TCC (SPRING 2021): p-ADIC MODULAR FORMS

PROBLEM SHEET 1

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This problem sheet is due at 11:59 PM on **Monday 22nd February 2021** (note the extended deadline). Please submit your work as a single PDF file (either typeset in LaTeX or a scan of legible handwriting) by email.

Problem 1. For this problem, you may use without proof the fact that the $\mathbf{Z}$-algebra of modular forms with Fourier coefficients in $\mathbf{Z}$ is generated by Q, R and Δ:

$$\bigoplus_{k\geq 4} M_{k,\mathbf{Z}} = \mathbf{Z}[Q, R, \Delta]$$

(so the same statement holds over $\mathbf{Z}_{(p)}$).

Let $p \in \{2, 3\}$.

(a) Show that the $\mathbf{F}_p$-algebra of mod p modular forms is generated by $\widetilde{\Delta}$, i.e.

$$\widetilde{M} = \mathbf{F}_p[\widetilde{\Delta}].$$

(b) In [Antwerp, P.197] it is claimed that when p is 2 or 3,

$$\textit{On a } \widetilde{M}_{k-2} \subset \widetilde{M}_k \textit{ et même } \widetilde{M}_{k-2} = \widetilde{M}_k \textit{ si } k \textit{ n'est pas divisible par } 12.$$

However this is not quite right, e.g. $\widetilde{M}_0 = \widetilde{M}_4 = \mathbf{F}_p$ but $\widetilde{M}_2 = 0$. Prove that:

(1) If $k \not\equiv 2 \pmod{12}$, then $\widetilde{M}_{k-2} \subset \widetilde{M}_k$.

(2) If $k \not\equiv 0, 4 \pmod{12}$, then $\widetilde{M}_{k-2} \supset \widetilde{M}_k$.

(c) (**NOT FOR SUBMISSION**) Complete the proof of théorème 1 [P.198] in the case when p is 2 or 3. The key point [P.200] is to show

$$\Delta \equiv \sum_{(n,p)=1} \sigma_{h-1}(n) q^n \pmod{p}$$

for any integer (resp. even integer) h when $p = 2$ (resp. $p = 3$).

Problem 2. Recall that the normalized Eisenstein series of weight 2

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad q = e^{2\pi i z}$$

satisfies the transformation law

$$P\left(-\frac{1}{z}\right) = z^2 P(z) + \frac{12z}{2\pi i}.$$

(a) Show that if f is a modular form of weight k, then $12\Theta f - kPf$ is a modular form of weight $k+2$.

(b) Show that $12\Theta P - P^2$ is a modular form of weight 4.

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(c) Deduce that

$$\begin{aligned}
\Theta P &= \frac{1}{12}(P^2 - Q), \\
\Theta Q &= \frac{1}{3}(PQ - R), \\
\Theta R &= \frac{1}{2}(PR - Q^2), \\
\Theta \Delta &= P\Delta.
\end{aligned}$$

Problem 3. Denote $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$. Using the fact that $\Theta : \widetilde{M}_k \to \widetilde{M}_{k+p+1}$, prove the following congruences:

(a) $\tau(n) \equiv n\sigma_5(n) \pmod 5$.

(b) $\tau(n) \equiv n\sigma_3(n) \pmod 7$.

Problem 4. Recall that the subspace $\mathbf{Z}_p[[q]] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \subset \mathbf{Q}_p[[q]]$ of formal power series with bounded coefficients is a p-adic Banach space under the norm

$$|f| = \sup_{n \geq 0} |a_n|, \quad f = \sum_{n=0}^{\infty} a_n q^n,$$

where $|\cdot| : \mathbf{Q}_p \to \mathbf{R}_{\geq 0}$ is the usual p-adic absolute value normalized such that $|p| = p^{-1}$.

Denote by $\mathbf{Q}_p\langle T \rangle$ the *Tate algebra*

$$\mathbf{Q}_p\langle T \rangle = \left\{ f = \sum_{n=0}^{\infty} a_n T^n \in \mathbf{Q}_p[[T]] : \lim_{n \to \infty} |a_n| = 0 \right\}$$

consisting of convergent power series on the closed unit disk.

Let $p = 5$. We have seen that $\frac{1}{j} = \frac{\Delta}{Q^3} \in M_0^\dagger$, i.e. is a 5-adic modular form of weight 0.

(a) Prove that $\mathbf{Q}_5 \left\langle \frac{1}{j} \right\rangle \subset M_0^\dagger$, and the norm of $f = \sum_{n=0}^{\infty} b_n \left(\frac{1}{j}\right)^n \in \mathbf{Q}_5 \left\langle \frac{1}{j} \right\rangle$ is given by

$$|f| = \sup_{n \geq 0} |b_n|.$$

(b) Prove that $\mathbf{Q}_5 \left\langle \frac{1}{j} \right\rangle = M_0^\dagger$.

In particular, $M_0^\dagger$ is infinite-dimensional.