *simple pole*.

A meromorphic functions is, loosely speaking, one that does not have essential singularities. We give the following slightly unorthodox definition.

Definition 8.5. Suppose $\Omega \subset \mathbb{C}$ is open and connected, and $f\colon \Omega \to \mathbb{C}_\infty$ is a continuous function that is not identically equal to ∞. We say that f is meromorphic if f is complex differentiable at every point $z_0 \in \Omega$ with $f(z_0) \neq \infty$, and $1/f$ is complex differentiable at every point $z_0 \in \Omega$ with $f(z_0) = \infty$.

As usual, we adopt the convention that $1/\infty = 0$.

Given a meromorphic function f as above, define $\mathcal{Z} := f^{-1}(0)$ to be the set of zeros of f in Ω, and define $\mathcal{P} := f^{-1}(\infty)$ to be the set of points that f sends to infinity. Because f is continuous, both $\mathcal{Z}$ and $\mathcal{P}$ are closed in Ω.

Unravelling Definition 8.5 a little we see that f will be holomorphic on $\Omega \setminus \mathcal{P}$, while $1/f$ will be holomorphic on $\Omega \setminus \mathcal{Z}$. If f is not identically zero then, being a holomorphic function on $\Omega \setminus \mathcal{P}$, its zeros are isolated, that is, $\mathcal{Z}$ is a *discrete*[6] subset of $\Omega \setminus \mathcal{P}$ and thus a discrete subset of the connected set Ω. A little topology exercise then confirms that $\Omega \setminus \mathcal{Z}$ is connected.

How big is the set $\mathcal{P}$?

If $z_0 \in \mathcal{P}$, then $1/f$ has a zero of finite order at z_0 since otherwise we would have $1/f \equiv 0$, i.e. f would be identically equal to ∞. Therefore, in some small ball $B_\varepsilon(z_0) \subset \Omega$, the function $1/f$ is zero only at z_0 and thus $f(z) \neq \infty$ for all $z \in B_\varepsilon(z_0) \setminus \{z_0\}$.

Thus the set $\mathcal{P}$ is a discrete subset of Ω.

We see that we could have defined a meromorphic function as a holomorphic function $f\colon \Omega \setminus \mathcal{P} \to \mathbb{C}$, where $\mathcal{P}$ is some closed discrete set in Ω, such that f has a pole at each $z_0 \in \mathcal{P}$.

[6] $\mathcal{Z}$ being discrete means that every element of $\mathcal{Z}$ is isolated, i.e. admits a neighbourhood within which there are no other elements of $\mathcal{Z}$.

8.3 Essential singularities; The Casorati-Weierstrass theorem

VIDEO: Essential singularities; The Casorati-Weierstrass theorem

Karl Theodor Wilhelm Weierstrass (1815 - 1897). Father of modern analysis.

Felice Casorati (1835 - 1890). Best known for this theorem.

If a holomorphic function $f\colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ has an isolated singularity at z_0 that is neither a removable singularity nor a pole, then by definition it cannot be extended to a continuous function $f : B_r(z_0) \to \mathbb{C}_\infty$. The following theorem shows that such essential singularities are much wilder than even that would suggest.

Theorem 8.6 (Casorati-Weierstrass theorem)**.** *Suppose that a holomorphic function $f\colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then however small we take $\delta \in (0, r)$, the image of the set $B_\delta(z_0) \setminus \{z_0\}$ under f is dense in $\mathbb{C}$.*

Proof. We prove the contrapositive: Suppose that it is *not* true that for every δ the image of f is dense. That is, there exist $\delta \in (0, r)$, $\varepsilon > 0$ and a point $w \in \mathbb{C}$ such that

$$|f(z) - w| \geq \varepsilon \qquad \text{for all} \qquad z \in B_\delta(z_0) \setminus \{z_0\}. \tag{8.3.1}$$

Our objective is to show that f has either a removable singularity or a pole at z_0. Consider the holomorphic function

$$h(z) = \frac{1}{f(z) - w}. \tag{8.3.2}$$

By assumption, h is bounded ($|h| \leq \frac{1}{\varepsilon}$) and is nonzero throughout $B_\delta(z_0) \setminus \{z_0\}$. By Riemann's removable singularity theorem 8.2, h can be extended to a holomorphic function on all of $B_\delta(z_0)$. If $h(z_0) \neq 0$, then we can rewrite

$$f(z) = \frac{1}{h(z)} + w, \tag{8.3.3}$$

to give a holomorphic extension of f to the whole of $B_\delta(z_0)$, and we see that f has a removable singularity at z_0. If, instead, $h(z_0) = 0$, then h must have a zero of *finite* order $n \in \mathbb{N}$ at z_0 (otherwise we would have $h \equiv 0$ by Theorem 7.4) and we can write

$$h(z) = (z - z_0)^n g(z)$$

for some nonzero holomorphic function $g : B_\delta(z_0) \to \mathbb{C}$. In that case we can write

$$f(z) = (z - z_0)^{-n} \frac{1}{g(z)} + w,$$

and we see that f has a pole at z_0. □

Remark 8.7. The Casorati-Weierstrass theorem is not the last word on essential singularities. The *Great Picard's theorem* states that however small a neighbourhood of an essential singularity you take, your holomorphic function attains every value in $\mathbb{C}$ with at most *one* exception! You can't do better than that because (for example) however small you take $\delta > 0$, the restriction of the function $f(z) = \exp(1/z)$ to $B_\delta(0) \setminus \{0\}$ attains every value in $\mathbb{C}$ except for 0. This theorem is trickier to prove.

8.4 Laurent series I: Double-ended series

VIDEO: Laurent series I: Double-ended series

Pierre Alphonse Laurent (1813 - 1854).

If we have a holomorphic function $f : B_r(0) \to \mathbb{C}$, then Taylor's theorem allows us to expand it as a power series $\sum_{k=0}^{\infty} a_k z^k$. But how can we expand a function as a power series at an isolated singularity?

For example, one could consider the function $f(z) = \frac{1}{z^2}$ or the function $f(z) = \frac{1}{\sin z}$, defined on (say) $D \setminus \{0\}$. There is no way we can write either function on the whole domain as a power series as above. Laurent's theorem will tell us that we can write such functions as what one might call a double-ended power series

$$\sum_{k \in \mathbb{Z}} a_k z^k, \tag{8.4.1}$$

allowing k to be negative. When we have such a double-ended power series representing a function on some annulus, then we call it a *Laurent series*.

In simple cases, the power series has terms with negative k, but not all the way down to $-\infty$, for example,

$$\frac{e^z}{z} = \sum_{k=-1}^{\infty} \frac{z^k}{(k+1)!}.$$

Let's check that we can make rigorous sense of these expressions when k *can* go down to $-\infty$.

Definition 8.8. A double-ended series $\sum_{k \in \mathbb{Z}} a_k$ is said to converge to $\ell \in \mathbb{C}$ if $\sum_{k=0}^{\infty} a_k$ converges to ℓ_+, $\sum_{k=1}^{\infty} a_{-k}$ converges to ℓ_-, and $\ell = \ell_+ + \ell_-$.

This allows us to make sense of a double-ended *power* series $\sum_{k \in \mathbb{Z}} a_k z^k$. Moreover, it highlights that we are really considering two normal power series here. The first is $f_+(z) = \sum_{k=0}^{\infty} a_k z^k$, and the second is $f_-(w) = \sum_{k=1}^{\infty} a_{-k} w^k$, where $w = 1/z$. In particular, we can invoke all the theory about power series from Section 3.1. We find that the first power series defines a holomorphic function for $|z| < R_+$, where R_+ is the first radius of convergence, and the second power series defines a holomorphic function for $|w| < R_-$, where R_- is the second radius of convergence, i.e. for $|z| > 1/R_-$. Provided $1/R_- < R_+$, the double-sided power series defines a holomorphic function on the annulus given by $1/R_- < |z| < R_+$, where it can be differentiated term by term, and where it converges locally uniformly.

8.5 Cauchy's integral formula for annuli

VIDEO: Cauchy's integral formula for annuli

In order to find Laurent series, we need a generalisation of Cauchy's integral formula.

Theorem 8.9 (Cauchy's integral formula for annuli)**.** *Let $\Omega \subset \mathbb{C}$ be open and let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Suppose that for some $a \in \mathbb{C}$ and radii $0 < R_1 < R_2 < \infty$, the closure of the annulus*

$$A = \{z \in \mathbb{C}\colon R_1 < |z-a| < R_2\} \tag{8.5.1}$$

is contained in Ω. Then for any $w \in A$ we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial A} \frac{f(z)}{z-w}\,dz. \tag{8.5.2}$$

Remark 8.10. As discussed in Section 3.4, the integral over ∂A in (8.5.2) is an integral over two boundary circles, taken in the appropriate direction. If f is holomorphic on the whole ball $B_{R_2}(a)$, then the inner integral vanishes and the formula reduces to the standard Cauchy integral formula (5.6.1).

Now that we have Riemann's removable singularity theorem 8.2 at our disposal, proving integral formulae becomes a lot easier, by virtue of the following observation.

Corollary 8.11 (Corollary of Riemann's removable singularity theorem 8.2)**.** *Suppose $\Omega \subset \mathbb{C}$ is open, $w \in \Omega$ and $f\colon \Omega \to \mathbb{C}$ is holomorphic. Then the function*

$$z \mapsto \frac{f(z)-f(w)}{z-w},$$

which is initially defined on $\Omega \setminus \{w\}$, extends to a holomorphic function on the whole of Ω.

Proof. The given function is clearly holomorphic on $\Omega \setminus \{w\}$, and by applying Riemann's removable singularity theorem 8.2 on a ball around w, we are done. □

Since the extension here must be unique, we will implicitly refer to this extension to the whole of Ω when we write $\frac{f(z)-f(w)}{z-w}$.

Proof of Theorem 8.9. Fix $w \in A$. By Corollary 8.11, the function $z \mapsto \frac{f(z)-f(w)}{z-w}$ is holomorphic throughout Ω. Using this function in the Cauchy theorem for annuli, i.e. Corollary 5.8, we obtain

$$\int_{\partial A} \frac{f(z)-f(w)}{z-w} dz = 0,$$

and so

$$\begin{aligned}\int_{\partial A} \frac{f(z)}{z-w}dz &= \int_{\partial A} \frac{f(w)}{z-w}dz \\ &= f(w)\left(\int_{\partial B_{R_2}(a)} \frac{dz}{z-w} - \int_{\partial B_{R_1}(a)} \frac{dz}{z-w}\right) \\ &= 2\pi i\, f(w)\Big(I(\partial B_{R_2}(a), w) - I(\partial B_{R_1}(a), w)\Big) \\ &= 2\pi i\, f(w)(1-0) \\ &= 2\pi i\, f(w),\end{aligned} \tag{8.5.3}$$

where we use the notation $\partial B_R(a)$ also to refer to the curve $t \mapsto a + Re^{it}$ for $t \in [0, 2\pi]$. The given values of the winding numbers are evident from a picture; we can justify that $I(\partial B_{R_2}(a), w) = 1$ using Q. 4.2, while $I(\partial B_{R_1}(a), w) = 0$ follows from Remark 4.5. □

It is tempting to ask why the original proof of Cauchy's integral formula was not as slick and quick as this. Why did we not use a result like Corollary 8.11? The answer is that we need the original Cauchy's integral formula along the way to prove Corollary 8.11.

8.6 Laurent series II: Laurent's theorem

VIDEO: Laurent series II: Laurent's theorem

Minor mistake at 26:15: I write $\mathbb{C} \setminus D$ when I mean $\mathbb{C} \setminus \overline{D}$.

We are now ready to prove that every holomorphic function defined on an annulus can be developed in a Laurent series.

Theorem 8.12 (Laurent's theorem)**.** *Suppose $0 \le r_1 < r_2$, $a \in \mathbb{C}$ and f is a holomorphic function on the annulus*

$$A = \{z \in \mathbb{C} \colon r_1 < |z-a| < r_2\}.$$

Then, for every $z \in A$, we have

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z-a)^k, \tag{8.6.1}$$

where for each $k \in \mathbb{Z}$ the coefficient a_k is given by

$$a_k = \frac{1}{2\pi i} \int_{\partial B_s(a)} \frac{f(w)}{(w-a)^{k+1}}\, dw, \tag{8.6.2}$$

for every $s \in (r_1, r_2)$.

The proof can be compared with that of Taylor's theorem 6.1.

Proof. Without loss of generality we may assume that $a = 0$. **After** fixing $z \in A$, choose numbers R_1, R_2 such that $r_1 < R_1 < |z| < R_2 < r_2$.

By Cauchy's theorem for annuli, Theorem 5.8, the integral of a holomorphic function on A around a circle $\partial B_s(0)$ does not depend on $s \in (r_1, r_2)$. In particular, the formula (8.6.2) given for a_k is independent of s, and indeed

$$a_k = \frac{1}{2\pi i} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w^{k+1}}\, dw = \frac{1}{2\pi i} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w^{k+1}}\, dw. \tag{8.6.3}$$

Cauchy's integral formula for annuli, Theorem 8.9, implies

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w - z}\, dw. \tag{8.6.4}$$

The first term on the right-hand side can be handled as in the proof of Taylor's theorem in Section 6.2, giving

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w - z}\, dw &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial B_{R_2}(0)} \frac{f(w)}{w^{k+1}}\, dw \right) z^k \\ &= \sum_{k=0}^{\infty} a_k z^k, \end{aligned} \tag{8.6.5}$$

by (8.6.3). To handle the second term on the right-hand side of (8.6.4) we write, for any $w \in \partial B_{R_1}(0)$,

$$-\frac{1}{w - z} = \frac{1}{z} \left(\frac{1}{1 - \frac{w}{z}} \right) = \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{w}{z} \right)^k, \tag{8.6.6}$$

where the power series converges because $|z| > R_1 = |w|$ and so $|\frac{w}{z}| < 1$. Hence

$$\begin{aligned} -\frac{1}{2\pi i} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w - z}\, dw &= \frac{1}{2\pi i} \int_{\partial B_{R_1}(0)} \sum_{k=0}^{\infty} \frac{f(w)}{z^{k+1}} w^k\, dw \\ &= \sum_{k=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\partial B_{R_1}(0)} \frac{f(w)}{w^{k+1}}\, dw \right) z^k \\ &= \sum_{k=-\infty}^{-1} a_k z^k. \end{aligned} \tag{8.6.7}$$

The summation and the integration can be switched here because the series converges uniformly as the integration variable w varies within $\partial B_{R_1}(0)$. □

Remark 8.13. Just as the Taylor coefficients are unique, the coefficients of Laurent series are uniquely determined by the function. Indeed, if $f : A \to \mathbb{C}$ is known to have a Laurent expansion (8.6.1), then one can divide by $(z - a)^{n+1}$ and integrate around $\partial B_s(a)$ to obtain (8.6.2).

Remark 8.14. Connecting with the discussion in Section 8.4, one can argue that the power series in (8.6.5) converges on $B_{r_2}(0)$, whereas the power series in (8.6.7) converges on $\mathbb{C} \setminus \overline{B_{r_1}(0)}$.

Example 8.15. The function $f(z) = \frac{1}{1-z}$ is holomorphic on the disc D and on the annulus $\mathbb{C} \setminus \overline{D}$. On D we have the usual Taylor series

$$f(z) = \sum_{k=0}^{\infty} z^k. \tag{8.6.8}$$

Beyond the radius of convergence, i.e. on $\mathbb{C} \setminus \overline{D}$, where $\frac{1}{|z|} < 1$, we instead have the Laurent series

$$f(z) = \frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{-1}(-1)\,z^k. \tag{8.6.9}$$

Of course, the simplest Laurent series would be about the centre point $a = 1$, and would be valid throughout $\mathbb{C} \setminus \{1\}$. There is only one term and it is simply $f(z) = -(z-1)^{-1}$!

As before in Corollary 6.6, the boundedness of f implies bounds on all of the Laurent coefficients:

Corollary 8.16. *Let f be as in the statement of Theorem 8.12. Suppose that for some $s \in (r_1, r_2)$ we have $|f(z)| \leq M$ for all $z \in \partial B_s(a)$. Then, for every $k \in \mathbb{Z}$, we have*

$$|a_k| \leq \frac{M}{s^k}. \tag{8.6.10}$$

8.7 Classifying singularities in terms of Laurent series

VIDEO: Classifying singularities in terms of Laurent series

Suppose we have a function f that is holomorphic on $B_r(a) \setminus \{a\} \subset \mathbb{C}$, for some $r > 0$ and $a \in \mathbb{C}$. We would like to relate the type of the isolated singularity at a, as in Definition 8.3, to the Laurent series

$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z-a)^k, \tag{8.7.1}$$

that is valid on $B_r(a) \setminus \{a\}$ by Laurent's theorem 8.12.

We would also like to extend the definition of the order $\operatorname{ord}(f,a)$ of a zero at a, as given in Definition 7.1, so that it applies not just to a zero of a holomorphic function f, but also to a general isolated singularity.

The first possibility is that $f \equiv 0$ in $B_r(a) \setminus \{a\}$. In this case we have a removable singularity. All the Laurent coefficients are zero, and we define the order $\operatorname{ord}(f,a) = \infty$.

If not all of the Laurent coefficients are zero, then we define the order of f at a to be

$$\operatorname{ord}(f,a) := \inf\{n \in \mathbb{Z} \;:\; a_n \neq 0\}.$$

Here we adopt the convention that if the set is not bounded below then the infimum is taken to be $-\infty$. If this order is not $-\infty$, then the Laurent series can be written $\sum_{k=n}^{\infty} a_k(z-a)^k$, where $n \in \mathbb{Z}$ is the order.

If $\mathrm{ord}(f, a) \geq 0$, then the Laurent series is really a Taylor series. Indeed, we have a removable singularity at a if and only if $\mathrm{ord}(f, a) \geq 0$ or $\mathrm{ord}(f, a) = \infty$. If $\mathrm{ord}(f, a) \geq 1$ then we have a zero of this order at a. Indeed, this coincides with the definition of ord given in Definition 7.1.

Similarly, $\mathrm{ord}(f, a) < 0$ (but $\mathrm{ord}(f, a) \neq -\infty$) if and only if f has a pole at a. This follows by considering the description of poles given in Theorem 8.4. For example, if f has a pole at a, then we can write $f(z) = (z-a)^{-n} g(z)$ with $g(a) \neq 0$ and then Taylor expand $g(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ to write

$$f(z) = \sum_{k=-n}^{\infty} a_{k+n} (z-a)^k$$

where $a_0 \neq 0$. Rather confusingly, the order $\mathrm{ord}(f, a)$ is minus the order of the pole at a.

The case that $\mathrm{ord}(f, a) = -\infty$, i.e. the Laurent series does not start at some finite n, then corresponds, by definition, to the case of an essential singularity.

8.8 Classification of injective entire functions

VIDEO: Classification of injective entire functions

There are many entire functions, but not so many *injective* entire functions.

Theorem 8.17 (Injective entire functions)**.** *If f is an* injective *entire function, then*

$$f(z) = \alpha z + \beta \tag{8.8.1}$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$.

Proof. The function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $g(z) = f(1/z)$ is clearly holomorphic and injective since both the functions $z \mapsto 1/z$ and $f(z)$ are holomorphic and injective.

What sort of isolated singularity does g have at 0? We would like to show that it is a pole, and we will do that by ruling out the other two possibilities, i.e. showing that it is neither removable nor essential.

If it were removable, then g would be bounded in, say, $\overline{D} \setminus \{0\}$, and therefore f would be bounded in $\mathbb{C} \setminus D$. But f is continuous and thus bounded on $\overline{D}$, so f would be bounded on the whole of $\mathbb{C}$. By Liouville's theorem (Corollary 6.7) f would then be constant, which is impossible for an injective function!

Suppose instead that 0 is an *essential* singularity for g. The Casorati-Weierstrass theorem 8.6 tells us that the image $g(D \setminus \{0\})$ would be dense, equivalently that the image $f(\mathbb{C} \setminus \overline{D})$ would be dense, in $\mathbb{C}$. But $f(D)$ is an open set by the open mapping theorem 7.11, and so there must be some intersection of $f(\mathbb{C} \setminus \overline{D})$ and $f(D)$, which is impossible for an injective function f. Contradiction!

We have shown that 0 is a pole for g. If we make a Taylor expansion

$$f(z)=\sum_{k=0}^{\infty}a_k z^k \tag{8.8.2}$$

then the (unique) Laurent expansion for g on $\mathbb{C}\setminus\{0\}$ must be

$$g(z)=f\big(1/z\big)=\sum_{k=-\infty}^{0}a_{-k}z^k, \tag{8.8.3}$$

and because g has a pole, let's say of order n, we must have $a_k=0$ for all $k>n$. This forces f to be a polynomial.

We conclude by observing that by the *Fundamental theorem of Algebra*, Corollary 6.8, or rather by its consequence that any polynomial of order at least one can be factorised, the only injective polynomials are of the form claimed in the theorem. Indeed, by injectivity only one point (say $z=a$) can be mapped to zero so the factorisation of the polynomial must be of the form $f(z)=\alpha(z-a)^n$ for $\alpha\neq 0$. But this polynomial is only injective for $n=1$, so we have

$$f(z)=\alpha z-\alpha a.$$

We conclude by setting $\beta=-\alpha a$. □

Remark 8.18. In Section 2.2 we mentioned briefly how to make sense of what it means for a continuous function $f:\mathbb{C}_\infty\to\mathbb{C}_\infty$ to be holomorphic. We are now in a position to classify all such functions that are bijections. Pick any Möbius transformation φ with the property that $\varphi(f(\infty))=\infty$. For example we could define

$$\varphi(z)=\frac{1}{z-f(\infty)}.$$

Then $\varphi\circ f$ will also be a holomorphic bijection from $\mathbb{C}_\infty$ to itself, this time sending ∞ to itself. Therefore the restriction of $\varphi\circ f$ to $\mathbb{C}$ will be a injective entire function, so Theorem 8.17 tells us that it must be of the form $z\mapsto\alpha z+\beta$ for some $\alpha\in\mathbb{C}\setminus\{0\}$ and $\beta\in\mathbb{C}$. In particular $\varphi\circ f$ is necessarily a Möbius transformation. We deduce that f also must be a Möbius transformation. We conclude that the holomorphic bijections from $\mathbb{C}_\infty$ to itself are precisely the Möbius transformations.

8.9 Line singularities

Most of Section 8 has concerned isolated singularities. Particularly relevant to this section was Riemann's removable singularity theorem 8.2 that told us (for example) that any bounded holomorphic function on $D\setminus\{0\}$ can be extended to a holomorphic function on the whole of D.

We would now like to imagine what happens if our function is holomorphic on a domain minus a line rather than on a domain minus a point. For example, we might have a holomorphic function $f\colon\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$. When can we deduce that f extends to an entire function? The direct analogue of Riemann's removable singularity theorem 8.2 fails. We could have $f\equiv 1$ on the upper half plane and

$f \equiv 0$ on the lower half plane. Such a function is bounded and holomorphic but does not extend to a holomorphic function on the whole of $\mathbb{C}$!

The point of this section is to establish that if we additionally assume that f extends to a continuous function, then this extension is indeed necessarily holomorphic.

Theorem 8.19. *Suppose $\Omega \subset \mathbb{C}$ is open with $\Omega \cap \mathbb{R} \neq \emptyset$. Suppose $f : \Omega \to \mathbb{C}$ is continuous and the restriction of f to $\Omega \setminus \mathbb{R}$ is holomorphic. Then f is holomorphic throughout Ω.*

The gain of regularity in this theorem will be powered by Morera's theorem 6.9.

In the proof below we will reuse the shorthand $H_{\Im>0} := \{z \in \mathbb{C} : \Im(z) > 0\}$ for the upper half-plane, and also write $H_{\Im\geq 0}$ for its closure and write $H_{\Im<0}$ and $H_{\Im\leq 0}$ for the lower half-plane analogues.

Proof. We need to show that f is complex differentiable at each point in $\Omega \cap \mathbb{R}$. It is not even immediately clear that any partial derivatives of f exist there.

To solve this problem it suffices to pick an arbitrary $z_0 \in \Omega \cap \mathbb{R}$, and $r > 0$ sufficiently small so that $\overline{B_r(z_0)} \subset \Omega$, and prove that f is holomorphic on $B_r(z_0)$. By Morera's theorem 6.9, it suffices to show that for every closed triangle $T \subset B_r(z_0)$ we have

$$\int_{\partial T} f(z)dz = 0. \tag{8.9.1}$$

If T lies within the half ball $B_r(z_0) \cap H_{\Im>0}$ then Goursat's theorem 5.3 does the job. Moreover, if T lies within the slightly larger set $B_r(z_0) \cap H_{\Im\geq 0}$ then we can consider the slightly shifted upwards triangle $T_\varepsilon := T + i\varepsilon$ with $\varepsilon > 0$ small, and use uniform continuity of f on $\overline{B_r(z_0)}$ to compute

$$\int_{\partial T} f(z)dz = \lim_{\varepsilon \downarrow 0} \int_{\partial T_\varepsilon} f(z)dz = 0.$$

In exactly the same way, we also have (8.9.1) for every triangle T within the lower half-ball $B_r(z_0) \cap H_{\Im\leq 0}$.

For a completely general triangle $T \subset B_r(z_0)$ we can divide T up into no more than three sub-triangles T_i, each of which lies in either $B_r(z_0) \cap H_{\Im\geq 0}$ or $B_r(z_0) \cap H_{\Im\leq 0}$, as in Figure 6, and write

$$\int_{\partial T} f(z)dz = \sum_i \int_{\partial T_i} f(z)dz = 0.$$

□

8.10 The Schwarz reflection principle

One situation in which a line singularity arises is when we reflect a holomorphic function across a line.

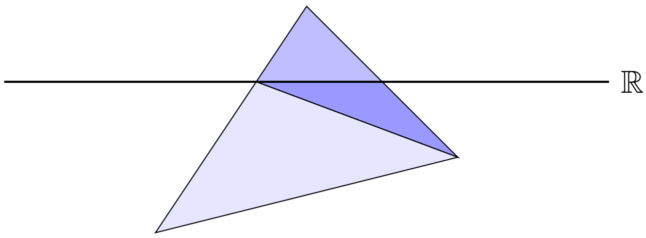


Figure 6: Triangles T_1, T_2 and T_3 making up triangle T

Theorem 8.20 (Schwarz reflection principle)**.** *Let $\Omega \subset \mathbb{C}$ be open and assume that Ω is invariant under complex conjugation (i.e. $z \in \Omega \Leftrightarrow \bar{z} \in \Omega$). Suppose that $f : \Omega \cap H_{\Im \geq 0} \to \mathbb{C}$ is a continuous function such that*

- *f is holomorphic on $\Omega \cap H_{\Im > 0}$*
- *f only attains real values on $\Omega \cap \mathbb{R}$.*

If we extend f to a function on the whole of Ω by setting

$$f(z) = \overline{f(\bar{z})} \qquad \text{for all } z \in \Omega \cap H_{\Im < 0},$$

then f is holomorphic on the whole of Ω.

You will want to pause for a moment to check that this extension actually makes sense!

Proof. Observe first that the extension as defined is continuous on the whole of Ω. In particular it is continuous even on $\Omega \cap \mathbb{R}$, because f only attains real values there.

To see that f is holomorphic on $\Omega \cap H_{\Im<0}$, observe first that it is *real* differentiable (being the composition of real differentiable functions $z \mapsto \bar{z}$, $z \mapsto f(z)$ and $z \mapsto \bar{z}$ again). We then just have to check that the Cauchy-Riemann equations hold there, i.e. that $\frac{\partial f}{\partial \bar{z}} = 0$. By definition, for $z \in \Omega \cap H_{\Im<0}$ we have

$$\frac{\partial}{\partial \bar{z}}(f(z)) = \frac{\partial}{\partial \bar{z}}(\overline{f(\bar{z})}) = \overline{\frac{\partial}{\partial z}(f(\bar{z}))}$$

(you should carefully unwind the definitions to verify this) while by the chain rule (1.2.5) (composing f with the function $g(z) = \bar{z}$) we know that

$$\frac{\partial}{\partial z}(f(\bar{z})) = f'(\bar{z})\frac{\partial \bar{z}}{\partial z} = 0$$

because $\frac{\partial \bar{z}}{\partial z} = 0$, as we saw in Q. 1.9.

We may as well assume that $\Omega \cap \mathbb{R} \neq \emptyset$ otherwise the proof is already complete. We are thus in the situation of Theorem 8.19 so f must be holomorphic throughout Ω. □

8.11 Exercises

8.1. The following functions are holomorphic on $D\setminus\{0\}$. In each case, identify whether the isolated singularity at 0 is removable, a pole, or essential.

(a) $f(z) = \frac{\sin z}{z}$
(b) $f(z) = \frac{\cos z}{z}$
(c) $f(z) = \frac{\exp z}{z}$
(d) $f(z) = z \exp \frac{1}{z}$
(e) $f(z) = e^{z^2} \exp \frac{1}{z}$

8.2. Suppose $w \in \mathbb{C} \setminus \{0\}$ and consider the meromorphic function $f(z) = \frac{1}{z-w}$.

(a) Give the Taylor series expansion of f centred at 0. What is the radius of convergence?
(b) Give the Laurent expansion of f, centred at 0, for $|z| > |w|$.

8.3. Consider the meromorphic function

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

By decomposing via partial fractions, find the following series expansions for f about 0:

(a) Taylor series for $|z| < 1$;
(b) Laurent series for $1 < |z| < 2$;
(c) Laurent series for $|z| > 2$.

8.4. Suppose $g, h \colon \Omega \to \mathbb{C}$ are two holomorphic functions on some open and connected subset $\Omega \subset \mathbb{C}$ such that h is not identically zero. Prove that $f \colon \Omega \to \mathbb{C}_\infty$ defined by $f(z) := \frac{g(z)}{h(z)}$ is a meromorphic function.

8.5. Suppose $\mathcal{S}$ is a finite subset of a domain $\Omega \subset \mathbb{C}$. Suppose that $f \colon \Omega \setminus \mathcal{S} \to \mathbb{C}$ is holomorphic, with a pole at each of the points in $\mathcal{S}$, and hence extends to a meromorphic function $\Omega \mapsto \mathbb{C}_\infty$. Prove that there exist holomorphic functions $g, h : \Omega \to \mathbb{C}$ such that $f(z) = \frac{g(z)}{h(z)}$ for all $z \in \Omega \setminus \mathcal{S}$.

8.6. Suppose $f : B_1(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0. Show that for every $n \in \mathbb{Z}$, the limit of $(z - z_0)^n f(z)$ as $z \to z_0$ does not exist.

8.7. Does there exist a holomorphic function $f \colon D \setminus \{0\} \to \mathbb{C}$ such that

$$|f(z)| \geq 2^{\frac{1}{|z|}}$$

throughout?

8.8. *Optional question.*

Suppose D^+ is the upper half disc, i.e. all the points in D with positive imaginary part. Suppose that $f : D^+ \to \mathbb{C}$ is holomorphic, and extends to a continuous function on $D^+ \cup (-1, 1)$, with $f(x) = 0$ for all $x \in (-1, 1)$. Prove that $f \equiv 0$ throughout D^+.

8.9. *Optional question.*

Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous, and the restriction of f to both D and $\mathbb{C} \setminus \overline{D}$ is holomorphic. Prove that f is entire.

9 The general form of Cauchy's theorem

The form of Cauchy's theorem that we proved in Theorem 5.7 was only able to handle individual closed curves in star-shaped domains. This was ideal at the time since it allowed us to develop the local theory of holomorphic functions and prove numerous amazing results. By now, however, we have the technology to prove a much more general form of Cauchy's theorem that will have the case of individual curves in simply connected domains as a special case.

A key application of this theory will be the Residue theorem 9.10. Amongst other things, this will give us a very powerful way of computing integrals.

9.1 Chains and cycles

VIDEO: Chains and cycles

In the last 90 seconds I mess up the sign of γ_2. Either I should take γ_2 going anticlockwise and consider $\gamma = \gamma_1 - \gamma_2$ (as in the lecture notes below) or I should take γ_2 going clockwise and consider $\gamma = \gamma_1 + \gamma_2$. Not a mixture of both!

Suppose $\Omega \subset \mathbb{C}$ is open, and $f : \Omega \to \mathbb{C}$ is a *continuous* function. If we are given two piecewise $\mathcal{C}^1$ curves $\gamma_1 : [a, b] \to \Omega$ and $\gamma_2 : [a', b'] \to \Omega$, we can make a formal definition

$$\int_{\gamma_1+\gamma_2} f(z)dz := \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

More generally, given finitely many piecewise $\mathcal{C}^1$ curves $\gamma_1, \ldots, \gamma_n$, and weights $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$, we can consider a formal linear combination

$$\gamma := \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n, \tag{9.1.1}$$

and define

$$\int_{\gamma} f(z)dz := \sum_{k=1}^{n} \alpha_k \int_{\gamma_k} f(z)dz. \tag{9.1.2}$$

Loosely speaking, you can consider this sum γ as a *chain*. Strictly speaking a chain would be an equivalence class of such things, where we say that two such formal linear combinations γ and $\tilde{\gamma}$ are equivalent if (9.1.2) gives the same value *whichever continuous function f we take*. This equivalence allows us to permute the curves (together with their weights!), it allows us to subdivide individual curves into finitely many sub-curves (so if $\gamma_3 : [a, b] \to \mathbb{C}$ is a curve that is split into two curves $\gamma_1 : [a, c] \to \Omega$ and $\gamma_2 : [c, b] \to \mathbb{C}$, then $\gamma_3 = \gamma_1 + \gamma_2$), it allows us to fuse pairs of individual curves into one if the first one starts where the second one ends (the opposite of the previous operation), it allows us to reparametrise individual curves, to combine multiple copies of the same γ_i by adding their weights, and to reverse the direction of a curve, provided we change the sign of the corresponding weight.

A *cycle* is, strictly speaking, one of these chains that can be represented in terms of curves $\gamma_1, \ldots, \gamma_n$ that are each *closed* curves.

For the purposes of this course, there is no merit in setting this up precisely (as a free abelian group) and the equivalence relation above is a bit of a technical distraction, so we simplify everything by permitting ourselves to take a cycle in Ω to mean simply a formal linear combination as in (9.1.1) with each γ_i a closed piecewise $\mathcal{C}^1$ curve within Ω, to define integration as in (9.1.2), and to define the winding number as

$$I(\gamma, w) := \sum_{k=1}^{n} \alpha_k I(\gamma_k, w).$$

See the lectures/video for some pictures!

As a small exception to this simplification, we will occasionally consider two piecewise $\mathcal{C}^1$ curves $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$ with the same start points, $\gamma_1(a) = \gamma_2(a)$, and the same end points, $\gamma_1(b) = \gamma_2(b)$, and it will be convenient to consider the cycle $\gamma_1 - \gamma_2$. Of course, this can be represented as a single closed curve that passes first along γ_1 and then returns along γ_2.

So much of the theory we have seen for closed piecewise $\mathcal{C}^1$ curves extends to cycles in a completely obvious way that we do not attempt to give a full list. Certainly, if $\Omega \subset \mathbb{C}$ is a star-shaped domain, and γ is a cycle in Ω then Theorem 5.7 will imply that for every holomorphic function $f : \Omega \to \mathbb{C}$ we have

$$\int_\gamma f(z)dz = 0. \tag{9.1.3}$$

In the next section we would like to understand for which cycles γ in a general open set $\Omega \subset \mathbb{C}$ do we still have (9.1.3). In preparation for that we make the following definition.

Definition 9.1. Let $\Omega \subset \mathbb{C}$ be open. A cycle γ in Ω is *homologous to zero* in Ω if for any $a \in \mathbb{C} \setminus \Omega$ we have

$$I(\gamma, a) = 0. \tag{9.1.4}$$

Example 9.2. As an example, if $\gamma_1, \gamma_2 : [0, 2\pi] \to \Omega := \mathbb{C} \setminus \{0\}$ are the curves given by $\gamma_1(\theta) = 2e^{i\theta}$ and $\gamma_2(\theta) = e^{i\theta}$, then the cycle $\gamma = \gamma_1 - \gamma_2$, is homologous to zero in Ω. In other words, the cycle that one might write as ∂A, where A is the annulus $\{z \in \mathbb{C} \ : \ 1 < |z| < 2\}$, is homologous to zero. Note that the individual curves γ_1 and γ_2 are **not** individually homologous to zero!

9.2 The homology version of Cauchy's theorem

VIDEO: The homology version of Cauchy's theorem

The pictures in the lectures/video are going to be particularly useful in this section.

Suppose $\Omega \subset \mathbb{C}$ is open. In this section we ask for which cycles γ in Ω do we have

$$\int_\gamma f(z)\, dz = 0 \qquad \text{for all holomorphic } f\colon \Omega \to \mathbb{C}?$$

Theorem 9.3 (Cauchy's theorem – homology version). *Let $\Omega \subset \mathbb{C}$ be open and $f \colon \Omega \to \mathbb{C}$ holomorphic. Then for any cycle γ that is* homologous to zero *in Ω we have*

$$\int_\gamma f(z)dz = 0. \tag{9.2.1}$$

The hypothesis that γ is homologous to zero is necessary since otherwise there exists some $a \in \mathbb{C} \setminus \Omega$ with $I(\gamma, a) \neq 0$, and we can define a holomorphic function $f \colon \Omega \to \mathbb{C}$ by $f(z) = \frac{1}{z-a}$, giving

$$\int_\gamma f(z)dz = \int_\gamma \frac{dz}{z-a} = 2\pi i I(\gamma, a) \neq 0.$$

Proof. We may assume without loss of generality that Ω is bounded. If it is not then we can replace Ω by $\Omega \cap B_R(0)$, where $R > 0$ is chosen large enough to ensure that all curves making up γ map within $B_R(0)$. By compactness, the distance of the image of γ to $\mathbb{C} \setminus \Omega$ is strictly positive - let us denote it by $2\delta > 0$.

Let's put γ aside for a moment and try to find a representation formula for f akin to the Cauchy integral formula, but where we write $f(w)$ for $w \in \Omega$ in terms of the integral of $\frac{f(z)}{z-w}$ over something more complicated than a circle or square surrounding w. Heuristically we would like to write it as an integral over all boundary components of Ω, but this does not make sense in general because f is only defined on Ω, and not on $\partial\Omega$, and because the boundary $\partial\Omega$ is not necessarily the image of a piecewise $\mathcal{C}^1$ curve. Instead we make a construction that will move the boundary components to sensible curves a little inside Ω made up of finitely many horizontal and vertical line segments.

I'll draw some pictures in the video!

Consider a grid of width δ on $\mathbb{C}$ made up of closed squares

$$\{x + iy \ : \ x \in [k\delta, (k+1)\delta] \text{ and } y \in [l\delta, (l+1)\delta]\}$$

for $k, l \in \mathbb{Z}$. Denote by $\{Q_j\}_{j=1}^J$ the finitely many such closed squares that are fully contained in Ω. They combine to make up an open set

$$\Omega_\delta := \text{interior}(\cup_{j=1}^J Q_j)$$

that is a slight shrinking of Ω.

Let w be an arbitrary point in the interior of some square Q_{j_0}.

By the version of Cauchy's integral formula in Theorem 5.10 that you proved in Q. 5.3, we have that

$$f(w) = \frac{1}{2\pi i} \int_{\partial Q_{j_0}} \frac{f(z)}{z-w} dz. \tag{9.2.2}$$

For any other square Q_j for $j \neq j_0$ we have

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)}{z-w} dz = 0, \tag{9.2.3}$$

by Cauchy's theorem 5.7 for star-shaped domains. (As star-shaped domain we can take an open square just a little larger than Q_j so that w lies outside its closure. Then $z \mapsto \frac{f(z)}{z-w}$ is holomorphic on this domain.) If we sum these identities over all squares making up Ω_δ then all the integrals over *interior* edges cancel because all interior edges are traversed twice with opposite directions, and we obtain

$$f(w) = \frac{1}{2\pi i} \sum_{j=1}^{J} \int_{\partial Q_j} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int_{\partial \Omega_\delta} \frac{f(z)}{z-w}\, dz. \tag{9.2.4}$$

Although we have asked for w to lie in the interior of some square Q_{j_0}, by continuity we see that this formula holds for any $w \in \Omega_\delta$ (i.e. also on the interior edges). This is then the representation formula that we sought.

Now let's bring the cycle γ back into the picture. By definition of δ, the image of the cycle γ is fully contained in Ω_δ. Also, for every $z \in \mathbb{C} \setminus \Omega_\delta$, and in particular for every $z \in \partial\Omega_\delta$, we have

$$I(\gamma, z) = 0, \tag{9.2.5}$$

as follows from Lemma 4.7 and the assumption that γ is homologous to zero.

Integrating the representation formula (9.2.4) over the cycle γ we obtain

$$\begin{aligned} \int_\gamma f(w)dw &= \int_\gamma \frac{1}{2\pi i} \left(\int_{\partial\Omega_\delta} \frac{f(z)}{z-w}\, dz \right) dw = \int_{\partial\Omega_\delta} f(z) \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{z-w}\, dw \right) dz \\ &= \int_{\partial\Omega_\delta} f(z) \left(-I(\gamma, z)\right) dz \\ &= 0 \end{aligned} \tag{9.2.6}$$

by (9.2.5). Here the interchange of integrals is justified by Fubini's theorem[7] because all the integrands involved are bounded and continuous. □

9.3 The general version of Cauchy's integral formula

VIDEO: The general version of Cauchy's integral formula

Around 1:55 the video forgets to stress that w in Corollary 9.4 should not lie on the image of γ

Corollary 9.4 (Cauchy's integral formula – general version). *Let $\Omega \subset \mathbb{C}$ be open and let γ be a cycle (e.g. a closed piecewise C^1 curve) in Ω that is homologous to zero in Ω. Then for any holomorphic function $f\colon \Omega \to \mathbb{C}$ and for any $w \in \Omega$ not lying in the image of γ, we have*

$$f(w)\, I(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz. \tag{9.3.1}$$

[7] If you have not taken Measure Theory then you can take this fact on trust.

The lectures/video may have some further context and justification

The proof is quite similar to the proof of Cauchy's integral formula for annuli given in Theorem 8.9.

Proof. By Corollary 8.11, the function

$$g(z) = \frac{f(z) - f(w)}{z - w},$$

which is holomorphic initially on $\Omega \setminus \{w\}$, can be extended to a holomorphic function throughout Ω. The homology form of Cauchy's theorem, Theorem 9.3, then implies that

$$\int_\gamma g(z)dz = 0 \tag{9.3.2}$$

and hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}dz = f(w)\, I(\gamma, w) \tag{9.3.3}$$

as claimed. □

9.4 The deformation theorem; Cauchy's theorem on simply connected domains

VIDEO: The deformation theorem; Cauchy's theorem on simply connected domains

In general open sets $\Omega \subset \mathbb{C}$, the homology version of Cauchy's theorem will only apply to special closed curves. However, when Ω is simply connected, it will apply to **all** closed curves, because by Corollary 4.11, in this case every closed curve has zero winding number about every point $w \in \mathbb{C} \setminus \Omega$, and thus every closed curve is homologous to zero. This then allows us to finally deduce Cauchy's theorem on simply connected domains, as given in Theorem 5.1.

One way that such a closed curve γ arises is if we have two curves $\gamma_1, \gamma_2 : [a, b] \to \Omega$, with the same start point and the same end point, in our simply connected domain Ω. We can then consider the cycle $\gamma_1 - \gamma_2$, which in the minimally abstract presentation we are giving would be represented by a closed curve that first follows γ_1 from $\gamma_1(a)$ to $\gamma_1(b) = \gamma_2(b)$, and then follows γ_2 in the reverse direction from $\gamma_2(b)$ back to $\gamma_2(a) = \gamma_1(a)$. Cauchy's theorem in this setting then immediately implies that integrating a holomorphic function along γ_1 gives the same value as integrating it along γ_2:

Theorem 9.5 (Deformation theorem on simply connected domains)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected open set, and let $f : \Omega \to \mathbb{C}$ be holomorphic. If $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are piecewise $\mathcal{C}^1$ curves that start at the same point $\gamma_1(a) = \gamma_2(a)$ and end at the same point $\gamma_1(b) = \gamma_2(b)$, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz. \tag{9.4.1}$$

Although this deformation result is the simplest result of this form, and is all we will need later on, the full Deformation theorem is nothing to do with simply connected domains. That Ω is simply connected is telling us that the closed curve we are writing $\gamma_1 - \gamma_2$ is homotopic to a constant curve, at which point we can apply the following lemma whose proof is a simple but messy exercise that belongs more in a topology course.

Lemma 9.6. *If $\Omega \subset \mathbb{C}$ is any open set and $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are paths that start at the same point $\gamma_1(a) = \gamma_2(a)$ and end at the same point $\gamma_1(b) = \gamma_2(b)$, then $\gamma_1 - \gamma_2$, viewed as a closed curve, is homotopic to a constant curve if and only if γ_1 is homotopic to γ_2.*

Thus, under the hypotheses of Theorem 9.5 the curves γ_1 and γ_2 are homotopic, and *that* is the essential point. Indeed, when $\Omega \subset \mathbb{C}$ is a *general* open set, and γ_1 and γ_2 are homotopic, then Lemma 9.6 tells us that $\gamma_1 - \gamma_2$, viewed as a closed curve, will be homotopic to a constant path within Ω. The second part of Theorem 4.10 then tells us that $I(\gamma, w) = 0$ for every $w \notin \Omega$, i.e. γ is homologous to zero, and we deduce the following from the homology version of Cauchy's theorem, i.e. Theorem 9.3.

Theorem 9.7 (Deformation theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $f\colon \Omega \to \mathbb{C}$ holomorphic. If $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are piecewise $\mathcal{C}^1$ curves that are homotopic in the sense of Definition 2.30, then*

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz. \tag{9.4.2}$$

Remark 9.8. Note that intuitively as we homotop the curve γ_1 to the curve γ_2, the integral along intermediate curves should remain constant. However, a homotopy does not retain the piecewise $\mathcal{C}^1$ nature of the curves, so integration is not immediately possible. The way we have developed the theory avoids all these technicalities by pushing everything we need about homotopies into statements about winding numbers rather than integrals. Other sources handle this issue in a number of different ways. It is possible to modify a homotopy so that it is more regular, so integration is possible, but this is quite painful technically. Another way is to generalise the definition of integration to handle merely continuous curves, which turns out to be possible as long as we're integrating *holomorphic* functions f.

9.5 The Residue theorem

VIDEO: The Residue theorem

The Residue theorem can be viewed as a generalisation of the homology version of Cauchy's theorem. It will give us a powerful method for computing integrals in terms of so-called residues.

Definition 9.9. Suppose that $f\colon B_\delta(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, for some $\delta > 0$, $z_0 \in \mathbb{C}$. The *residue* of f at z_0 is defined to be

$$\operatorname{res}(f, z_0) := \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_0)} f(z)\, dz, \tag{9.5.1}$$

for any $\varepsilon \in (0, \delta)$.

By Corollary 5.8, the integral in (9.5.1) is independent of the choice of $\varepsilon \in (0, \delta)$.

Theorem 9.10 (Residue theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open. Assume that f is holomorphic on $\Omega \setminus \mathcal{S}$, where $\mathcal{S} \subset \Omega$ is a discrete set that is closed in Ω. Let γ be a closed piecewise C^1 curve in $\Omega \setminus \mathcal{S}$ (or more generally a cycle in $\Omega \setminus \mathcal{S}$) that is homologous to zero in Ω.*

Then there are finitely many points $a \in \mathcal{S}$ such that $I(\gamma, a) \neq 0$, and we have

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{a \in \mathcal{S}} I(\gamma, a) \operatorname{res}(f, a). \tag{9.5.2}$$

Recall that $\mathcal{S}$ being a discrete set means that for all $a \in \mathcal{S}$, there exists $\varepsilon > 0$ such that a is the only point in both $\mathcal{S}$ and $B_\varepsilon(a)$. That is, each point in $\mathcal{S}$ is isolated. By asking for $\mathcal{S}$ to be closed in Ω we rule out the possibility of an accumulation point within Ω.

In the case that $\mathcal{S} = \emptyset$, the Residue theorem 9.10 recovers the homology version of Cauchy's theorem, i.e. Theorem 9.3.

Proof. We begin by showing that γ winds around at most finitely many points in $\mathcal{S}$, that is,

$$\mathcal{A} := \{a \in \mathcal{S} \colon I(\gamma, a) \neq 0\}$$

is finite. Suppose instead that is not the case. Then we can pick a sequence a_n within $\mathcal{A}$ with pairwise distinct elements. By Q. 4.3, the set $\mathcal{A}$ is bounded and so we can pass to a subsequence so that $a_n \to a_\infty \in \overline{\Omega}$. We cannot have $a_\infty \in \Omega$ since then we would have $a_\infty \in \mathcal{S}$ by closedness of $\mathcal{S}$ in Ω, and then $\mathcal{S}$ would fail to be discrete since a_∞ would be an accumulation point. Therefore we have $a_\infty \in \partial\Omega$, and in particular $a_\infty \in \mathbb{C} \setminus \Omega$ and so $I(\gamma, a_\infty) = 0$ by definition of γ being homologous to zero. By Lemma 4.7, $I(\gamma, a) = 0$ for all a in some neighbourhood of a_∞, and hence $I(\gamma, a_n) = 0$ for sufficiently large n, giving a contradiction.

At this point we can write $\mathcal{A} = \{a_1, \ldots, a_N\}$, and choose $\varepsilon > 0$ small so that $B_{2\varepsilon}(a_k) \setminus \{a_k\} \subset \Omega \setminus \mathcal{S}$ for every $k \in \{1, \ldots, N\}$. Writing $\gamma_k : [0, 1] \to \Omega \setminus \mathcal{S}$ for the curve

$$\gamma_k(t) = a_k + \varepsilon e^{i2\pi t}, \tag{9.5.3}$$

we notice that $I(\gamma_k, a_k) = 1$, while $I(\gamma_k, a) = 0$ for all $a \in \mathcal{S} \setminus \{a_k\}$.

Define $n_k := I(\gamma, a_k)$ and consider the cycle

$$\Gamma = \gamma - n_1\gamma_1 - n_2\gamma_2 \ldots - n_N\gamma_N. \tag{9.5.4}$$

By construction, the cycle Γ does not wind around any point in $\mathcal{S}$ in the sense that $I(\Gamma, a) = 0$ for all $a \in \mathcal{S}$. Moreover, $I(\Gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$. Hence by the general Cauchy theorem, Theorem 9.3, applied on $\Omega \setminus \mathcal{S}$, we have that $\int_\Gamma f(z)\, dz = 0$, i.e.

$$\int_\gamma f(z)\, dz = \sum_{k=1}^N n_k \int_{\gamma_k} f(z)\, dz = \sum_{k=1}^N I(\gamma, a_k) 2\pi i \operatorname{res}(f, a_k). \tag{9.5.5}$$

□

9.6 Evaluation of residues

VIDEO: Evaluation of residues

Now that the Residue theorem 9.10 gives us a way of evaluating integrals in terms of residues, we had better be able to compute these residues!

Given an explicit function f with an isolated singularity at z_0, it may be hard to compute the residue directly from the definition, i.e. from the integral in (9.5.1). Luckily we have a collection of tricks up our sleeves to make this easier in practice.

Removable singularities:

Let's get one trivial case out of the way: If f has a removable singularity at z_0 then we can remove it and apply Cauchy's theorem 5.7 to deduce that $\operatorname{res}(f, z_0) = 0$.

Example 9.11. The residue of $\frac{\sin z}{z}$ at $z_0 = 0$ is zero.

Simple poles:

The first nontrivial case to consider is when f has a simple pole (i.e. a pole of order 1) at z_0. We know from Theorem 8.4 that we can then write $f(z) = \frac{g(z)}{z-z_0}$ for some holomorphic $g\colon B_\delta(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$, and substituting into (9.5.1) gives

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_0)} \frac{g(z)}{z - z_0} dz = g(z_0) \tag{9.6.1}$$

by Cauchy's integral formula (5.6.1).

Example 9.12. To compute the residue of $f(z) = \frac{1}{z^2-1}$ at $z_0 = 1$, we rewrite $f(z) = \frac{1}{(z-1)(z+1)}$. We are then in the situation above with $g(z) = \frac{1}{1+z}$, and so $\operatorname{res}(f, 1) = g(1) = \frac{1}{2}$.

Ratios, with at worst a simple pole:

If $f(z) = \frac{h(z)}{k(z)}$, where $h, k\colon B_\delta(z_0) \to \mathbb{C}$ are holomorphic with $k(z_0) = 0$ but $k'(z_0) \neq 0$, i.e. k has a zero of order one at z_0, then we can write

$$f(z) = \frac{g(z)}{z - z_0} \qquad \text{for } g(z) = \frac{h(z)}{\left(\frac{k(z)}{(z-z_0)}\right)}.$$

Then g has a removable singularity at z_0, which we can remove by setting

$$g(z_0) = \lim_{z \to z_0} g(z) = \frac{h(z_0)}{k'(z_0)}.$$

Thus we conclude that

$$\operatorname{res}(f, z_0) = \frac{h(z_0)}{k'(z_0)}.$$

Example 9.13. The residue of $\frac{1}{\sin z}$ at $z_0 = 0$ is $\frac{1}{\cos 0} = 1$.

Pole of general order:

If f has a pole of general order n, then Theorem 8.4 now tells us that $f(z) = (z - z_0)^{-n} g(z)$ for some holomorphic $g\colon B_\delta(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$. Substituting into (9.5.1) gives

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_0)} \frac{g(z)}{(z - z_0)^n} dz = \frac{g^{(n-1)}(z_0)}{(n-1)!} \tag{9.6.2}$$

according to the formula from Corollary 6.5. We can rewrite this in terms of f, giving

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \Big((z - z_0)^n f(z) \Big), \tag{9.6.3}$$

where we take care not to evaluate f at z_0 where it is not defined!

Example 9.14. The residue of $\frac{\cos z}{z^3}$ at $z_0 = 0$ is $\frac{\cos''(0)}{2!} = -\frac{1}{2}$.

General case

For essential singularities, or when we can find the Laurent series of f explicitly, then we can appeal to the formula (8.6.2) for the Laurent coefficients a_k of f to find that

$$\operatorname{res}(f, z_0) = a_{-1}.$$

Example 9.15. The function $f(z) = z^n \exp\left(\frac{1}{z}\right)$ has an essential singularity at 0, but we can expand it as

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k}$$

and the coefficient of z^{-1} is the term in which $k = n + 1$, so $\operatorname{res}(f, 0) = \frac{1}{(n+1)!}$.

9.7 Computation of real integrals and series using the Residue theorem

One of the many delights of Complex Analysis is the ability it gives you to quickly and precisely compute integrals and series that you might otherwise have great difficulty with. We already saw an example in Q. 5.4, and you will see further examples in Q. 9.3 and Q. 9.4.

Here we will cover just one of the various available techniques, geared to computing the integral over $[0, 2\pi]$ of rational functions of $\sin\theta$ and $\cos\theta$. Rather than trying to quote a general theorem, we illustrate the idea with an example.

We would like to prove that

$$I := \int_0^{2\pi} \frac{4\sin^2\theta}{5 + 4\cos\theta} d\theta = \pi. \tag{9.7.1}$$

We can reverse engineer this into a complex integral over the C^1 curve we've been calling ∂D, i.e. $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = e^{i\theta}$, so $\gamma'(\theta) = ie^{i\theta} = i\gamma(\theta)$. Note that

$$\int_\gamma \frac{f(z)}{iz} dz = \int_0^{2\pi} \frac{f(\gamma(\theta))}{i\gamma(\theta)} \gamma'(\theta) d\theta = \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

We would like to pick f so that this integral equals the integral in (9.7.1). But by definition we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}, \qquad \text{and} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2}$$

so we obtain f by replacing every instance of $\sin\theta$ in (9.7.1) by $\frac{1}{2i}(z - \frac{1}{z})$ and every instance of $\cos\theta$ by $\frac{1}{2}(z + \frac{1}{z})$. In this case

$$f(z) = \frac{4\left(\frac{z - \frac{1}{z}}{2i}\right)^2}{5 + 4\left(\frac{z + \frac{1}{z}}{2}\right)} = \frac{-z\left(z - \frac{1}{z}\right)^2}{(2z+1)(z+2)}$$

(we multiplied top and bottom by z) and thus

$$I = \int_\gamma F(z) dz, \qquad \text{where } F(z) = \frac{i\left(z - \frac{1}{z}\right)^2}{(2z+1)(z+2)}.$$

We evaluate this integral using the Residue theorem. Within D, we have a double pole at 0, and a simple pole at $z = -\frac{1}{2}$.

To compute the residue at $z = -\frac{1}{2}$, it may help to write

$$F(z) = \left[\frac{i\left(z - \frac{1}{z}\right)^2}{2(z+2)}\right] \frac{1}{(z - (-\frac{1}{2}))}.$$

Then

$$\operatorname{res}(F(z), -\frac{1}{2}) = \left[\frac{i\left(-\frac{1}{2} + 2\right)^2}{2(-\frac{1}{2} + 2)}\right] = \frac{3i}{4}.$$

To compute the residue at $z = 0$, first expand brackets in the numerator: $\left(z - \frac{1}{z}\right)^2 = z^2 - 2 + z^{-2}$. This splits the function F into three terms, but the first two are holomorphic near $z = 0$ so do not contribute to the residue. Therefore

$$\operatorname{res}(F(z), 0) = \operatorname{res}\left(\frac{i}{z^2(2z+1)(z+2)}, 0\right).$$

It is more annoying to compute residues at double poles. We could expand using partial fractions and take the a_{-1} Laurent coefficient. Alternatively we could write

$$\frac{i}{z^2(2z+1)(z+2)} = \frac{1}{z^2} g(z), \qquad \text{for } g(z) = \frac{i}{(2z+1)(z+2)}$$

and use (9.6.2) to find that the residue would be

$$\frac{1}{(2-1)!}g'(0) = g'(0) = i(-2^{-2} + -2.\frac{1}{2}) = -\frac{5i}{4}.$$

By the Residue theorem 9.10 we then have

$$I = 2\pi i\left(\frac{3i}{4} - \frac{5i}{4}\right) = \pi.$$

9.8 The argument principle

VIDEO: The argument principle

The video uses the concept of *simple* closed path, as defined in Definition 9.16 without defining it (because in an earlier version of the lectures notes we covered this in an earlier section).

In this section we will develop a way of counting the number of zeros, minus the number of poles, of a meromorphic function $f\colon \Omega \to \mathbb{C}_\infty$ within an appropriate subset $A \subset \Omega$. The subset A will be described as some sort of interior of a closed path, and for that to make sense we should ensure that the path does not cross itself in the following sense.

Definition 9.16. We say that a closed continuous path $\gamma : [a, b] \to \mathbb{C}$ is **simple** if its restriction to $[a, b)$ is injective.

Although we will neither prove it nor use it, the Jordan curve theorem tells us that for *any* simple closed continuous path $\gamma : [a, b] \to \mathbb{C}$, for $a < b$, the set $\mathbb{C} \setminus \gamma([a, b])$ *necessarily* has exactly two connected components, one bounded and one unbounded. Moreover, if we call the bounded one A then either $I(\gamma, z) = 1$ for every $z \in A$ or $I(\gamma, z) = -1$ for every $z \in A$. We know that the sign of $I(\gamma, z)$ can be flipped by reversing the parametrisation of γ. This motivates the following definition, which we will tend to consider for explicit paths γ, e.g. bounding a disc, thus avoiding the need for the Jordan curve theorem.

Definition 9.17. A simple closed continuous path $\gamma : [a, b] \to \mathbb{C}$ is said to bound an open set $A \subset \mathbb{C}$ in a positive direction if $\mathbb{C} \setminus \gamma([a, b])$ has two connected components, one of which is A, and $I(\gamma, z) = 1$ for every $z \in A$.

The connected component other than A will necessarily be unbounded, and we know from Q. 4.3 that $I(\gamma, z) = 0$ for every z in that component.

Suppose now $f\colon \Omega \to \mathbb{C}_\infty$ is a meromorphic function on a domain $\Omega \subset \mathbb{C}$ that is not identically zero. Let $\mathcal{P} \subset \Omega$ be the set of poles of f, and let $\mathcal{Z} \subset \Omega$ be the set of zeros of f. We know that both $\mathcal{P}$ and $\mathcal{Z}$ are discrete and closed in Ω. (See Section 8.2.)

Given any $A \subset \Omega$ that is bounded in a positive direction by a simple closed curve γ in the sense of Definition 9.17, define $\mathcal{Z}_A(f)$ and $\mathcal{P}_A(f)$ to be the number of zeros and poles of f in A, respectively, counting multiplicity. That is,

$$\mathcal{Z}_A(f) = \sum_{z \in \mathcal{Z} \cap A} \operatorname{ord}(f, z), \qquad \mathcal{P}_A(f) = \sum_{z \in \mathcal{P} \cap A} [-\operatorname{ord}(f, z)],$$

where we recall that if f has a pole of order n at z then $\operatorname{ord}(f, z) = -n$ (see Section 8.7).

Our objective is to try to count the number of zeros minus the number of poles, i.e.

$$\mathcal{Z}_A(f) - \mathcal{P}_A(f) = \sum_{z \in (\mathcal{Z} \cup \mathcal{P}) \cap A} \operatorname{ord}(f, z),$$

in terms of f on the image of γ.

Theorem 9.18 (Argument principle)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f\colon \Omega \to \mathbb{C}_\infty$ be a meromorphic function that is not identically zero. Let $\gamma : [a, b] \to \Omega \setminus (\mathcal{P} \cup \mathcal{Z})$ be a piecewise $\mathcal{C}^1$ simple closed curve that bounds an open set $A \subset \Omega$ in a positive direction in the sense of Definition 9.17. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \mathcal{Z}_A(f) - \mathcal{P}_A(f). \tag{9.8.1}$$

Proof. Consider the behaviour of the logarithmic derivative $\frac{f'(z)}{f(z)}$, as first considered in Section 7.3, near a pole or a zero of f at z_0. In either case we can write

$$f(z) = (z - z_0)^n g(z), \tag{9.8.2}$$

for z in a neighbourhood of z_0, where g is a holomorphic function with $g(z_0) \neq 0$ and $0 \neq n = \operatorname{ord}(f, z_0)$ is positive in the case of a zero and negative in the case of a pole. For z sufficiently close to z_0 so that $g(z) \neq 0$, the logarithmic derivative then yields

$$\frac{f'(z)}{f(z)} = \frac{1}{f(z)} \left(n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z) \right) = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}. \tag{9.8.3}$$

As $g(z_0) \neq 0$ we can conclude that $\frac{f'}{f}$ has a simple pole with residue n at z_0, and so

$$\operatorname{res}\left(\frac{f'(z)}{f(z)}, z_0 \right) = \operatorname{ord}(f, z_0).$$

The Argument principle then follows immediately from the Residue theorem 9.10. □

Note that we apply the Residue theorem with $\mathcal{S} = \mathcal{Z} \cup \mathcal{P}$. We also observe that γ is necessarily homologous to zero: it only winds around points in A, and all of those are within Ω.

If we unwind the definition of the integral in the Argument principle, using the chain rule (1.2.4), we find that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt = \int_a^b \frac{(f \circ \gamma)'(t)}{f \circ \gamma(t)} dt = \int_{f \circ \gamma} \frac{1}{z}\, dz.$$

Dividing by $2\pi i$, the argument principle then implies that $\mathcal{Z}_A(f) - \mathcal{P}_A(f)$ is the winding number of $f \circ \gamma$ about 0.

Corollary 9.19. *With Ω, f, A and γ as in the Argument principle, Theorem 9.18, we have*

$$\mathcal{Z}_A(f) - \mathcal{P}_A(f) = I(f \circ \gamma, 0).$$

Example 9.20. Let's cross-check this corollary in the simple case that $f(z) = z^n$ and A is the unit disc D. The function f has a zero of order n at 0, but no poles, and so $\mathcal{Z}_D(f) - \mathcal{P}_D(f) = n$. The boundary of D is given by the curve $\gamma(t) = e^{2\pi i t}$ for $t \in [0, 1]$ and so $f \circ \gamma(t) = e^{2\pi i n t}$. This curve winds exactly n times around 0 as required.

Already in the case of holomorphic f, this interpretation of the number of zeros as the winding number of $f \circ \gamma$ leads to the so-called Rouché theorem.

9.9 Rouché's theorem

VIDEO: Rouché's theorem

We have seen that the number of zeros of a holomorphic function $G\colon \Omega \to \mathbb{C}$ over an open set $A \subset \Omega$ bounded by a piecewise $\mathcal{C}^1$ simple closed curve $\gamma : [a, b] \to \Omega \setminus \mathcal{Z}$ in a positive direction, is equal to the winding number of the curve $G \circ \gamma$ around 0. If we perturb G, then the curve $G \circ \gamma$ will be perturbed, but if we don't change G very much then the winding number should stay the same. Thus the number of zeros should stay the same. This is the content of Rouché's theorem:

Theorem 9.21 (Rouché's theorem)**.** *Let $g, G : \Omega \to \mathbb{C}$ be two holomorphic functions on some domain $\Omega \subset \mathbb{C}$, and let $\gamma : [a, b] \to \Omega$ be a piecewise $\mathcal{C}^1$ simple closed curve that bounds an open set $A \subset \Omega$ in a positive direction in the sense of Definition 9.17. Suppose that $|g(z)| < |G(z)|$ for every z in the image of γ. Then G and $G + g$ have the same number of zeros in A.*

Note that the assumption $|g(z)| < |G(z)|$ implies that neither G nor $G + g$ can have any zeros on the image of the boundary curve γ.

Proof. Because of our hypothesis that $|g(z)| < |G(z)|$ for every z in the image of γ, we can apply the dog walking lemma 4.6 with γ there equal to $G \circ \gamma$ here, and $\tilde{\gamma}$ there equal to $(G + g) \circ \gamma$ here. This implies that

$$I(G \circ \gamma, 0) = I\left((G + g) \circ \gamma, 0\right).$$

By the argument principle, in the form given by Corollary 9.19, the left-hand side is the number of zeros of G in A, while the right-hand side is the number of zeros of $G + g$ in A. □

Example 9.22. Let's illustrate one use of Rouché's theorem by showing that the equation $z^5 + 15z + 1 = 0$ has precisely four solutions in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

First we apply Rouché with $\Omega = \mathbb{C}$, $A = B_2(0)$, $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = 2e^{i\theta}$, $G(z) = z^5$ and $g(z) = 15z + 1$. Then on the image of γ, i.e. on $\partial B_2(0)$, we have $|G(z)| = 2^5 = 32$, but $|g(z)| =$

$|15z+1| \leq 15|z|+1 = 31$. Thus $|g(z)| < |G(z)|$, and so by Rouché's theorem $P(z) = G(z)+g(z)$ has the same number of roots in $B_2(0)$ as $G(z)$ does, i.e. five.

Next we can apply Rouché with $\Omega = \mathbb{C}$, $A = B_1(0)$, $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = e^{i\theta}$, $G(z) = 15z$ and $g(z) = z^5+1$. Then on the image of γ, i.e. on $\partial B_1(0)$, we have $|G(z)| = 15$, but $|g(z)| = |z^5+1| \leq |z|^5+1 = 2$. Thus $|g(z)| < |G(z)|$, and so by Rouché's theorem $P(z) = G(z)+g(z)$ has the same number of roots in $B_1(0)$ as $G(z)$ does, i.e. one.

That leaves four roots in the given annulus.

We can improve the inner radius 1 of this annulus a lot. The same argument works for any inner radius r so that $15r > r^5+1$. For example, we could take $r = 1.95$, so the four roots are quite close to the circle $\partial B_2(0)$. On the other hand, we could also take $r = 0.07$, so the fifth root is actually very close to the origin!

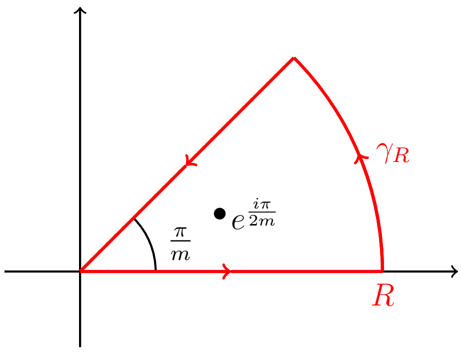


Figure 7: Another useful cake contour

9.10 Exercises

9.1. *A consequence of Theorem 4.10 is that if $\Omega \subset \mathbb{C}$ is open then any piecewise C^1 closed curve $\gamma : [a, b] \to \Omega$ that is homotopically trivial (within Ω) is necessarily homologous to zero. In this exercise you see that the converse is not true.*

Let $\Omega \subset \mathbb{C}$ be a domain constructed by taking the unit disc D and deleting two distinct points. Find a closed curve within Ω that is homologous to zero, but not homotopic to a constant curve.

9.2. Suppose $f : D \setminus \{0\} \to \mathbb{C}$ is a holomorphic function with zero residue at 0, i.e. $\mathrm{res}(f, 0) = 0$. Is the singularity at 0 necessarily removable?

9.3. Use the Residue theorem 9.10, integrating over the boundary of a large half disc $\{z \in B_R(0) : \Im(z) > 0\}$, to explicitly verify that

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi.$$

Of course, you already knew how to do this one at school because $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.

9.4. Let's generalise the previous question and verify that for any $m \in \mathbb{N}$ we have

$$\int_{-\infty}^{\infty} \frac{ds}{1 + s^{2m}} = \frac{\pi}{m \sin(\frac{\pi}{2m})}.$$

To do this, integrate around a piece of cake as in Figure 7. That way you should just have one residue to compute, and it corresponds to a simple pole at the point I've marked on the figure. If you prefer, you could specialise to the case that $m = 2$ or $m = 3$.

You could integrate around the boundary of a large half disc as in Q. 9.3 to get the same answer. However this creates extra work because you end up with m residues to compute rather than one.

9.5. *Here is another type of integral you can do using complex integration.*

Verify that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi$$

by turning it into an integral of a holomorphic function over the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = e^{i\theta}$. It may help to recall that writing $z = e^{i\theta}$ we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}, \qquad \text{and} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2},$$

and that $\gamma'(\theta) = ie^{i\theta} = iz$. We can then write

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \int_\gamma \left[\frac{1}{3 - (z + \frac{1}{z}) + \frac{1}{2i}(z - \frac{1}{z})}\right] \frac{dz}{iz}$$

and we are heading towards a situation in which we can apply the residue theorem.

9.6. In this question we use Rouché's theorem to find the rough location of the six roots of the polynomial $P(z) = z^6 + 10z + 1$. In each part, we take $G(z)$ to be one of the terms in the polynomial, and $g(z)$ to be the remaining two terms.

(a) Suppose $R > 0$ is sufficiently large so that $R^6 > 10R + 1$. (For example, $R = 1.605$.) Show that $P(z)$ has all six roots in $B_R(0)$.

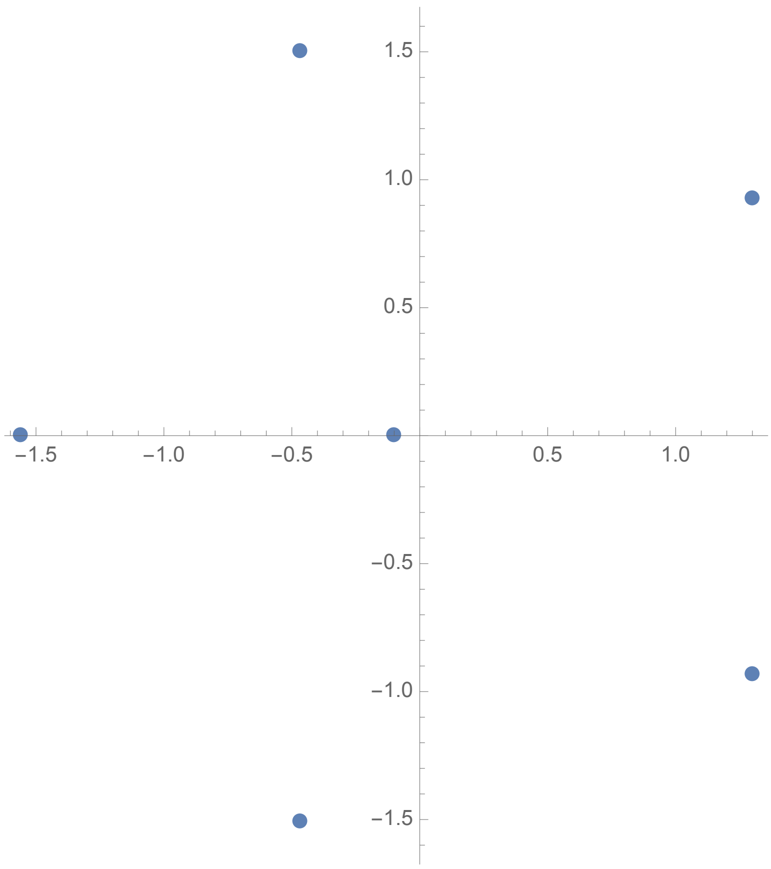
(b) Suppose $r > 0$ is sufficiently small so that $r^6 + 10r < 1$. (For example, $r = 0.0999$.) Show that $P(z)$ has no roots in $B_r(0)$.

(c) Suppose $s > 0$ satisfies $10s > s^6 + 1$. (For example, $s = 0.10001$, or $s = 1.564$.) Show that $P(z)$ has one root in $B_s(0)$.

By conjugating the equation $P(z) = 0$, we see that if z is a root then $\bar{z}$ is also a root. In particular, since there is only one root within $B_s(0)$, it must be real! Clearly it can't be positive (because $P(x) > 0$ for $x > 0$) so the root must be around -0.1. The other five roots are in an annulus with inner radius 1.564 and outer radius 1.605. Around this zone, the dominant terms of the polynomial $P(z)$ are z^6 and $10z$, so these five roots should be near the roots of $z^5 + 10 = 0$.

This Mathematica code will find and plot the roots:

```
f[z_] := z^6 + 10z + 1;
ComplexListPlot[z /. Solve[f[z]==0,z], PlotStyle->PointSize[Large]]
```


1.5
1.0
0.5
−1.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0
−1.5

10 Sequences of holomorphic functions

10.1 Weierstrass convergence theorem

VIDEO: Weierstrass convergence theorem

At 17:10 I missed a bar over the $B_\delta(z)$.

If a sequence $f_n : [a,b] \to \mathbb{R}$ of continuous functions converges uniformly to a function $f : [a,b] \to \mathbb{R}$, then we know that the limit function f is continuous.

However, if we additionally assume that the functions f_n are smooth (i.e. infinitely differentiable) then the limit need not be even once differentiable. In fact, *any* continuous function $f : [a,b] \to \mathbb{R}$ is the uniform limit of a sequence of smooth functions f_n. According to the Weierstrass approximation theorem, we can even ask for these approximating functions f_n to be polynomials!

As usual, the case of holomorphic functions is very different.

Theorem 10.1 (Weierstrass convergence theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $f_n \colon \Omega \to \mathbb{C}$ be a sequence of holomorphic functions on Ω. If f_n converges locally uniformly to a function $f \colon \Omega \to \mathbb{C}$, then*

(i) f is holomorphic, and

(ii) also the higher derivatives converge: For every $k \in \mathbb{N}$ we have

$$f_n^{(k)} \to f^{(k)}$$

locally uniformly as $n \to \infty$.

Recall that the sequence f_n is said to converge locally uniformly to f if for every compact set $K \subset \Omega$ the sequence of restricted functions $f_n|_K$ converges uniformly to $f|_K$. For example, the sequence $f_n \colon D \to \mathbb{C}$ defined by $f_n(z) = z^n$ converges *locally* uniformly to $f \equiv 0$, but the convergence is not uniform.

Implicit in Part (ii) of the theorem is that the limit function f is infinitely differentiable, as follows from f being known to be holomorphic, by Corollary 6.4. The convergence of Part (ii) would generally be referred to as *smooth local convergence of f_n to f*.

Proof. Because f is the local uniform limit of a sequence of continuous functions, we know that it is continuous. By Morera's theorem 6.9 in order to establish Part (i) it suffices to prove that for every closed triangle $T \subset \Omega$ we have

$$\int_{\partial T} f(z)dz = 0. \qquad (10.1.1)$$

This is true because by Goursat's theorem 5.3 and the uniform convergence $f_n \to f$ on ∂T, we know that

$$0 = \int_{\partial T} f_n(z)dz \to \int_{\partial T} f(z)dz.$$

To prove Part (ii), suppose $K \subset \Omega$ is the compact set on which we would like the uniform convergence $f_n^{(k)} \to f^{(k)}$. For this K, we choose $\delta > 0$ sufficiently small so that for every $z \in K$ we have $B_{2\delta}(z) \subset \Omega$. Here we are using the compactness of K. If we then define K_δ to be the set of points whose distance from K is no more than δ, i.e.

$$K_\delta := \cup_{z\in K} \overline{B_\delta(z)},$$

then we have another compact subset of Ω.

See the lectures/video for some pictures!

We will turn the uniform convergence $f_n \to f$ on K_δ into uniform convergence $f_n^{(k)} \to f^{(k)}$ on the smaller set K. By the formula for higher derivatives, akin to Cauchy's integral formula, given in Corollary 6.5, for every $z \in K$ and $k \in \mathbb{N}$ we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial B_\delta(z)} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} dw,$$

and so by (3.4.3) we have

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leq \frac{k!}{2\pi}(2\pi\delta) \sup_{w\in\partial B_\delta(z)} \frac{|f_n(w) - f(w)|}{\delta^{k+1}} \leq \frac{k!}{\delta^k} \sup_{w\in K_\delta} |f_n(w) - f(w)|.$$

The final term is independent of $z \in K$, and converges to zero by the uniform convergence $f_n \to f$ on K_δ. □

10.2 Hurwitz's Theorem

VIDEO: Hurwitz's Theorem

Adolf Hurwitz (1859 - 1919).

In the Weierstrass convergence theorem 10.1, the limit function f does not just inherit the property of being holomorphic from the approximating functions f_n. It also inherits upper bounds on the number of zeros.

Theorem 10.2 (Hurwitz's theorem). *Let $\Omega \subset \mathbb{C}$ be open and connected and suppose that $f_n\colon \Omega \to \mathbb{C}$ is a sequence of holomorphic functions that converges locally uniformly to a limit f, which is necessarily holomorphic by the Weierstrass convergence theorem 10.1. Suppose that, for some $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, each function f_n has no more than k zeros (counting multiplicity). Then either $f \equiv 0$ or f also has at most k zeros (counting multiplicity).*

In particular, if the f_n have no zeros, then either f does not have any zeros either, or $f \equiv 0$.

Remark 10.3. The number of zeros can decrease in the limit because they can wander out of the domain. For example, if $f_n : D \to \mathbb{C}$ is defined by $f_n(z) := (z - 1 + \frac{1}{n})$, then for any $n \in \mathbb{N}$ the function f_n has a zero at $1 - \frac{1}{n} \in D$. But f_n converges locally uniformly to $f(z) = z - 1$, which does not have a zero in D.

Proof. Suppose that the theorem is not true. Then we can find a situation satisfying the hypotheses of the theorem where the function f has strictly more than k zeros without being identically zero. All the zeros of f must be of finite order, since otherwise Theorem 7.4 would tell us that f would vanish throughout the *connected* open set Ω. Therefore each zero is isolated, and if we pick enough of them, say at distinct points $z_1, z_2, \ldots, z_K$, with orders $m_1, m_2, \ldots, m_K$ respectively, we can arrange that $\sum m_i > k$.

Because there are finitely many points z_i, and each is isolated in the set of zeros, we can pick a small radius $\delta > 0$ so that the K closed balls $\overline{B_\delta(z_i)}$ are pairwise disjoint sets lying within Ω, and so that there are no zeros of any type in any of the sets $\overline{B_\delta(z_i)} \setminus \{z_i\}$. Consider now the union of circles

$$\Sigma := \cup_{i=1}^{K} \partial B_\delta(z_i).$$

By compactness of Σ and continuity of $|f|$, we can define

$$\varepsilon := \min_{z \in \Sigma} |f(z)| > 0.$$

By the assumed uniform convergence of f_n to f on Σ, after deleting finitely many terms in the sequence f_n, we may assume that

$$|f_n(z) - f(z)| < \varepsilon$$

for all $z \in \Sigma$. Rouché's theorem 9.21, applied with $G(z) = f(z)$ and $g(z) = f_n(z) - f(z)$, then implies that each f_n also has exactly m_i zeros in the ball $B_\delta(z_i)$ for each i. The total number of zeros of each f_n is then strictly larger than k, giving a contradiction. □

Corollary 10.4. *Any function f arising as a local uniform limit of* injective *holomorphic functions f_n, defined on an open connected set $\Omega \subset \mathbb{C}$, is either constant or injective.*

Proof. Suppose that f is neither constant nor injective. Because it is not injective, there exist $z_1 \neq z_2$ with $f(z_1) = f(z_2) =: w$.

We would like to apply the Hurwitz theorem to the functions $f_n(z) - w$, which have $f(z) - w$ as a local uniform limit. For each n, the function $f_n(z) - w$ has at most one zero because f_n is injective. Because f is not constant, the function $f(z) - w$ is not identically zero. Hurwitz's theorem 10.2 then implies that $f(z) - w$ has at most one zero. However, we already found zeros at two distinct points z_1 and z_2, giving a contradiction. □

10.3 Compactness: Montel's theorem

I have dropped the video link because changes to the lecture notes take them too far from the original recording.

Paul Antoine Aristide Montel (1876 - 1975).

Recall that a sequence of functions $f_n \colon \Sigma \to \mathbb{C}$ from a set $\Sigma \subset \mathbb{C}$ is said to be uniformly bounded if there exists $M < \infty$ such that $|f_n(z)| \leq M$ for all $z \in \Sigma$ and for all $n \in \mathbb{N}$.

A weaker condition that is often easier to work with is the following.

Definition 10.5. Let $\Omega \subset \mathbb{C}$ be open. A sequence of functions $f_n \colon \Omega \to \mathbb{C}$ is said to be *locally* uniformly bounded if for all compact $K \subset \Omega$, the restricted functions $f_n|_K : K \to \mathbb{C}$ are uniformly bounded.

Suppose we have such a sequence f_n. Given that for each fixed $z \in \Omega$ the sequence $f_n(z)$ is a bounded sequence in $\mathbb{C}$, and we can pass to a subsequence to obtain a limit, one might think that one can take one subsequence and obtain local uniform convergence of f_n to some limit function. However, in general this fails. Even when considering functions $f_n \colon \mathbb{R} \to \mathbb{R}$, one could take the uniformly bounded sequence $f_n(x) = \sin nx$. These functions have no chance of converging uniformly, whichever subsequence we take!

According to the Ascoli-Arzelà theorem, this problem is fixed if we assume that the functions f_n are uniformly equicontinuous.

Definition 10.6. Let $K \subset \mathbb{C}$ be compact. A sequence of functions $f_n \colon K \to \mathbb{C}$ is said to be uniformly equicontinuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $z, w \in K$ with $|z - w| < \delta$, we have $|f_n(z) - f_n(w)| < \varepsilon$.

Thus each f_n is uniformly continuous, and the constant $\delta > 0$ does not depend on n.

Theorem 10.7 (Ascoli-Arzelà)**.** *If $K \subset \mathbb{C}$ is compact and $f_n \colon K \to \mathbb{C}$ is a uniformly bounded and uniformly equicontinuous sequence of functions, then a subsequence converges uniformly to a continuous function $f \colon K \to \mathbb{C}$.*

The key point of this section is that when we take a locally uniformly bounded sequence of holomorphic functions, we will be able to show that they are uniformly equicontinuous on any compact subset, and by appealing to the Ascoli-Arzelà theorem we will obtain:

Theorem 10.8 (Montel's theorem)**.** *Every locally uniformly bounded sequence of holomorphic functions $f_n \colon \Omega \to \mathbb{C}$ on an open set $\Omega \subset \mathbb{C}$ has a locally uniformly convergent subsequence.*

Proof. The main content of Montel's theorem will be contained in the following apparently weaker claim.

Claim: *For every ball $\overline{B_{2r}(a)} \subset \Omega$, we can pass to a subsequence so that f_n is uniformly convergent on the smaller ball $\overline{B_r(a)}$.*

Proof of claim. By rescaling and dilating we may assume that $\overline{B_2(0)} \subset \Omega$ and ask that f_n is uniformly convergent on $\overline{D}$.

By the local uniform boundedness, we can pick $M < \infty$ so that $|f_n| \leq M$ for all n, throughout $\overline{B_2(0)}$ and in particular on $\partial B_2(0)$.

Suppose $z_1, z_2 \in \overline{D}$. Cauchy's integral formula (5.6.1) tells us that for any n we have

$$\begin{aligned} f_n(z_1) - f_n(z_2) &= \frac{1}{2\pi i}\int_{\partial B_2(0)} \frac{f_n(w)}{w - z_1}dw - \frac{1}{2\pi i}\int_{\partial B_2(0)} \frac{f_n(w)}{w - z_2}dw \\ &= \frac{z_1 - z_2}{2\pi i}\int_{\partial B_2(0)} \frac{f_n(w)}{(w - z_1)(w - z_2)}dw. \end{aligned}$$

For $w \in \partial B_2(0)$, where we are integrating, we have $|w - z_1| \geq 1$ and $|w - z_2| \geq 1$, and so

$$\left|\frac{1}{(w - z_1)(w - z_2)}\right| \leq 1.$$

Therefore by (3.4.3) we have

$$|f_n(z_1) - f_n(z_2)| \leq \frac{|z_1 - z_2|}{2\pi}(2\pi 2)M = 2M|z_1 - z_2|.$$

We have shown that when restricted to $\overline{D}$, the functions f_n are uniformly equicontinuous. (We have even shown that the restricted functions have a uniform bound on their Lipschitz constants, which is much stronger still.)

Indeed, given any $\varepsilon > 0$ we can take $\delta = \frac{\varepsilon}{2M}$.

We can then apply the Ascoli-Arzelà theorem 10.7 to obtain the desired uniform convergence on $\overline{D}$. This completes the proof of the claim. □

To see how Montel's theorem follows from the claim, we pick a countable sequence of centres $a_i \in \Omega$ and radii $r_i > 0$ such that $\overline{B_{2r_i}(a_i)} \subset \Omega$ for every $i \in \mathbb{N}$ and so that the sets

$$K_m = \bigcup_{i=1}^{m} \overline{B_{r_i}(a_i)}$$

exhaust the whole of Ω, that is, for every compact $K \subset \Omega$ we have $K \subset K_m$ for large enough m.[8]

For each i in turn we use the claim to pass to a subsequence so that f_n is uniformly convergent on $\overline{B_{r_i}(a_i)}$. It remains to pass to a diagonal subsequence. □

[8] There are several ways we could find these centres a_i and radii r_i. For example, we could start by taking any sequence $\tilde{K}_k$ of compact sets in Ω that exhaust Ω (e.g. we could define $\tilde{K}_k$ to be all the points $z \in \overline{B_k(0)}$ such that $B_{\frac{1}{k}}(z) \subset \Omega$). By compactness, we can cover each $\tilde{K}_k$ by a finite number of balls $B_{a_i}(r_i)$, for $a_i \in \tilde{K}_k$ and $r_i > 0$, with the property that $\overline{B_{2r_i}(a_i)} \subset \Omega$. We can then list all these balls for $k = 1, 2, 3, \ldots$ in turn.

11 The Riemann mapping theorem

11.1 Riemann mapping theorem - statement and final ingredients

VIDEO: Riemann mapping theorem - statement + final ingredient

The video does not cover the final part of this subsection, which was added later.

Georg Friedrich Bernhard Riemann (1826 - 1866).

Back in Section 2.9, we introduced the notion of biholomorphic maps/functions and the notion of conformal equivalence of domains. With the developments of Theorem 7.14, we improved this a little to say that two domains[9] $\Omega_1, \Omega_2 \subset \mathbb{C}$ are conformally equivalent if there exists a bijective holomorphic function $\varphi : \Omega_1 \to \Omega_2$.

Also in Section 2.9 we discussed a handful of examples of domains that were conformally equivalent to the unit disc D. The only simply connected domain that we saw that was *not* conformally equivalent to D was the whole plane $\mathbb{C}$. Our objective is to prove that *every other simply connected domain is conformally equivalent to D*. This is an amazing and very powerful result. We are not just considering some very special domains as in Section 2.9. Our domains could be arbitrarily complicated. Imagine the interior of a Koch snowflake (if you know what that is).

Theorem 11.1 (Riemann mapping theorem). *Let $\Omega \subset \mathbb{C}$ be any simply connected domain other than the whole of $\mathbb{C}$. Then Ω is conformally equivalent to the unit disc D.*

Remark 11.2. The true power of this theorem is in the existence of a *biholomorphic* function from Ω to D. However, the vastly weaker consequence that we can find a *homeomorphism* from Ω to D is already significant. When coupled with the observation that $\mathbb{C}$ (which is prohibited in the Riemann mapping theorem) is homeomorphic to D, one deduces the topological implication that every simply connected $\Omega \subset \mathbb{C}$ is homeomorphic to the unit disc.

Remark 11.3. The map φ giving the conformal equivalence in the Riemann mapping theorem will never be unique because we can always compose it with a Möbius transformation that maps the disc D bijectively to itself, as in Remark 2.22. However, this then describes all the possible maps φ. Indeed, if we have two such maps $\varphi_1, \varphi_2 : \Omega \to D$, then $\varphi_1 \circ \varphi_2^{-1}$ is a biholomorphic map from the disc D to itself, and Corollary 7.16 classifies these.

The following lemma, generalising Corollary 7.10, allows us to take the square root of certain functions.

Lemma 11.4. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and let $g\colon \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic. Then there exists a holomorphic function $\ell\colon \Omega \to \mathbb{C}$ such that*

$$g(z) = e^{\ell(z)} \qquad \text{for all } z \in \Omega. \tag{11.1.1}$$

[9] recall that a domain is a nonempty connected open subset of $\mathbb{C}$

In particular, for $k \in \mathbb{N}$, the function $\psi(z) := e^{\frac{1}{k}\ell(z)}$ is a holomorphic function on Ω whose kth power is $g(z)$. Furthermore, if g is injective then ψ is injective.

Remark 11.5. If $\Omega \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain then we can apply the lemma to the holomorphic function $g(z) = z$ to give a well-defined logarithm function on Ω that is automatically holomorphic. Note that this would fail for $\Omega = \mathbb{C} \setminus \{0\}$ because it is not simply connected. When working on a simply connected domain such as the slit plane on which we have already defined a logarithm, the new logarithm must agree with the old, modulo the usual additive constant $2\pi i n$ for $n \in \mathbb{Z}$. This is because by (3.3.2) and (11.1.1) with $g(z) = z$, we have

$$e^{\ell(z)} = z = e^{\log z}$$

for all $z \in \Omega$.

Proof. By Lemma 7.9, all we need to do to find ℓ is to find an anti-derivative $F(z)$ of the function $f(z) := \frac{g'(z)}{g(z)}$.

Fix an arbitrary $z_0 \in \Omega$. For any other point $z \in \Omega$, let γ be a piecewise $\mathcal{C}^1$ curve in Ω that connects z_0 to z. The existence of a *continuous* path connecting z_0 to z is ensured by the connectedness of Ω. That continuous path can be modified into a nearby piecewise $\mathcal{C}^1$ curve (even a piecewise linear curve) γ by dividing it up into tiny portions and replacing each by a line segment.

I'll draw some pictures in the video!

Define

$$F(z) = \int_{g\circ\gamma} \frac{dw}{w} = \int_\gamma \frac{g'(w)}{g(w)} dw. \tag{11.1.2}$$

By the Deformation theorem 9.5, because Ω is simply connected, the integral defining F will be independent of the choice of γ, and so F is well-defined.

Moreover, F will be holomorphic with $F'(z) = \frac{g'(z)}{g(z)}$. To see this, for given $z \in \Omega$, fix a curve from z_0 to z and then append a line segment $[z, z+h]$ to reach an arbitrary point $z+h$ in a neighbourhood $B_\delta(z) \subset \Omega$ of z. Then

$$F(z+h) = F(z) + \int_{[z,z+h]} \frac{g'(w)}{g(w)} dw,$$

and differentiating with respect to h and evaluating at $h = 0$, using Corollary 5.6, we deduce that $F'(z) = \frac{g'(z)}{g(z)}$ as required.

Finally, suppose g is injective. Then whenever we have $\psi(z_1) = \psi(z_2)$ for $z_1, z_2 \in \Omega$, taking the kth power of both sides we obtain $g(z_1) = g(z_2)$ and thus $z_1 = z_2$ by the injectivity of g. Thus ψ is injective. □

Recall that the Schwarz lemma tells us that a holomorphic function $H : D \to D$ with $H(0) = 0$ satisfies $|H'(0)| \leq 1$, i.e. H cannot stretch too much at 0. The following lemma says that if we shrink the domain a bit in the right way then there is space for some stretching.

Lemma 11.6 (Stretching lemma)**.** *Suppose that $U \subset D$ is a simply connected domain with $0 \in U$ and $U \neq D$. Then there exists an injective holomorphic function $H : U \to D$ with $H(0) = 0$ such that $|H'(0)| > 1$.*

Note that although it is useful intuition that H stretches the domain near 0 more than would be possible if U were the whole of D, we are not claiming that $H(U)$ contains the original domain U so the idea that H stretches the domain U has to be treated with caution.

Proof. Recall from Example 2.21 that for each $w \in D$, the Möbius transformation

$$\varphi_w(z) = \frac{z - w}{\bar{w}z - 1}$$

gives a biholomorphic function from D to itself, interchanging 0 and w, and with the property that φ_w is its own inverse. Pick any $w_0 \in D \setminus U$, and choose either possible $w_1 \in D$ with $w_1^2 = w_0$. We can then define a holomorphic function $h : D \to D$, with $h(0) = 0$, as the composition of φ_{w_1}, then $z \mapsto z^2$, then φ_{w_0}, i.e.

$$h(z) = \varphi_{w_0}([\varphi_{w_1}(z)]^2).$$

Clearly h is not injective because $z \mapsto z^2$ is not injective. Indeed, every point in D other than w_0 is hit twice. Therefore the Schwarz lemma, i.e. Theorem 7.15, implies that $|h'(0)| < 1$ (otherwise h would have to be a rotation, which would be injective).

Because $0 \notin \varphi_{w_0}(U)$, we can invoke Lemma 11.4, with g there equal to φ_{w_0} here, to give an injective holomorphic 'square root' function $\psi : U \to \mathbb{C}$ such that $\psi(z)^2 = \varphi_{w_0}(z)$ for all $z \in U$. In fact, $\psi : U \to D$, i.e. it takes values in $D \subset \mathbb{C}$. Because

$$\psi(0)^2 = \varphi_{w_0}(0) = w_0 = w_1^2,$$

by flipping the sign of ψ throughout if necessary we may assume that $\psi(0) = w_1$.

We can now define our injective holomorphic function $H : U \to D$ with $H(0) = 0$ by

$$H = \varphi_{w_1} \circ \psi.$$

Observe that

$$\begin{aligned} h \circ H(z) &= \varphi_{w_0}([\varphi_{w_1} \circ \varphi_{w_1} \circ \psi(z)]^2) \\ &= \varphi_{w_0}([\psi(z)]^2) \\ &= \varphi_{w_0} \circ \varphi_{w_0}(z) \\ &= z. \end{aligned} \tag{11.1.3}$$

Differentiating using the chain rule gives

$$h'(H(z))H'(z) = 1,$$

and evaluating at $z = 0$ allows us to conclude that

$$|H'(0)| = \frac{1}{|h'(0)|} > 1.$$

□

11.2 Proof of the Riemann mapping theorem

We give the proof due to Koebe, which uses a large number of the results that we have proved in the course. We divide the proof into three separate claims. The first is a much weaker version of the full claim.

Claim 11.7. *The domain Ω is conformally equivalent to some open subset of D.*

Proof of Claim 11.7. To get a feel for what is being claimed, observe that the claim is trivial if Ω is bounded, i.e. fully contained in $B_R(0)$ for some R. In that situation the function $\varphi(z) = \frac{z}{R}$ gives the conformal equivalence. More generally, if Ω omits some small ball, i.e. we can find some $w_0 \in \mathbb{C}$ and $\delta > 0$ such that $B_\delta(w_0) \cap \Omega = \emptyset$, then we can simply set $\varphi(z) = \frac{\delta}{z - w_0}$. Consequently, it is sufficient to show that there exists an injective holomorphic function $\psi : \Omega \to \mathbb{C} \setminus B_\delta(w_0)$ for some $\delta > 0$, $w_0 \in \mathbb{C}$.

An example of a domain we still have to worry about would be the plane minus a slit, e.g. $\mathbb{C}\setminus(-\infty, 0]$. Such a domain does not omit any open ball. However, we can make some space for an open ball by mapping it to a half space using the square root function $re^{i\theta} \mapsto \sqrt{r}e^{\frac{i\theta}{2}}$ for $\theta \in (-\pi, \pi)$. The following argument essentially does this in general.

Because $\Omega \neq \mathbb{C}$, after possibly translating Ω we may assume that $0 \notin \Omega$. Applying Lemma 11.4 in the case that $g(z) = z$ gives an injective holomorphic branch of the square root $\psi\colon \Omega \to \mathbb{C}$ defined by $\psi(z) = e^{\frac{1}{2}\ell(z)}$ so $\psi(z)^2 = z$ for all $z \in \Omega$. We now show essentially that the image of Ω under the square root function ψ consists of less than half of $\mathbb{C}$. Indeed, if $w \in \psi(\Omega)$ then we cannot have $-w \in \psi(\Omega)$ because the only point that can map to either is $z := w^2 = (-w)^2$. If we pick any $w_1 \in \psi(\Omega)$ then the open mapping theorem 7.11 tells us that the image $\psi(\Omega)$ must contain some ball $B_\delta(w_1)$. Therefore the ball $B_\delta(-w_1)$ is disjoint from the image $\psi(\Omega)$. By setting $w_0 = -w_1$ we have completed the proof of Claim 11.7. □

By Claim 11.7, we may assume from now on that $\Omega \subset D$. We may as well assume also that $0 \in \Omega$ because if $0 \notin \Omega$ then we can always shrink Ω by a factor two and then translate it. Consider the set of functions

$$\mathcal{F} = \{f\colon \Omega \to D : f \text{ is holomorphic and injective, and } f(0) = 0\}. \tag{11.2.1}$$

The function $f(z) = z$ lies in $\mathcal{F}$, so $\mathcal{F}$ is nonempty. Our goal, the Riemann mapping theorem, will be realised if we can show that $\mathcal{F}$ contains at least one surjective function.

Claim 11.8. *If $f \in \mathcal{F}$ is* not *surjective, then there exists a different function $F \in \mathcal{F}$ with $|f'(0)| < |F'(0)|$.*

Proof of Claim 11.8. If $f \in \mathcal{F}$ is not surjective then we can apply the stretching lemma 11.6 with $U = f(\Omega)$ to give an injective holomorphic function $H : U \to D$ with $H(0) = 0$ and $|H'(0)| > 1$. We can then define an injective holomorphic function $F : \Omega \to D$ with $F(0) = 0$ by

$$F = H \circ f.$$

The chain rule gives

$$F'(z) = H'(f(z))f'(z)$$

so

$$|F'(0)| = |H'(0)|.|f'(0)| > |f'(0)|,$$

as required. □

The Riemann mapping theorem will then follow if we can find $f \in \mathcal{F}$ with maximal $|f'(0)|$. That is precisely what we claim now.

Claim 11.9. *There exists* $f \in \mathcal{F}$ *such that*

$$|f'(0)| = \mathcal{S} := \sup\{|g'(0)| \ : \ g \in \mathcal{F}\}. \tag{11.2.2}$$

Proof of Claim 11.9. The supremum $\mathcal{S}$ is strictly positive, indeed at least 1, because $g(z) = z$ lies in $\mathcal{F}$. Moreover we have that $\mathcal{S} < \infty$ because if we pick δ small enough so that $B_\delta(0) \subset \Omega$, then by the Schwarz lemma, or rather its consequence Q. 7.7 applied with $r = \delta$ and $s = 1$, we find that

$$|g'(0)| \leq \frac{1}{\delta}$$

for all $g \in \mathcal{F}$.

Let f_n be a sequence in $\mathcal{F}$ with $|f_n'(0)| \uparrow \mathcal{S}$. By Montel's theorem, Theorem 10.8, after passing to a subsequence we have local uniform convergence to a limit function $f \colon \Omega \to \mathbb{C}$.

We claim that $f \in \mathcal{F}$ with $|f'(0)| = \mathcal{S}$, as required.

By Part (i) of the Weierstrass convergence theorem 10.1, the limit f is a holomorphic function. By the local uniform convergence we have $f(0) = \lim_{n\to\infty} f_n(0) = 0$.

By Part (ii) of the Weierstrass convergence theorem 10.1 the derivatives f_n' converge locally uniformly to f', and in particular pointwise at $z = 0$, and so $|f'(0)| = \mathcal{S}$. Since $\mathcal{S}$ is positive, one consequence of this is that f cannot be constant and hence, by Hurwitz's theorem, Theorem 10.2, or rather Corollary 10.4, f is injective. Finally, as f is the locally uniform limit of the f_n we have $|f(z)| \leq 1$ for all $z \in \Omega$, i.e. $f(\Omega) \subset \overline{D}$. But by the Maximum modulus principle of Corollary 7.12, we must have $|f(z)| < 1$ for all $z \in \Omega$, i.e. $f(\Omega) \subset D$. Hence, $f \in \mathcal{F}$ with $|f'(0)| = \mathcal{S}$, as claimed. □

This finishes the proof of the Riemann mapping theorem.