Algebraic Geometry II project
Hölder-Hahn embedding theorem

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The goal of these notes is to give a self-contained proof of the Hölder-Hahn embedding theorem, which says that any commutative group equipped with a complete ordering, can be embedded in an order-preserving way, into a lexicographically ordered function space with values in the reals. Hausner-Wendel gave in [HW52] a proof of this theorem in the case of ordered vector space, and later Clifford showed in [Cli54] that this proof also applies to the case of ordered groups. This is the case which Hahn originally showed in 1907, and the case which we are interested in. In these notes we will follow the proof given by Ambrus Pal as part of the TCC course *Algebraic Geometry II*, held in 2020, which contains some simplifications compared to the original proofs in [HW52], [Cli54].

Definition 0.1. An *ordered group* is a pair $(G, <)$ consisting of a commutative group G and a complete ordering $<$ on the elements of G, such that $a < b$ implies $a + c < b + c$ for $a, b, c \in G$. If we have an ordered group $(G, <)$ we will by $(G, <^{op})$ denote the ordered group with underlying group G and ordering given by $a <^{op} b$ if and only if $a > b$.

Given an ordered group $(G, <)$ and a completely ordered set Ω we want to consider the group G^Ω consisting of those functions $f : \Omega \to G$ where

$$\operatorname{supp}(f) := \{\alpha \in \Omega \,|f(\alpha) \neq 0\}$$

is well-ordered. We can then equip this group with the *Lexicographical ordering*: $f < g$ when

$$f(\alpha) < g(\alpha), \;\; \alpha = \min \{\beta \in \Omega \mid f(\beta) \neq g(\beta)\} \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g).$$

Then $(G^\Omega, \text{lexicographical ordering})$ is an ordered group.

We want to equip ordered groups with a so called *"rank"*, for which we need the following definitions:

Definition 0.2. Let $(G, <)$ be an ordered group and $x, y \in G$, $x, y \neq 0$. Then we say that x and y are *Archimedean equivalent*, written $x \sim y$, if there exists $m, n \in \mathbb{N}_{>0}$ such that

$$|x| < m|y|, \;\; |y| < n|x|.$$

It is clear that Archimedean equivalence is symmetric and reflective. To see that it is also transitive, assume that $x \sim y$ and $y \sim z$. Then there exists $m, n \in \mathbb{N}_{>0}$ such that $|x| < n|y|$ and $|y| < m|z|$, hence

$$|x| < n|y| < nm|z|$$

and similarly for the other inequality. Hence $x \sim z$, so the Archimedean equivalence is an equivalence relation.

Definition 0.3. Let $(G, <)$ be an ordered group and $a \in G$, $a > 0$. We then say $g \in G$, $g > 0$, is *dominated* by a, if $mg < a$ for all $m \in \mathbb{N}_{>0}$. We denote this by $g \ll a$.

Definition 0.4. Given an ordered group $(G, <)$ we define its *rank* to be the ordered set Ω consisting of the Archimedean equivalence classes on G equipped with the complete ordering $\ll$.

We now want to see that $\ll$ indeed induces a complete ordering on the set Ω consisting of the Archimedean equivalence classes on an ordered group $(G, <)$. First we see that $\ll$ descends to an order on Ω, so assume that $x \sim x'$ and $y \sim y'$. Note that we may assume without loss of generality, that all the elements are positive, since $x \sim |x|$. We will therefore omit $|-|$. Then there exists $m_x, n_x, m_y, n_y \in \mathbb{N}_{>0}$ such that

$$x < m_x x',\ x' < n_x x,\ y < m_y y',\ y' < n_y y.$$

Further assume $x \ll y$, then $mx < y$ for all $m \in \mathbb{N}_{>0}$. So for any $m' \in \mathbb{N}_{>0}$ we have that

$$m' m_y x' < m' m_y n_x x < y < m_y y'$$

which implies $m'x' < y'$. Hence $x \ll y'$, so $\ll$ does indeed descend to an order on Ω. That this is anti reflective is clear, so only missing to show that this is a total ordering on Ω, i.e. we want to show that for all $x, y \in \Omega$ either $x \ll y$ or $y \ll x$ or $x \sim y$. Assume that x is not dominated by y and that y is not dominated by x. Then there exists $m, n \in \mathbb{N}_{>0}$ such that $y < mx$ and $x < ny$, hence $x \sim y$. So we conclude that $\ll$ does indeed give a complete ordering on Ω.

The main goal of these notes is to prove the following theorem:

Theorem 0.5 (Hahn)**.** *Let $(G, <)$ be an ordered group with rank Ω. Then there exists an order-preserving embedding*

$$G \hookrightarrow \mathbb{R}^{\Omega^{op}},$$

where $\mathbb{R}^{\Omega^{op}}$ has the lexicographical ordering.

We will first prove this in the case where the rank is a singleton, which is known as Hölder's theorem:

Theorem 0.6 (Hölder)**.** *Let $(G, <)$ be an ordered group with rank $|\Omega| = 1$. Then there exists an order-preserving embedding*

$$G \hookrightarrow \mathbb{R}.$$

Proof. We first note that $(x/n) \sim x$ for all n so we get that $\Omega_G \cong \Omega_{G \otimes_{\mathbb{Z}} \mathbb{Q}}$. We further note that any ordered group is torsion free, since if we assume $x \in G$ different from 0 is torsion, i.e. $nx = 0$ for some positive integer n, we reach the contradiction that $0 = n|x| > n \cdot 0 = 0$. This gives us that we have an order-preserving embedding $G \hookrightarrow G \otimes_{\mathbb{Z}} \mathbb{Q}$, so using that the rank of these two ordered groups are isomorphic, we may assume that G is a $\mathbb{Q}$-vector space. Fix $x \in G \backslash \{0\}$ and write $\mathbb{Q}x$ for the subspace generated by x. We then have a map

$$\begin{aligned} c : G &\to P(\mathbb{Q}x) \\ \alpha &\mapsto c(\alpha) := \{\beta \in \mathbb{Q}x \mid \beta < \alpha\}, \end{aligned}$$

where $P(\mathbb{Q}x)$ denotes the power set. Further let $\varphi : \mathbb{Q}x \to \mathbb{Q}$ denote the canonical map, and note that this is order-preserving. We use this to define the map

$$\begin{aligned} &G \to \mathbb{R} \\ &\alpha \mapsto \sup(\varphi(c(\alpha))), \end{aligned}$$

which is our candidate for the order-preserving embedding.

Since we have assumed that the rank of G is 1, we have that $\alpha \sim x$ for any $\alpha \in G$. This means that there exists $m \in \mathbb{N}_{>0}$ such that $\alpha < mx$ which then implies that any element in $c(\alpha)$ is less than mx, hence we get that $c(\alpha)$ is bounded. Therefore it makes sense for us take $\sup(-)$ in the above.

We need to prove that this map satisfies the following 3 properties:

1. It is order preserving
2. It is injective
3. It is a group homomorphism.

We will omit φ to ease our notation.

Proof of 1: Assume that $\alpha \leq \beta$ in G. Then $c(\alpha) \subseteq c(\beta)$ which implies $\sup(c(\alpha)) \leq \sup(c(\beta))$.

Proof of 2: Let $\alpha < \beta$ in G. We first wish to prove that $(\alpha, \beta) \cap \mathbb{Q}x \neq \emptyset$, where (α, β) denotes the open interval. Since $\beta - \alpha > 0$ and $\beta - \alpha \sim x$, we have that there exists $m \in \mathbb{N}_{>0}$ such that $m(\beta - \alpha) > x$, hence $\frac{1}{m}x < \beta - \alpha$. Since $\alpha \in G$ we know that $\alpha \sim \frac{1}{m}x$, hence there exists $n \in \mathbb{N}_{>0}$ such that $\alpha < \frac{n}{m}x$. Assume that n is the smallest such number and assume for contradiction that $\frac{n}{m}x \geq \beta$. Then

$$\beta - \frac{1}{m}x \leq \frac{n-1}{m}x,$$

so since $\frac{1}{m}x < \beta - \alpha$, we get that

$$\beta - (\beta - \alpha) = \alpha < \beta - \frac{1}{m}x \leq \frac{n-1}{m}x.$$

This is a contradiction, hence $\frac{n}{m}x < \beta$. This gives us that

$$\alpha < \frac{n}{m}x < \beta$$

so $\frac{n}{m}x \in (\alpha, \beta) \cap \mathbb{Q}x \neq \emptyset$ as wanted. This further gives us that $\frac{n}{m}x \notin c(\alpha)$ but $\frac{n}{m}x \in c(\beta)$, so we conclude that

$$\sup(c(\alpha)) < \frac{n}{m}x \leq \sup(c(\beta)).$$

Proof of 3: We want to show that $c(\alpha + \beta) = c(\alpha) + c(\beta)$.

" $\supseteq$ ": Noting that

$$c(\alpha) + c(\beta) = \{y + z | y \in c(\alpha), z \in c(\beta)\}$$

we get that we can write $w \in c(\alpha) + c(\beta)$ as $w = y + z$ for $y, z \in \mathbb{Q}x$, $y < \alpha$, $z < \beta$. Hence

$$w = y + z < \alpha + \beta \Rightarrow w \in c(\alpha + \beta).$$

" $\subseteq$ ": Let $r \in c(\alpha + \beta)$, i.e. $r \in \mathbb{Q}x$, and $r < \alpha + \beta$. Then $r - \beta < \alpha$, and as we did when we proved that the map is injective, we can find $r_1 \in \mathbb{Q}x$ such that

$$r - \beta < r_1 < \alpha \Rightarrow r < r_1 + \beta < \alpha + \beta.$$

Consider the inequality $r < \ r_1 + \beta$, which implies $r - r_1 < \beta$. So we get that we can write $r = r_1 + (r - r_1)$ with $r - r_1 < \beta$ and $r < \alpha$, hence we get $r \in c(\alpha) + c(\beta)$.

Using this we get that

$$\sup(c(\alpha + \beta)) = \sup(c(\alpha) + c(\beta)) = \sup(c(\alpha)) + \sup(c(\beta)).$$

□

Before we continue with the proof of Hahn's embedding theorem we will need the following algebraic constructions.

Definition 0.7. Let $(G, <)$ be an ordered group and $a \in G$, $a > 0$. We then define the subset

$$G_{\ll a} := \{g \in G \mid |g| \ll a\} \subset G.$$

Definition 0.8. Let $(G, <)$ be an ordered group. A subgroup $C \subseteq G$ is called *convex* if $x < y < z$, for $x, z \in C$ and $y \in G$, implies that $y \in C$ as well.

Lemma 0.9. *$G_{\ll a}$ is a convex subgroup of G.*

Proof. First we prove that $G_{\ll a}$ is a subgroup of G. Let $x, y \in G_{\ll a}$ such that $m|x| < \frac{a}{2}$ and $m|y| < \frac{a}{2}$ for some $m \in \mathbb{N}_{>0}$. Then

$$m|x + y| \leq m|x| + m|y| < \frac{a}{2} + \frac{a}{2} = a.$$

To further see it is convex, assume that $x < h < y$ for $x, y \in G_{\ll a}$ and $h \in G$. Then $|h| < \max\{|x|, |y|\}$, which implies that for all $m \in \mathbb{N}_{>0}$:

$$m|h| < m \cdot \max\{|x|, |y|\} < a,$$

hence $h \in G_{\ll a}$. $\square$

We want to be able to form the quotient $G/G_{\ll a}$ in the category of ordered groups, for which we will need the following lemma.

Lemma 0.10. *Let $\varphi : G \to H$ be an order preserving homomorphism between two ordered groups $(G, <_G)$, $(H, <_H)$. Then ker(φ) is a convex subgroup of G. If $C \subseteq G$ is a convex subgroup, then there exists a unique ordering on G/C such that the quotient map $\varphi : G \twoheadrightarrow G/C$ is order preserving.*

Proof. To show the first claim, let $y <_G x <_G z$ in G and assume $y, z \in \ker(\varphi)$. Since φ is order preserving we then have

$$0 = \varphi(y) <_H \varphi(x) <_H \varphi(z) = 0,$$

so $\varphi(x) = 0$, i.e. $x \in \ker(\varphi)$, so $\ker(\varphi)$ is convex.

To show the second claim, let $a, b \in G/C$, $a \neq b$. Since the quotient map is surjective we can choose $x, y \in G$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Since $a \neq b$ we have that either $x < y$ or $y < x$. We then see that the only possible ordering we can equip G/C with, which will make φ order preserving, is by saying that $x <_G y$ implies $a <_{G/C} b$ and $y <_G x$ implies $b <_{G/C} a$.

We need to show this is well-defined, so assume $a <_{G/C} b$ and $b <_{G/C} a$. Then there exists $x_1 <_G y_1$ and $y_2 <_G x_2$ in G with

$$\varphi(x_1) = \varphi(x_2) = a, \;\; \varphi(y_1) = \varphi(y_2) = b.$$

Then $x_1 + x' = x_2$ and $y_1 + y' = y_2$ for some $y', x' \in C$. So we have that

$$y' <_G y' + y_1 - x_1 = y_2 - x_1 <_G x_2 - x_1 = x',$$

and using that C is convex, we get that $y_2 - x_1 \in C$, so $a = b$ as desired. $\square$

So we can form $G/G_{\ll a}$ in the category of ordered groups.

Definition 0.11. Let $(G, <)$ be an ordered group and $a \in G$, $a > 0$. We then define the subset

$$G_a := \{x \in G \mid \exists m \in \mathbb{N}_{>0}, \; |x| < ma\} \subseteq G.$$

Note that $G_{\ll a} \subseteq G_a \subseteq G$.

Lemma 0.12. *G_a is a convex subgroup of G.*

Proof. First we show that G_a is a subgroup of G, so let $x, y \in G_a$. Then there exists $m, n \in \mathbb{N}_{>0}$ such that $|x| < ma$ and $|y| < na$. Hence

$$|x + y| \leq |x| + |y| < (n + m)a,$$

so since $(n + m)$ still is in $\mathbb{N}_{>0}$ we get that $x + y \in G_a$. To see that G_a is a convex subgroup, assume further that we have $h \in G$ such that $x < h < y$ with x, y as before. Then

$$|h| < \max\{|x|, |y|\} \cdot a,$$

so $h \in G_a$ as desired. □

This gives us that we can form the quotient $G_a/G_{\ll a}$ in the category of ordered groups.

Note that for every $x \in G_a$, we have that either $x \sim a$ or $x \ll a$, since otherwise we would have that $x \gg a$ which would imply that $ma < |x|$ for all $m \in \mathbb{N}_{>0}$, which is a contradiction.

Before we begin the proof of the main theorem, we finally recall the following map.

Definition 0.13. Let Ω be the rank of some ordered group and $t \in \Omega$. Then the *cut operator* at t is the map

$$c : \mathbb{R}^{\Omega^{op}} \to \mathbb{R}^{\Omega^{op}}$$

which on $\underline{x} \in \mathbb{R}^{\Omega^{op}}$, $\underline{x} = \prod_{a \in \Omega^{op}} x_a$, is given by

$$c(\underline{x})_a = \begin{cases} x_a, & a < t \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Hahn's theorem. As in the proof of Hölder's theorem we can assume that G is a $\mathbb{Q}$-vector space. In that case we get an exact sequence

$$G_{\ll a} \hookrightarrow G_a \twoheadrightarrow G_a/G_{\ll a},$$

for $a \in G$, $a > 0$, and writing $K_a := G_a/G_{\ll a}$ this gives us that

$$G_a \cong G_{\ll a} \oplus K_a.$$

Note that we consider $(K_a, <)$ as an ordered group by forming the quotient $G_a/G_{\ll a}$ in the category of ordered groups. Further note that this identification above gives us a choice of inclusion $K_a \hookrightarrow G$ for any positive $a \in G$.

Since any $x \in G_a$ is either $x \sim a$ or $|x| \ll a$, we get that every $x \in G_a/G_{\ll a}$ is $x \sim a$, hence K_a has rank 1. This allows us to apply Hölder's embedding theorem, which tells us that there exists an order-preserving embedding

$$\varphi_a : K_a \hookrightarrow \mathbb{R}.$$

Since $G_a = G_b$ when $a \sim b$ we will simply write G_a for $a \in \Omega$ and similarly for $G_{\ll a}$ and K_a. Writing $K := \oplus_{a\in\Omega} K_a$ for $a \in \Omega$, we get an order-preserving embedding

$$\varphi_0 : K \hookrightarrow \bigoplus_{a\in\Omega} \mathbb{R} \hookrightarrow \mathbb{R}^{\Omega^{op}}$$

by assembling these φ_a. We note that for each $a \in \Omega$ we can fix $ta \in G$ such that $[ta] = a$ and we may assume that $\varphi_0(ta)_s$ is 1 for $s = a$ and 0 otherwise. We want to extend this map φ_0 to the desired order-preserving embedding $G \hookrightarrow \mathbb{R}^{\Omega^{op}}$. We will do this by using Zorn's lemma. Consider the class of extensions $\psi : V \to \mathbb{R}^{\Omega^{op}}$ for $V \subseteq G$ a $\mathbb{Q}$-vector space with $K \subseteq V$, which satisfies

1. $\psi|_K = \varphi_0$
2. ψ is injective
3. Every cut operator c satisfies that $c(\mathrm{Im}(\psi)) \subseteq \mathrm{Im}(\psi)$
4. ψ is order-preserving.

It is clear that (K, φ_0) satisfies $1, 2$ and 4. To see that the pair also satisfies number 3 we note that something in K has the form $\sum_{a\in\Omega} x_a$ which by φ_0 is mapped to the map $f : \Omega \to \mathbb{R}$, with $f(a) = \varphi_a(x_a)$. Then $c(f)$ is just f except that $c(f)(a) = 0$ when $a \geq t$, which is $\varphi_0(\sum_a y_a)$ where $y_a = x_a$ when $a < t$ and $y_a = 0$ otherwise.

Now, let P denote the poset of pairs (V, ψ) where V and ψ are as above, and we say that $(V, \psi) \leq (V', \psi')$ if $V \subseteq V'$ and $\psi'|_V = \psi$. We see that the union of any chain of such pairs will still be contained in P and contains each pair, hence we get by Zorn's lemma that P contains a maximal element, call this (V, φ). Assume that $V \subsetneq G$ and that $x \in G \backslash V$, then if we have an extension $\psi : V \oplus \mathbb{Q}x \to \mathbb{R}^{\Omega^{op}}$, we get a contradiction since we assumed (V, φ) maximal. Hence it is enough for us to show that given this $\varphi : V \to \mathbb{R}^{\Omega^{op}}$ and some fixed $x \in G \backslash V$, we can extend to

$$\psi : V \oplus \mathbb{Q}x \to \mathbb{R}^{\Omega^{op}},$$

which satiesfies the 4 properties.

First we define for this fixed x the set

$$S = \{a \in \Omega^{op} \mid \exists\, z_a \in V \text{ such that } x - z_a \in G_{\ll a}\},$$

which we consider with the induced ordering from Ω^{op}. Note that if $a, b \in S$ and $a > b$, then

$$tb \gg ta \gg x - z_a \in G_{\ll b}.$$

Claim: S has no maximal element.

Proof: Assume for contradiction that $a \in S$ is maximal. Then there exist $z_a \in V$ such that $x - z_a \ll ta$. This gives us that $[x - z_a] := b \in \Omega$ where $b \gg^{op} a$.

We now want to show that there exists $y \in K_b$ such that $x - z_a - y \ll tb$. We first note that $x - z_a \in G_b$. Using that

$$G_b \cong G_{\ll b} \oplus K_b$$

we get that we can write

$$x - z_a = y' + y, \quad \text{with } y' \in G_{\ll b},\ y \in K_b.$$

Hence we get that

$$x - z_a - y = y' + y - y = y' \ll tb,$$

which gives us that $b \in S$, but this contradicts the maximality of a.

We will now start constructing this ψ by first describing it on $\mathbb{Q}x$. We have that $\psi(x)$ is a function $\Omega^{op} \to \mathbb{R}$ and we define it such that $\operatorname{supp}(\psi(x)) \subseteq S$, i.e. we define $\psi(x)_t = 0$ for $t \notin S$. Further, if $t \in S$ we put

$$\psi(x)_t = \varphi(z_{t'})_t$$

where $t' > t$ in S and $z_{t'} \in V$ such that $x - z_{t'} \ll t'$. Using that S has no maximal element, we get that we can always find such an t'.

Next, we wish to prove that ψ is well-defined.

Claim: $\psi(x)$ is well-defined.

Proof: We need to prove that

$$\varphi(z_a)_t = \varphi(z_b)_t \ \ \forall\, t < \min(a, b)$$

and that the support is well-ordered. If we assume that the above equality holds and that $a \leq b$ we see that

$$\operatorname{supp}(\varphi(z_a)) \cap \{t < a\} = \operatorname{supp}(\varphi(z_b)) \cap \{t < a\}$$

is well-ordered, which gives us that

$$\operatorname{supp}(\psi(x)) = \bigcup_{a \in S} \operatorname{supp}(\varphi(z_a)) \cap \{t < a\}$$

is well-ordered as well, since the union of a chain of well-ordered subsets which are initial segments of each other are again well-ordered.

To show the equality, let $a, b \in S$ and assume that $a \leq b$ in S. Then

$$x - z_a \in G_{\ll a}, \ \ x - z_b \in G_{\ll b},$$

and assume $z_b > z_a$. Since $a \leq b$ implies that $ta \gg tb$, or equal if $a = b$, we get that $G_{\ll b} \subseteq G_{\ll a}$, hence

$$(x - z_a) - (x - z_b) = z_b - z_a \in G_{\ll a}.$$

This implies that $z_b - z_a \ll ta$, so using that φ is order-preserving, we get that

$$\varphi(z_b) - \varphi(z_a) \ll \varphi(ta).$$

Note that $\varphi(ta) : \Omega^{op} \to \mathbb{R}$ is give by

$$\varphi(ta)_k = \begin{cases} 1, & a = k \\ 0, & a \neq k. \end{cases}$$

Let k_0 be the smallest element in Ω^{op} such that $\varphi(z_a)_{k_0} \neq \varphi(z_b)_{k_0}$. We know that such an element exists since the support of φ is well-ordered. Consider

$$\varphi(z_b) - \varphi(z_a) \geq 0.$$

The first spot where this is non-zero is the k_0 coordinate, in which case the difference is positive by the assumption that $z_b \geq z_a$ and the fact that φ is order-preserving. If $k_0 < a$, then

$$\varphi(ta) \leq \varphi(z_b) - \varphi(z_a),$$

which is a contradiction, hence we get that $k_0 \geq a$ so

$$\varphi(z_a)_k = \varphi(z_b)_k \; \forall k < a \leq b.$$

If $z_a > z_b$ we can do the argument similarly by starting with considering

$$(x - z_b) - (x - z_a) = z_a - z_b \in G_{\ll a}$$

instead.

We can now define ψ on all of $V \oplus \mathbb{Q}x$

$$\begin{aligned} \psi : V \oplus \mathbb{Q}x : &\to \mathbb{R}^{\Omega^{op}} \\ v + \lambda x &\mapsto \varphi(v) + \lambda\psi(x), \end{aligned}$$

which we see is well-defined and $\mathbb{Q}$-linear. We want to show that this map satisfies the 4 properties we claimed above, i.e. we want to show that ψ satisfies:

1. $\psi|_K = \varphi_0$,
2. ψ is injective,
3. Every cut operator c satisfies that $c(\mathrm{Im}(\psi)) \subseteq \mathrm{Im}(\psi)$,
4. ψ is order-preserving.

Proof of 1: By construction we have that $\psi|_V = \varphi$, hence $\psi|_K = \varphi|_K = \varphi_0$.

Proof of 2: Assume that $v + \lambda x \in \ker(\psi)$. We want to show that this implies $\lambda = v = 0$. Assume first that $\lambda = 0$, then $\psi(v) = \varphi(v) = 0$. Since φ is injective we get that $v = 0$. Now assume $\lambda \neq 0$, then

$$\frac{\psi(\lambda x + v)}{\lambda} = \psi(x + \lambda^{-1} v) = 0,$$

which implies

$$\psi(x) = \psi(-\lambda^{-1} v) = \varphi(-\lambda^{-1} v).$$

So to derive a contradiction, it is enough to prove that

$$\psi(x) \neq \varphi(v) \ \ \forall v \in V.$$

Let $a \in S$. Using that S has no maximal element we get that there exists $b \in S$ such that $a < b$, hence $tb \ll ta$. We know that $\varphi(z_b)_{a'} = \psi(x)_{a'}$ for $a' < b$, so by assuming for contradiction that $\psi(x)_{a'} = \varphi(v)_{a'}$ we get that

$$\varphi(z_b) - \varphi(v) \ll \varphi(tb),$$

which implies that $z_b - v \ll tb$ since φ is order preserving. Using this together with $x - z_b \ll tb$ we get that $x - v \ll tb$. Since this holds for any $b \in S$ we get that $[x - v] > s$ for all $s \in S$. We now claim that $[x - v] \in S$, since then we would get that $[x - v]$ is a maximal element of S, which we have showed does not exist, hence we reach a contradiction.

To show $[x - v] \in S$ we note that $x - v \in G_{[x-v]} = G_{\ll x-v} \oplus K_{[x-v]}$. Write $x - v = y' + y$ with $y' \in G_{\ll x-v}$ and $y \in K_{[x-v]}$. Then $y' \ll x - v$ and y is in particular an element in V. Now

$$x - (v + y) = x - v - y = y' + y - y + y' \ll x - v.$$

So $v + y$ is our desired $z_{[x-v]}$.

Proof of 3: We first note that since any cut operator c is $\mathbb{Q}$-linear we get that

$$\begin{aligned} c(\psi(v + \lambda x)) &= c(\psi(v)) + \lambda c(\psi(x)) \\ &= c(\varphi(v)) + \lambda c(\psi(x)) \end{aligned}$$

for $v + \lambda x \in V \oplus \mathbb{Q}x$. So we just need to show that $c(\psi(x)) \subseteq \mathrm{Im}(\psi)$. Assume that the cut is at $t \in \Omega^{op}$. We then have two different cases. First, if $t \in S$ then $c(\psi(x)) = c(\varphi(z_t))$ since $c(\psi(x))_a = \psi(x)_a$ for $a < t$. Then $c(\psi(x)) = c(\varphi(z_t)) = \varphi(v)$ for some $v \in V$, since $c(\mathrm{Im}(\varphi)) \subseteq \mathrm{Im}(\varphi)$. This implies $\varphi(v) = \psi(v) \in \mathrm{Im}(\psi)$, so $c(\psi(x)) \in \mathrm{Im}(\psi)$ as desired.

In the other case, if $t \notin S$, then $t > a$ for all $a \in S$. Note that

$$c(\psi(x))_a = \begin{cases} \psi(x)_a, & a < t \\ 0, & a \geq t. \end{cases}$$

Since $a \geq t$ would imply $a \notin S$, we get that in this case $\psi(x)_a = 0$, hence

$$c(\psi(x))_a = \psi(x)_a, \quad \forall a,$$

so we conclude that $c(\psi(x)) = \psi(x) \in \mathrm{Im}(\psi)$ as desired.

Proof of 4: We first want to prove the following claim;

Claim: Let $v \in V$ and assume that $\mathrm{supp}(\varphi(v))$ contains an element in Ω^{op} which is bigger than any element of $\mathrm{supp}(\psi(x))$. Since $\mathrm{supp}(\varphi(v))$ is well-ordered, we can let t_0 denote the smallest such element. Then there exists $l < t_0$ such that $\varphi(v)_l \neq \psi(v)_l$.

Proof: Assume for contradiction that such an element l does not exists, i.e. $\psi(x)_l = \varphi(v)_l$ for all $l < t_0$. So if $s \in \mathrm{supp}(\psi(x))$, then $s < t_0$ which implies $\psi(x) = c(\varphi(v))$ if we cut at t_0. By property 3, we know that $c(\varphi(v)) \in \mathrm{Im}(\varphi)$, which implies that $\psi(x) = \varphi(v')$ for some $v' \in V$, but this is a contradiction.

We will now continue the proof of property 4. Let $y \in V$ such that $y < x$. Assume for contradiction that $\varphi(y) > \psi(x)$ and let t_0 be the first coordinate where $\varphi(y)$ and $\psi(x)$ differs. We then want to show that there exists an $s \in S$ such that $t_0 < s$. Assume for contradiction that $s < t_0$ for all $s \in S$. Then in particular $s < t_0$ for all $s \in \mathrm{supp}(\psi(x))$ since $\mathrm{supp}(\psi(x)) \subseteq S$. Since $\psi(x)_{t_0} \neq \varphi(y)_{t_0}$, we get that $t_0 \in \mathrm{supp}(\varphi(y))$, hence we can apply the claim above, which gives us that there exists $l < t_0$ such that $\psi(x)_l \neq \varphi(y)_l$. This is a contradiction since t_0 is the minimal such element. Hence there exists $s \in S$ such that $t_0 < s$. This means that

$$\psi(x)_{t_0} = \varphi(z_s)_{t_0} < \varphi(y)_{t_0},$$

since we assumed that $\psi(x) < \varphi(y)$ and t_0 is the first coordinate where they differ. Using that for all $l < t_0$ we have that $\psi(x)_l = \varphi(y)_l$ and $\psi(x)_l = \varphi(z_s)_l$, we get that $\varphi(z_s) < \varphi(y)$ with respect to the lexiographical ordering. Since φ is order preserving this implies

$$0 < z_s < y < x.$$

We then get that

$$0 < y - z_s < x - z_s \ll ts,$$

so using that φ is an order preserving group homomorphism we get that

$$0 < \varphi(y) - \varphi(z_s) \ll \varphi(ts).$$

Note that $ts \in K_s$ implies that $ts \in \oplus_{[ta]\in\Omega} K_a$, so $\varphi(ts)_k = 1$ if $s = k$ and 0 if $s \neq k$. If $l < s$ and $\varphi(y)_l - \varphi(z_s)_l > 0$, then

$$\varphi(y) - \varphi(z_s) > \varphi(ts).$$

But this is a contradiction, since we have just showed that $\varphi(y) - \varphi(z_s) \ll \varphi(ts)$. So

$$\varphi(y)_l = \varphi(z_s)_l \ \ \forall\, l < s.$$

But this is again a contradiction, since $t_0 < s$ and if this was true, then

$$\varphi(y)_{t_0} = \varphi(z_s)_{t_0} = \psi(x)_{t_0},$$

which contradicts that t_0 is the smallest element such that $\varphi(y)$ and $\psi(x)$ differ. So we conclude that

$$x > y \Rightarrow \psi(x) > \varphi(y), \quad y \in V.$$

In the same way it can be shown that

$$x < y \Rightarrow \psi(x) < \varphi(y), \quad y \in V.$$

Since both φ and ψ are injective, we get that the two relations above actually are if and only if.

We wish to use this to show that

$$v' + \lambda' x > v + \lambda x \Rightarrow \psi(v' + \lambda' x) > \psi(v + \lambda x)$$

for $v, v' \in V$ and $\lambda, \lambda' \in \mathbb{Q}$. Assume $\lambda > \lambda'$. We first note that

$$\begin{aligned} v' + \lambda' x > v + \lambda x &\Leftrightarrow v' - v + (\lambda' - \lambda)x > 0 \\ &\Leftrightarrow v' - v > (\lambda - \lambda')x \\ &\Leftrightarrow \frac{v' - v}{\lambda - \lambda'} > x. \end{aligned}$$

Since V is a $\mathbb{Q}$-vector space we get that $\frac{v'-v}{\lambda-\lambda'} \in V$, so the inequality which we just showed above gives us that

$$\psi\left(\frac{v' - v}{\lambda - \lambda'}\right) > \psi(x).$$

Using that ψ is $\mathbb{Q}$-linear we get that

$$\begin{gathered} \psi\left(\frac{v' - v}{\lambda - \lambda'}\right) = \frac{1}{\lambda - \lambda'}\psi(v' - v) > \psi(x) \\ \Leftrightarrow \\ \psi(v') - \psi(v) = \psi(v' - v) > \psi(x)(\lambda - \lambda') = \psi(\lambda x) - \psi(\lambda' x) \\ \Leftrightarrow \\ \psi(v' + \lambda' x) = \psi(v') + \psi(\lambda' x) > \psi(v) + \psi(\lambda x) = \psi(v + \lambda x) \end{gathered}$$

as desired.

If instead $\lambda' > \lambda$, then we would get $\lambda - \lambda' < 0$, and we would need to flip our inequality when we divided by $\lambda - \lambda'$, but since we later multiply with the same term, we can ignore this detail.

$\square$

References

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