TCC (SPRING 2021): p-ADIC MODULAR FORMS

PROBLEM SHEET 2

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This problem sheet is due at 11:59 PM on **Monday 8th March 2021**. Please submit your work as a single PDF file (either typeset in LaTeX or a scan of legible handwriting) by email.

Problem 0. (**NOT FOR SUBMISSION**) This problem collects a few analytic properties of p-adic weights and divisor sums that have been used without proof.

(a) Check that for each $d \in \mathbf{Z}_p^\times$, the evaluation map

$$\mathfrak{X} \to \mathbf{Q}_p^\times$$
$$k \mapsto d^k$$

is continuous with respect to the p-adic topology.

(b) Let $k \in \mathfrak{X}$ be a p-adic weight and $k_i \in \mathbf{Z}$ be a sequence of integers with $k_i \to k$ in $\mathfrak{X}$ and $k_i \to \infty$ in $\mathbf{R}$. Check that the convergence

$$\sigma_{k_i}(n) \to \sigma_k^*(n)$$

is uniform in $n \in \mathbf{Z}_{\geq 1}$.

(c) Let $k \in \mathfrak{X}$ and $k_i \in \mathfrak{X}$ be a sequence with $k_i \to k$. Check that the convergence

$$\sigma_{k_i}^*(n) \to \sigma_k^*(n)$$

is uniform in $n \in \mathbf{Z}_{\geq 1}$.

Problem 1. This problem concerns the action of Hecke operators on the p-adic Eisenstein series

$$G_k^* = \frac{1}{2}\zeta^*(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}^*(n) q^n \in M_k^\dagger,$$

which is defined for $k \in \mathfrak{X} - \{0\}$ even. Show that:

(a) $G_k^* | T_\ell = (1 + \ell^{k-1}) G_k^*$.

(b) $G_k^* | U_p = G_k^*$.

(c) $G_k^* = G_k | (1 - p^{k-1} V_p)$ for $k \in \mathbf{Z}_{\geq 2}$ even.

Problem 2.

(a) Prove that $\zeta^*(1-k)$ is continuous for $k \in \mathfrak{X} - \{0\}$ even. (Hint: You may use without proof the results of Problem 0.)

(b) Prove that if $k \in \mathbf{Z}_{\geq 2}$ is even, then

$$\zeta^*(1-k) = (1 - p^{k-1})\zeta(1-k).$$

(c) Conclude that if $k_i \in \mathbf{Z}_{\geq 2}$ is any sequence of even integers convergent to $k \in \mathfrak{X} - \{0\}$ (not necessarily with $k_i \to \infty$ in $\mathbf{R}$), then

$$\zeta^*(1-k) = \lim_{i \to \infty} (1 - p^{k_i - 1})\zeta(1 - k_i).$$

Last updated: February 22, 2021. Please send questions and comments to Pak-Hin.Lee@warwick.ac.uk.

Problem 3. Prove that

$$P := E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is a p-adic modular form of weight 2. (Hint: Use Problem 1(c) and construct an inverse operator to $1 - p^{k-1} V_p$.)

Problem 4. Let $k \in \mathfrak{X} - \{0\}$ with $k \equiv 4, 6, 8, 10, 14 \pmod{p-1}$.

(a) Prove that

$$\zeta^*(1-k) \neq 0.$$

(Hint: Use the Clausen–von Staudt theorem and Kummer congruence.)

(b) (**NOT FOR SUBMISSION**) Extend théorème 7 [P.215] for general p with $k \equiv 4, 6, 8, 10, 14 \pmod{p-1}$.

Problem 5. Recall the spectral decomposition

$$\widetilde{M}^{\alpha} = \widetilde{S}^{\alpha} \oplus \widetilde{N}^{\alpha}$$

for the action of U_p on mod p modular forms.

(a) Show that if $\tilde{f}$ is a mod p modular form in $\widetilde{N}^{\alpha}$, then $\tilde{f}$ is a mod p cusp form (i.e. $a_0(\tilde{f}) = 0$ in $\mathbf{F}_p$).

(b) For $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$, show that $\widetilde{\Delta} \in \widetilde{N}^{\alpha}$ if and only if $\tau(p) \equiv 0 \pmod{p}$. Hence give a counterexample to the converse of (a).

(c) (**NOT FOR SUBMISSION**) As part of your work on Problem 4(b), lemme 3 [P.216] holds for *cusp* forms $f \in M_k^{\dagger}$ and general p under the condition $k \equiv 4, 6, 8, 10, 14 \pmod{p-1}$. Discuss how the proof fails[1] for *non-cusp* forms f satisfying $\tilde{f} \in \widetilde{N}^{\alpha}$ (where $k \not\equiv 4, 6, 8, 10, 14 \pmod{p-1}$ necessarily). Can you find an explicit counterexample?

[1]contrary to what I said in Lecture 5