Introduction to topological Hochschild homology

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Abstract

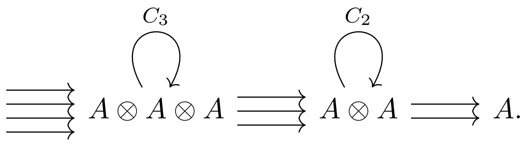
The goal of these notes is to introduce the necessary preliminary notions and results to get a basic introduction to topological Hochschild homology THH, on $\mathbb{E}_1$-ring spectra, and discuss their cyclotomic structure maps. We further give a more explicit description of these maps when we consider THH on $\mathbb{E}_\infty$-ring spectra.

Contents

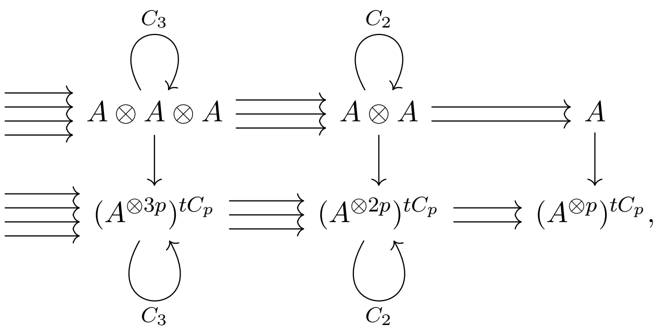
Introduction

We will introduce the ∞-categorical construction of topological Hochschild homology as a cyclotomic spectrum on $\mathbb{E}_1$-ring spectra. We will follow [NS18] "*On topological cyclic homology*", mainly part $I.3, III.1-3$ and $IV.2$.

For $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ the diagram



can formally be described as a specific cyclic spectra, i.e. a functor $\Lambda \to \mathrm{Sp}$. Taking the colimit of a certain functor constructed from this cyclic spectra gives us an object $\mathrm{THH}(A)$ in $\mathrm{Sp}^{B\mathbb{T}}$ which is called the *topological Hochschild homology*. This is defined formally in definition 2.10. In section 2.2 "*THH and cyclotomic spectra*" we describe the cyclotomic structure maps of this $\mathbb{T}$-spectra. To do this we need to construct a map of cyclic spectra



where the geometric realization of the lower cyclic spectrum will be shown to be exactly $\mathrm{THH}(A)^{tC_p}$. To be able to describe this map, we need to take a step back and discuss the Tate-construction $(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$, with G some finite group.

We show in theorem 1.1 that $(-)^{tC_p}$ admits an unique lax symmetric monoidal structure which also makes the natural transformation $(-)^{hG} \to (-)_{hG}$ lax symmetric monoidal. This arguments relies heavily on some finer structural analysis of Verdier quotients of stable ∞-categories. This is an important result when we wish to describe the cyclotomic structure maps on THH. One of the main applications is that it directly gives us a lax symmetric monoidal structure on the exact functor

$$\begin{aligned} T_p : \mathrm{Sp} &\to \mathrm{Sp} \\ X &\mapsto (X^{\otimes p})^{tC_p}, \end{aligned}$$

which is given in proposition 1.8. We use this functor to define the natural transformation

$$\begin{aligned} \Delta_p : id_{\mathrm{Sp}} &\to T_p \\ X &\mapsto (X^{\otimes p})^{tC_p}, \end{aligned}$$

which, under the equivalence

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp})}(id_{\mathrm{Sp}}, F) \to \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, F(\mathbb{S})) = \Omega^\infty F(\mathbb{S})$$

proven in proposition 1.9, corresponds to the map $\mathbb{S} \to T_p(\mathbb{S}) = \mathbb{S}^{tC_p}$, which is the composite $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$, with trivial C_p-action. This is called the *Tate Diagonal*, and the before mentioned result is used to prove in lemma 1.12 that this is the unique lax symmetric monoidal transformation $id_{\mathrm{Sp}} \to T_p$. Another interesting property of the Tate diagonal, which we show in theorem 1.15, is that for a spectra X which is bounded below, it exhibits $(X^{\otimes p})^{tC_p}$ as the p-completion. The main purpose of the Tate diagonal for us, is that we extend it to the map of cyclic spectra, which is necessary to construct the cyclotomic structure maps on $\mathrm{THH}(A)$. In the case where A is an $\mathbb{E}_\infty$-ring spectrum we have an easier description of the cyclotomis structure on $\mathrm{THH}(A)$. We show in theorem 2.12 that $\mathrm{THH}(A)$ can be equipped with cyclotomic structure $\mathbb{E}_\infty$-maps. We end these notes by showing that for $A \in \mathrm{Alg}_{\mathbb{E}_\infty}$ the *Frobenius map* φ_p is the unique $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-map which makes the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{THH}(A) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \varphi_p} \\ (A^{\otimes p})^{tC_p} & \longrightarrow & \mathrm{THH}(A)^{tC_p} \end{array}$$

of $\mathbb{E}_\infty$-rings, commute.

1 The Tate construction

Let G be a finite group, acting on an abelian group M. The algebraic norm-map Nm_G lets us splice together cohomology and homology to obtain *Tate cohomology* $\hat{H}^i(G,M)$ which is defined as

$$\hat{H}^i(G,M) = \begin{cases} H^i(G,M) & 1 \leq i \\ \operatorname{coker}(Nm_G) & i = 0 \\ \ker(Nm_G) & i = -1 \\ H_{-i-1}(G,M) & i \leq -2. \end{cases}$$

The Tate construction is the cofiber of the ∞-categorical analogue of the norm-map. See [NS18, I.1-I.2] for the description and a discussion of this construction. For $X \in \mathrm{Sp}^{BG}$ we have the following spectral sequence

$$E_2^{i,j} = \hat{H}^i(G, \pi_{-j}(X)) \Rightarrow \pi_{-i-j}(X^{tG}),$$

with differentials $d_r : E_r^{i,j} \to E_r^{i+r,j-r+1}$, which relates the Tate construction to Tate cohomology.

1.1 Multiplicativity of the Tate construction

The main goal of this section is to show that for a finite group G, the Tate construction

$$(-)^{tG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$$

admits a unique lax symmetric monoidal structure which makes the natural transformation $(-)^{hG} \to (-)^{tG}$ lax symmetric monoidal. More precisely we wish to prove the following theorem:

Theorem 1.1. *The space consisting of all pairs of lax symmetric monoidal structure on the functor $(-)^{tG} : Sp^{BG} \to Sp$ together with a lax symmetric monoidal refinment of the natural transformation $(-)^{hG} \to (-)^{tG}$ is contractible.*

We start by recalling some definitions.

Definition 1.2.

i) A *stable subcategory* of a stable ∞-category $\mathcal{C}$, is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that $\mathcal{D}$ is stable and the inclusion $\mathcal{D} \subseteq \mathcal{C}$ is exact.

ii) We say that $\mathcal{C}$ is *stably symmetric monoidal* if the tensor product is exact in each variable seperately.

iii) Assume that $\mathcal{C}$ is a stably symmetric monoidal stable ∞-category. A stable subcategory $\mathcal{D} \subseteq \mathcal{C}$ is a $\otimes$*-ideal* if for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, one has that $X \otimes Y \in \mathcal{D}$.

To prove the main theorem we will need to use some finer structural analysis of the Verdier quotient $\mathcal{C}/\mathcal{D}$ by first showing that it is equivalent to a specific Dwyer-Kan localization $\mathcal{C}[\mathcal{W}^{-1}]$. In general when we say something is a *localization* we mean in the sense of [Lur09, 5.2.7.2], i.e. a functor which admits a fully faithful right adjoint. We will explicitly write *Dwyer-Kan localization* to emphasise when we are talking about this localization instead.

Theorem 1.3. *Let $\mathcal{C}$ be a small, stable ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a stable subcategory. Let $\mathcal{W}$ be the collection of all arrows in $\mathcal{C}$ whose cone lies in $\mathcal{D}$.*

i) The Dwyer-Kan localization $\mathcal{C}[\mathcal{W}^{-1}]$ is a stable ∞-category and the canonical map $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is an exact functor.

If $\mathcal{E}$ is another stable ∞-category, then composition with the exact functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ induces an equivalence between $Fun^{ex}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{E})$ and the full subcategory $Fun^{ex}(\mathcal{C}, \mathcal{E})' \subseteq Fun^{ex}(\mathcal{C}, \mathcal{E})$ consisting of those functors which send all objects of $\mathcal{D}$ to 0.

ii) Let $X, Y \in \mathcal{C}$ and write $\overline{X}, \overline{Y}$ for their images in $\mathcal{C}[\mathcal{W}^{-1}]$ under $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$. The mapping space in $\mathcal{C}[\mathcal{W}^{-1}]$ is given by

$$Map_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{X}, \overline{Y}) \simeq colim_{Z \in \mathcal{D}_{/Y}} Map_{\mathcal{C}}(X, cofib(Z \to Y))$$

where the colimit is filtered. In particular, the Yoneda functor

$$\begin{aligned} \mathcal{C}[\mathcal{W}^{-1}] &\xrightarrow{\iota} Fun(\mathcal{C}^{op}, \mathscr{S}) \\ \overline{Y} &\mapsto (X \mapsto Map_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{X}, \overline{Y})), \end{aligned}$$

factors over the Ind-category $Ind(\mathcal{C})$. This functor $\mathcal{C}[\mathcal{W}^{-1}] \to Ind(\mathcal{C})$ is an exact functor of stable ∞-categories which sends the image $\overline{Y} \in \mathcal{C}[\mathcal{W}^{-1}]$ of $Y \in \mathcal{C}$ to the formal colimit

$$\overline{Y} \mapsto colim_{Z \in \mathcal{D}_{/Y}} cofib(Z \to Y) \in Ind(\mathcal{C}).$$

iii) Assume that $\mathcal{E}$ is a presentable stable ∞-category. Then the full inclusion $Fun^{ex}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{E}) \subseteq Fun^{ex}(\mathcal{C}, \mathcal{E})$ is right adjoint to a localization. This localization functor is given by

$$\begin{aligned} Fun^{ex}(\mathcal{C}, \mathcal{E}) &\to Fun^{ex}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{E}) \subset Fun^{ex}(\mathcal{C}, \mathcal{E}) \\ (F : \mathcal{C} \to \mathcal{E}) &\mapsto (\mathcal{C}[\mathcal{W}^{-1}] \to Ind(\mathcal{C}) \xrightarrow{Ind(F)} Ind(\mathcal{E}) \to \mathcal{E}) \end{aligned}$$

where the first map is the one obtained in 1.3.ii and the last map comes by taking the colimit in $\mathcal{E}$.

We see that 1.3.i gives us that $\mathcal{C}[\mathcal{W}^{-1}] = \mathcal{C}/\mathcal{D}$ is the Verdier quotient, so after proving this part of the theorem we will mainly use the notation $\mathcal{C}/\mathcal{D}$. So 1.3.iii gives us that any exact functor $F : \mathcal{C} \to \mathcal{E}$ has a universal approximation which factors over $\mathcal{C}/\mathcal{D}$, i.e. we have a diagram of the form

$$\begin{array}{ccccc} \mathcal{C} & & \xrightarrow{\quad F \quad} & & \mathcal{E} \\ \downarrow & & & & \uparrow \\ \mathcal{C}/\mathcal{D} & \longrightarrow & \mathrm{Ind}(\mathcal{C}) & \longrightarrow & \mathrm{Ind}(\mathcal{E}) \end{array}$$

where we have a natural transformation from the lower composition to F.

Before we prove this theorem, we wish to consider mapping spaces in $\mathcal{C}[\mathcal{W}^{-1}]$. Let $\mathcal{C}$ be a small ∞-category and $\mathcal{W}$ a collection of arrows in $\mathcal{C}$. Since the universal property of the localization $\mathcal{C}[\mathcal{W}^{-1}]$ comes with a functor

$$\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

which maps $\mathcal{W}$ to equivalences, we can consider $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$ as the full subcategory $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of those functors sending $\mathcal{W}$ to equivalences. Assume that $\mathcal{D}$ has all small limits and consider the diagram

$$\begin{array}{ccc} \overline{F} & \longmapsto & F \\ & & \\ \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\ \simeq \big| & \nearrow & \\ \mathrm{Fun}'(\mathcal{C}, \mathcal{D}). & & \end{array}$$

We see that the inclusion $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ preserves all limits. If we further assume that $\mathcal{D}$ is presentable, then if follows by [Lur09, 5.5.3.6] that both $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are again presentable. This lets us apply the adjoint functor theorem [Lur09, 5.5.2.9] to the inclusion functor, which gives us a left adjoint $L : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D})$. We wish to consider this left adjoint in the case $\mathcal{D} = \mathscr{S}$, where $\mathscr{S}$ denotes the infinity category of spaces, which we know is presentable by [Lur09, 5.5.1.8]. In the lemma below, let $\mathcal{C}$ and $\mathcal{W}$ be as above and let L denote the just constructed left adjoint.

Lemma 1.4.

i) For every object $X \in \mathcal{C}$ with image $\overline{X} \in \mathcal{C}[\mathcal{W}^{-1}]$ under $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, the functor $Map_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{X}, -)$ is given by $L(Map_{\mathcal{C}}(X, -))$.

ii) The following diagram of ∞-categories commutes, where the horizontal maps are Yoneda embeddings:

$$\begin{array}{ccc} \mathcal{C}^{op} & \longrightarrow & Fun(\mathcal{C}, \mathscr{S}) \\ \downarrow & & \downarrow L \\ \mathcal{C}[\mathcal{W}^{-1}]^{op} & \longrightarrow & Fun(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S}). \end{array}$$

Proof. Let $X \in \mathcal{C}$ and $F \in \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$. Then we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}],\mathscr{S})}(L(\mathrm{Map}_{\mathcal{C}}(X,-), F) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathscr{S})}(\mathrm{Map}_{\mathcal{C}}(X,-), F|_{\mathcal{C}}) \\ &\simeq F(\overline{X}), \end{aligned}$$

where the first equivalence follows from the adjunction and the second is due to Yoneda in $\mathcal{C}$. This proves statement *i*).

To prove *ii*) let $f : X \to Y$ be in $\mathcal{W}$ such that $\overline{X} \xrightarrow{\simeq} \overline{Y}$. Then it follows by the Yoneda lemma that

$$L(\mathrm{Map}_{\mathcal{C}}(Y,-)) \to L(\mathrm{Map}_{\mathcal{C}}(X,-))$$

is an equivalence in $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$. Since we can view $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$ as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathscr{S})$ consisting of those functors which send $\mathcal{W}$ to equivalences, we get that the composite

$$\mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathscr{S}) \xrightarrow{L} \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$$

factors uniquely over $\mathcal{C}[\mathcal{W}^{-1}]^{op}$. Hence we get the desired commutative square. To see that the horizontal maps are Yoneda embeddings, we first note that due to the universal property of $\mathcal{C}[\mathcal{W}^{-1}]^{op}$ it is sufficient to show it when restricted to $\mathcal{C}^{op}$. Hence it is sufficient to show that the upper horizontal map is the Yoneda embedding. Recall that the Yoneda embedding in $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$ is given by sending X to $\mathrm{Map}_{\mathcal{C}}(-, X)$, so the desired follows from the first equation in this proof. □

We are now ready to prove theorem 1.3, which will be cut into two proofs. We will still let L denote the left adjoint to the inclusion as above.

Proof of theorem 1.3.ii. First we wish to give a better description of the left adjoint L, by forming another functor and show it has to be this localization. Let $F \in \mathrm{Fun}(\mathcal{C}, \mathscr{S})$ and define the functor

$$\begin{aligned} L'(F) : \mathcal{C} &\to \mathscr{S} \\ X &\mapsto \mathrm{colim}_{Y \in \mathcal{D}_{/X}} F(\mathrm{cofib}(Y \to X)). \end{aligned}$$

Since $\mathcal{D}$ in particular is stable we know that $\mathcal{D}$ has all finite colimits. By [Lur09, 1.2.13.8(1)] we have that $\mathcal{D}_{/X}$ has all small colimits, so in particular it is filtered. Now, if we have $f : Y \to X$ in $\mathcal{C}$ we get in a canonical way a morphism $F(X) \to \mathrm{colim}_{Y \in \mathcal{D}_{/X}}(F(\mathrm{cofib}(f)))$. This gives us a natural transformation $F \to L'(F)$ which is functorial in F.

Now, assume that F sends arrows in $\mathcal{W}$ to equivalences, i.e. $F \in \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$. Let $f : Y \to X$ be in $\mathcal{D}_{/X}$ and consider the successive pushout diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) & \longrightarrow & \mathrm{cofib}(g). \end{array}$$

Since $\mathcal{D}$ is stable and the outer diagram is a pushout as well, it follows that $\mathrm{cofib}(g) = \Sigma Y \in \mathcal{D}$, so $g \in \mathcal{W}$, hence sent to an equivalence by F. So in this case the natural transformation $F \to L'(F)$ is an equivalence, so in general $L'(F)$ sends arrows in $\mathcal{W}$ to equivalences. So we get that L' considered as an endofunctor on $\mathrm{Fun}(\mathcal{C}, \mathscr{S})$ has essential image given by $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S})$. This means that [Lur09, 5.2.7.4(3)] is satiesfied, so it follows that L' is left adjoint to

$$L'(\mathrm{Fun}(\mathcal{C}, \mathscr{S})) \simeq \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathscr{S}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathscr{S}),$$

so $L' \simeq L$. This lets us apply lemma 1.4, which gives us

$$\mathrm{Map}_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{X}, \overline{Y}) \simeq L(\mathrm{Map}_{\mathcal{C}}(X, Y)) \simeq \mathrm{colim}_{Z \in \mathcal{D}_{/X}} \mathrm{Map}_{\mathcal{C}}(X, \mathrm{cofib}(Z \to Y))$$

which proves theorem 1.3.ii. □

Proof of theorem 1.3.i and iii. First we will show that $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is exact. Let $K \to \mathcal{C}$, $i \mapsto x(i)$ be a finite diagram. Then we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{(\mathrm{colim} x(i))}, \overline{Y}) &\simeq \mathrm{colim}_{Z \in \mathcal{D}_{/Y}}(\mathrm{colim}\ x(i), \mathrm{cofib}(Z \to Y)) \\
&\simeq \mathrm{colim}_{Z \in \mathcal{D}_{/Y}}(\lim_{i \in K} \mathrm{Map}_{\mathcal{C}}(x(i), \mathrm{cofib}(Z \to Y))) \\
&\simeq \lim_{i \in K}(\mathrm{colim}_{Z \in \mathcal{D}_{/Y}} \mathrm{Map}_{\mathcal{C}}(x(i), \mathrm{cofib}(Z \to Y))) \\
&\simeq \lim_{i \in K}(\mathrm{Map}_{\mathcal{C}[\mathcal{W}^{-1}]}(\overline{x(i)}, \overline{Y})) \\
&\simeq \mathrm{Map}_{\mathcal{C}[\mathcal{W}^{-1}]}(\mathrm{colim}_{i \in K} \overline{x(i)}, \overline{Y}),
\end{aligned}$$

where we have used that filtered colimits commutes with finite limits in $\mathscr{S}$ and theorem 1.3(ii). The Yoneda lemma then implies that

$$\overline{\mathrm{colim}\ x(i)} \simeq \mathrm{colim}\ \overline{x(i)},$$

hence $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ commutes with finite colimits. By considering the opposite categories we get that it also commutes with finite limits, hence we conclude that it is exact.

We can now use this to prove that $\mathcal{C}[\mathcal{W}^{-1}]$ is a stable ∞-category. We know by assumption that $\mathcal{C}$ is stable, so using that $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is exact, we can show that $\mathcal{C}[\mathcal{W}^{-1}]$ is pointed, preadditive and additive. This is done by considering the structure up in $\mathcal{C}$ and then mapping it down to $\mathcal{C}[\mathcal{W}^{-1}]$.

Since $\mathcal{C}$ admits all pushouts and these are preserved by $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, we get that $\mathcal{C}[\mathcal{W}^{-1}]$ admits all pushouts aswell, so it follows by [Lur09, 4.4.2.4] that it has all finite colimits. Dually we get that $\mathcal{C}[\mathcal{W}^{-1}]$ has all finite limits. That means that both the loop space functor Ω and the suspension functor Σ are defined on $\mathcal{C}[\mathcal{W}^{-1}]$. Since $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ commutes with all finite limits and colimits, we get that both Σ and Ω on $\mathcal{C}[\mathcal{W}^{-1}]$ commutes with the loop space and suspension functors on $\mathcal{C}$. To see that Σ and Ω are equivalences in $\mathcal{C}[\mathcal{W}^{-1}]$, we note that this is the case in $\mathcal{C}$, and that it is sufficient to prove it when restricted to $\mathcal{C}$.

Now, the last part of theorem 1.3.i follows by the universal property of $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$.

The statement in theorem 1.3.iii follows by checking that it satiesfies [Lur09, 5.2.7.4(3)]. □

Using this theorem we will from now on use the notation $\mathcal{C}/\mathcal{D}$ instead of $\mathcal{C}[\mathcal{W}^{-1}]$, and when we mention the functor $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ we refer to the canonical functor from the Dwyer-Kan localization.

Theorem 1.5. *Let $\mathcal{C}$ be a small, stably symmetric monoidal stable ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a $\otimes$-ideal.*

i) There is a unique way to simultaneously endow $\mathcal{C}/\mathcal{D}$ and $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ with a symmetric monoidal structure. If $\mathcal{E}$ is another symmetric monoidal stable ∞-category, then composition with $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ induces an equivalence between $Fun^{ex}_{Lax}(\mathcal{C}/\mathcal{D}, \mathcal{E})$ and the full subcategory of $Fun^{ex}_{Lax}(\mathcal{C}, \mathcal{E})$ of those functors which sends all objects of $\mathcal{D}$ to 0.

ii) Assume that $\mathcal{E}$ is a presentably symmetric monoidal stable ∞-category. Then the full inclusion $Fun^{ex}_{Lax}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subseteq Fun^{ex}_{Lax}(\mathcal{C}, \mathcal{E})$ admits a left adjoint, which hence is a localization. The corresponding localization functor

$$Fun^{ex}_{Lax}(\mathcal{C}, \mathcal{E}) \to Fun^{ex}_{Lax}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subseteq Fun^{ex}_{Lax}(\mathcal{C}, \mathcal{E})$$

is given by

$$(F : \mathcal{C} \to \mathcal{E}) \mapsto (\mathcal{C}/\mathcal{D} \to Ind(\mathcal{C}) \xrightarrow{Ind(F)} Ind(\mathcal{E}) \to \mathcal{E}),$$

where the functor $\mathcal{C}/\mathcal{D} \to Ind(\mathcal{C})$ comes from theorem 1.3(ii) and the functor $Ind(\mathcal{E}) \to \mathcal{E}$ is taking the colimit in $\mathcal{E}$. Both of these two functors are canonically lax symmetric monoidal functors.

Proof.

i) We wish to apply [NS18, A.5]. Using that $\mathcal{C}/\mathcal{D} \simeq \mathcal{C}[\mathcal{W}^{-1}]$ we need to prove that for any $f \in \mathcal{W}$ the tensor product $f \otimes X$ for any $X \in \mathcal{C}$ is again in $\mathcal{W}$. This is true since we have assumed that $\mathcal{C}$ is stably symmetric monoidal, so

$$\mathrm{cone}(f \otimes X) \simeq \mathrm{cone}(f) \otimes X.$$

Since $\mathcal{D}$ is a $\otimes$-ideal this is an object of $\mathcal{D}$, so $f \otimes X \in \mathcal{W}$. So Proposition $A.5$ gives us a symmetric monoidal structure on $\mathcal{C}/\mathcal{D}$ and a symmetric monoidal refinment on $\mathcal{C} \to \mathcal{C}/\mathcal{D}$. Using the universal property of $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ and the equivalence given by $A.5(v)$ we further get the uniqueness of the refinment. We also get that this refinment satiesfies that for every symmetric monoidal ∞-category $\mathcal{E}$, the functor

$$\mathrm{Fun}_{\mathrm{Lax}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{Lax}}(\mathcal{C}, \mathcal{E})$$

is fully faithful with essential image those functors which send $\mathcal{W}$ to equivalences in $\mathcal{E}$. If $\mathcal{E}$ is stable, we see by theorem 1.3(i) that this functor induces equivalences between the respective full subcategories of exact functors, since we can check exactness after forgetting the lax symmetric monoidal structures.

ii) First we wish to show that the functor

$$\mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Lax}}(\mathcal{C},\mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Lax}}(\mathcal{C}/\mathcal{D},\mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Lax}}(\mathcal{C},\mathcal{E})$$

is well-defined, by proving that all functors in the composition

$$\mathcal{C}/\mathcal{D} \to \mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{Ind}(F)} \mathrm{Ind}(\mathcal{E}) \to \mathcal{E}$$

are naturally lax symmetric monoidal. The main tool for this is [Lur17, 7.3.2.7], which gives us that since $\mathrm{Ind}(\mathcal{C}/\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$ is right adjoint to the symmetric monoidal projection $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}/\mathcal{D})$ by [NS18, I.3.5], we have that it is lax symmetric monoidal. [Lur17, 7.3.2.7] further gives us that both

$$\mathcal{C}/\mathcal{D} \to \mathrm{Ind}(\mathcal{C}), \quad \mathrm{Ind}(\mathcal{E}) \to \mathcal{E}$$

are lax symmetric monoidal, so since composition of lax symmetric monoidal functors are again lax symmetric monoidal, it follows that the functor is well-defined.

To show that this is a localization we wish to prove that it satiesfies condition (3) of [Lur09, 5.2.7.4]. But it suffices to check this condition without the lax symmetric monoidal structure, since lax symmetric monoidal transformation is an equivalence if and only if the underlying natural transformation is an equivalence. By theorem 1.3(iii) and [Lur09, 5.2.7.4] we know that it holds in that case, hence it is the localization as desired.

□

This means that given any lax symmetric monoidal functor $F : \mathcal{C} \to \mathcal{E}$, the universal approximation over $\mathcal{C}/\mathcal{D}$ admits a unique lax symmetric monoidal structure, such that the relevant natural transformation is lax symmetric monoidal.

Now, we wish to apply the above theory about the Verdier quotient to the setup of theorem 1.1, so to $\mathcal{C} = \mathrm{Sp}^{BG}$.

Definition 1.6. Let $\mathrm{Sp}^{BG}_{\mathrm{Ind}} \subseteq \mathrm{Sp}^{BG}$ be the stable subcategory generated by spectra of the form $\oplus_{g\in G} X$ with permutation G-action, where $X \in \mathrm{Sp}$. Spectra of the form $\oplus_{g\in G} X$ are called *induced spectra*.

So $\mathrm{Sp}^{BG}_{\mathrm{Ind}}$ is the smallest subcategory of Sp^{BG} which contains all induced spectra and is stable, hence contains all induced spectra and cones of maps between these. We will need the following properties:

Lemma 1.7. *Let $X \in Sp^{BG}$.*

i) If $X \in Sp^{BG}_{Ind}$, then $X^{tG} \simeq 0$.

ii) For all $Y \in Sp^{BG}_{Ind}$, the tensor product $X \otimes Y \in Sp^{BG}_{Ind}$, so $Sp^{BG}_{Ind} \subseteq Sp^{BG}$ is a $\otimes$-ideal.

iii) The natural maps

$$colim_{Y \in (Sp^{BG}_{Ind})_{/X}} Y \to X$$
$$colim_{Y \in (Sp^{BG}_{Ind})_{/X}} (cofib(Y \to X))^{hG} \to colim_{Y \in (Sp^{BG}_{Ind})_{/X}} (cofib(Y \to X))^{tG} \simeq X^{tG}$$

are both equivalences.

The proof will be omitted here, but can be found in [NS18, I.3.8]. We are now ready to prove the main theorem of this section, but ignoring the possible set-theoretic issues that might arise from the fact that Sp^{BG} is not a small ∞-category. The rest of this proof can be found in [NS18, I.3.1.]

Proof of theorem 1.1. Using that $\mathrm{Sp}^{BG}_{\mathrm{Ind}}$ is a $\otimes$-ideal of Sp^{BG} by lemma 1.7(ii) we may take $\mathcal{C} = \mathrm{Sp}^{BG}$, $\mathcal{D} = \mathrm{Sp}^{BG}_{\mathrm{Ind}}$ in theorem 1.5. Using that $X^{tG} \simeq 0$ for $X \in \mathrm{Sp}^{BG}_{\mathrm{Ind}}$ by lemma 1.7(i) we get that $(-)^{tG}$ factors over another functor, which we will still denote $(-)^{tG} : \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \to \mathrm{Sp}$, i.e.

$$\begin{array}{ccc} \mathrm{Sp}^{BG} & \xrightarrow{(-)^{tG}} & \\ \downarrow & \searrow & \\ \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} & \xrightarrow[(-)^{tG}]{} & \mathrm{Sp}. \end{array}$$

Given a lax symmetric monoidal structure on $(-)^{tG}$ with a lax symmetric monoidal transformation

$$(-)^{hG} \Rightarrow (-)^{tG}$$

we get by theorem 1.5(i) that this gives rise to an exact lax symmetric monoidal functor

$$H : \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \to \mathrm{Sp},$$

such that the natural transformation from $(-)^{hG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$ to the composition

$$\mathrm{Sp}^{BG} \to \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \xrightarrow{H} \mathrm{Sp}$$

is a lax symmetric monoidal transformation.

On the other hand, take

$$F := (-)^{hG} : \mathcal{C} = \mathrm{Sp}^{BG} \to \mathrm{Sp} = \mathcal{E}$$

in theorem 1.5(ii), then we get that there is a universal lax symmetric monoidal functor $H' : \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \to \mathrm{Sp}$ with a natural lax symmetric monoidal transformation from $(-)^{hG}$ to the composition

$$\mathrm{Sp}^{BG} \to \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \xrightarrow{H'} \mathrm{Sp}.$$

We wish to show that H and H' agree. Since the localization functors of 1.5(ii) and 1.3(iii) are compatible, it is sufficient to check it without the multiplicative structure. Using the universal property of H' we get that there is a unique natural transformation $H' \to (-)^{tG}$, which we want to show is an equivalence. By theorem 1.3(iii) we have the following description of the localization functor:

$$\begin{aligned} \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{BG}, \mathrm{Sp}) &\to \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}}, \mathrm{Sp}) \subseteq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{BG}, \mathrm{Sp}) \\ (F : \mathrm{Sp}^{BG} \to \mathrm{Sp}) &\mapsto (\mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} \xrightarrow{i} \mathrm{Ind}(\mathrm{Sp}^{BG}) \xrightarrow{\mathrm{Ind}(F)} \mathrm{Ind}(\mathrm{Sp}) \to \mathrm{Sp}), \end{aligned}$$

where we know by theorem 1.3(ii) that i is given by

$$\begin{aligned} \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{Ind}} &\to \mathrm{Ind}(\mathrm{Sp}^{BG}) \\ \overline{Y} &\mapsto \mathrm{colim}_{Z \in (\mathrm{Sp}^{BG}_{\mathrm{Ind}})_{/Y}} (\mathrm{cofib}(Z \to Y)). \end{aligned}$$

By lemma 1.7(iii) we get the desired equivalence. □

1.2 The Tate Diagonal

Throughout this section, let p be a prime and for $X \in \mathrm{Sp}$ let $X^{\otimes p}$ denote the p-fold self tensor product. In general there is no natural diagonal map in Sp in the sense, that for $X \in \mathrm{Sp}$ there is not necessarily a symmetric map $X \to X \otimes X$. In this section we wish to introduce a substitute for this map, in the stable case. This map is called the Tate diagonal and it can be shown to admit a unique lax symmetric monoidal structure. An interesting property of the Tate diagonal is further that it has a deep connection to p-completion, which we will show in the end of this section.

Proposition 1.8. *The functor*

$$\begin{aligned} T_p : Sp &\to Sp \\ X &\mapsto (X^{\otimes p})^{tC_p}, \end{aligned}$$

is exact, where $X^{\otimes p}$ has C_p-action given by cyclic permutation of the factors.

Proof. First we show the weaker statement that T_p preserves sums. Using that $(-)^{tC_p}$ is exact we have

$$\begin{aligned} T_p(X_0 \oplus X_1) &\simeq \left(\oplus_{(i_1,\dots,i_p)\in\{0,1\}^p} X_{i_1} \otimes \cdots \otimes X_{i_p}\right)^{tC_p} \\ &\simeq (X_0^{\otimes p})^{tC_p} \oplus (X_1^{\otimes p})^{tC_p} \oplus \left(\bigoplus_{(i_1,\dots,i_p)\in\{0,1\}^p\setminus\{(0,\dots,0),(1,\dots,1)\}} X_{i_1} \otimes \cdots \otimes X_{i_p} \right)^{tC_p} \\ &\simeq T_p(X_0) \oplus T_p(X_1) \oplus \bigoplus_{[i_1,\dots,i_p]} \left(\bigoplus_{(i_1,\dots,i_p)\in[1_i,\dots,i_p]} X_{i_1} \otimes \cdots \otimes X_{i_p} \right)^{tC_p}, \end{aligned}$$

where in the last sum $[i_1, ..., i_p]$ runs through a set of representatives of orbits of the cyclic C_p action on the set $S := \{0,1\}^p \backslash \{(0,...,0),(1,...,1)\}$. Since p is prime, these orbits are all isomorphic to C_p, so each

$$\bigoplus_{(i_1,...,i_p)\in[i_1,...,i_p]} X_{i_1} \otimes \cdots \otimes X_{i_p}$$

is a C_p-spectrum induced from $\{1\} \subseteq C_p$. From lemma 1.7(i) we know that the Tate construction vanishes on induced spectra, so projection to the first two summands gives us an equivalence

$$T_p(X_0 \oplus X_1) \simeq T_p(X_0) \oplus T_p(X_1).$$

Next we wish to show that T_p commutes with extensions. Consider a fiber sequence

$$X_0 \to \tilde{X} \to X_1$$

in Sp. This gives us a fibration sequence

$$X_0^{\otimes p} \to \bigoplus_{(i_1,...,i_p)\in\mathbb{I}_1} X_{i_1} \otimes \cdots \otimes X_{i_p} \to \cdots \to \tilde{X}^{\otimes p} \to X_1^{\otimes p},$$

where $\mathbb{I}_n = \{[i_1, ..., i_p] \in S/C_p | \sum_{k=1}^p i_k = n\}$ for $1 \leq n \leq p-1$. Using that $(-)^{tC_p}$ is exact and vanishes on induced spectra we get that it kills all imediate steps $\oplus_{(i_1,...,i_p)\in\mathbb{I}_n} X_{i_1} \otimes \cdots \otimes X_{i_p}$, so we get a fiber sequence

$$(X_0^{\otimes p})^{tC_p} \to (\tilde{X}^{\otimes p})^{tC_p} \to (X_1^{\otimes p})^{tC_p},$$

hence T_p commutes with extensions, which is sufficient. □

Proposition 1.9. *Consider the full subcategory $Fun^{ex}(Sp, Sp) \subseteq Fun(Sp, Sp)$ of exact functors, and let $id_{Sp} \in Fun^{ex}(Sp, Sp)$ denote the identity functor. For any $F \in Fun^{ex}(Sp, Sp)$, evaluation on the sphere spectrum $\mathbb{S} \in Sp$ induces an equivalence*

$$Map_{Fun^{ex}(Sp,Sp)}(id_{Sp}, F) \to Map_{Sp}(\mathbb{S}, F(\mathbb{S})) = \Omega^\infty F(\mathbb{S}).$$

Proof. Using that $\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp},\mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp})$ we get an equivalence

$$\begin{aligned} \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp}) &\xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp},\mathscr{S}) \\ (F : \mathrm{Sp} \to \mathrm{Sp}) &\mapsto (\mathrm{Sp} \xrightarrow{F} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathscr{S}) \end{aligned}$$

by [Lur17, 1.4.2.23]. So we get that

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp})}(id_{\mathrm{Sp}}, F) &\simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp},\mathscr{S})}(\Omega^\infty \circ id_{\mathrm{Sp}}, \Omega^\infty \circ F) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp},\mathscr{S})}(\Omega^\infty, \Omega^\infty \circ F) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathscr{S})}(\Omega^\infty, \Omega^\infty \circ F), \end{aligned}$$

where we have used that $\mathrm{Fun}^{\mathrm{Lex}}(\mathrm{Sp},\mathscr{S})$ is a full subcategory of $\mathrm{Fun}(\mathrm{Sp},\mathscr{S})$. Using the Yoneda map and the fact that Ω^∞ is corepresented by $\mathbb{S}$, i.e. $\Omega^\infty \simeq \mathrm{Map}(\mathbb{S}, -)$, we get that this is equivalent to $\Omega^\infty F(\mathbb{S})$ as desired. □

Corollary 1.10. *The space of natural transformations from id_{Sp} to T_p, considered as endofunctors on Sp, is equivalent to $\Omega^\infty T_p(\mathbb{S}) = Hom_{Sp}(\mathbb{S}, T_p(\mathbb{S}))$.*

Proof. By proposition 1.8 we know that $T_p : \mathrm{Sp} \to \mathrm{Sp}$ is exact, so it follows by proposition 1.9. □

Definition 1.11. We define the *Tate diagonal* as the natural transformation

$$\begin{aligned}\Delta_p : id_{\mathrm{Sp}} &\to T_p \\ X &\mapsto (X^{\otimes p})^{tC_p},\end{aligned}$$

which under the equivalence of proposition 1.9 corresponds to the map

$$\mathbb{S} \to T_p(\mathbb{S}) = \mathbb{S}^{tC_p}$$

which is the composition $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$, where C_p acts trivially on $\mathbb{S}$.

Note that using theorem 1.1 we can endow T_p with a canonical lax symmetric monoidal structure.

Lemma 1.12. *There is a unique lax symmetric monoidal transformation $\Delta_p : id_{Sp} \to T_p$. The underlying transformation of functors is given by the Tate diagonal.*

Proof. By [Nik16, 6.9(1)] we know that $id_{\mathrm{Sp}} : \mathrm{Sp} \to \mathrm{Sp}$ is initial among exact lax symmetric endofunctors on Sp, hence we get that there exists an unique lax symmetric monoidal transformation $id_{\mathrm{Sp}} \to T_p$. Using the proof [NR12, 6.9], we see that the underlying transformation of this unique lax symmetric monoidal transformation corresponds to $\mathbb{S} \to T_p(\mathbb{S})$, which also corresponds to Δ_p. □

A very interesting property of the Tate diagonal is that it exhibits $(X^{\otimes p})^{tC_p}$ as the p-completion of X, so before we move on we wish to give a brief introduction to p-complete spectra. For details see [Lur10] or [Bau07]. Fix some spectrum E.

Definition 1.13. We say that $X \in \mathrm{Sp}$ is *E-acyclic* if $X \otimes E \simeq 0$. We further say that X is *E-local* if for every E-acyclic spectra Y every map $Y \to X$ is nullhomotopic.

Let $\mathrm{Sp}_{E-\mathrm{acyc}}$ denote the full subcategory of Sp consisting of E-acyclic spectra. Then it can be shown that the inclusion $\mathrm{Sp}_{E-\mathrm{acyc}} \subseteq \mathrm{Sp}$ admits a right adjoint $G_E X$. So we have a counit $\varepsilon_X : G_E X \to X$, which gives us a cofiber sequence

$$G_E X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} \mathrm{cofib}(\varepsilon_X) =: L_E X.$$

For every $A \in \mathrm{Sp}_{E-\mathrm{acyc}}$ we get that

$$\mathrm{map}_{\mathrm{Sp}}(A, G_E X) \to \mathrm{map}_{\mathrm{Sp}}(A, X)$$

is a homotopy equivalence, so $\mathrm{map}_{\mathrm{Sp}}(A, L_E X) \simeq 0$. So $L_E X$ is E-local. This cofiber sequence is esentially unique, in the sense that there is a unique $\eta_X : X \to L_E(X)$ such that $L_E(X)$ is E-local and the fiber $\mathrm{fib}(\eta_X) = G_E X$ is E-acyclic. To see this, assume that we have two such cofiber sequences

$$G_E X \to X \xrightarrow{\eta_X} L_E X, \quad G \to X \xrightarrow{\eta'_X} L.$$

We can then form the following diagram

$$\begin{array}{ccccc}
\mathrm{fib}(f) & \longrightarrow & 0 & \longrightarrow & \mathrm{fib}(g) \\
\downarrow & & \downarrow & & \downarrow \\
G_E X & \longrightarrow & X & \xrightarrow{\eta_X} & L_E X \\
\downarrow f & & \| & & \downarrow g \\
G & \longrightarrow & X & \xrightarrow{\eta'_X} & L
\end{array}$$

by taking the fiber of the two lower fiber sequences, which gives us the upper fiber sequence. We get that $\mathrm{fib}(f)$ is also E-acyclic, so since $\mathrm{fib}(g) \simeq \Sigma\mathrm{fib}(f)$ we get that $\mathrm{fib}(g)$ is also E-acyclic. Hence $\mathrm{fib}(g) \to L_E X$ is 0, from where we get that $\mathrm{fib}(g) \simeq 0$, so $L_E X \simeq L$.

The functor L_E is called the *Bousfield localization* with respect to E. By [Lur10, 9] we get that L_E preserves homotopy colimits, so in particular we get that Bousfield localization is exact.

Definition 1.14. A morphism $f : X \to Y$ in Sp is said to be an *E-equivalence* if $E \otimes f : E \otimes X \to E \otimes Y$ is an equivalence in Sp. Given both E and X in Sp we define an *E-localization of* X to be a pair $(L_E X, X \to L_E X)$ which consist of an E-local spectrum $L_E X$ and an E-equivalence $X \to L_E X$.

It can be shown that the map $\eta_X : X \to L_E(X)$ from the cofiber above is the universal map to the E-acyclic spectrum $L_E X$ which is an E-equivalence. Hence η_X is an E-localization. Let p be a prime, then the Bousfield localization with respect to the Moore spectrum $M(\mathbb{Z}_{(p)})$ is called the *p-localization* $X_{(p)} := L_{M(\mathbb{Z}_{(p)})} X$. It can be shown that this is in fact something called a *smashing localization*, which means that we have

$$L_{M(\mathbb{Z}_{(p)})} X \simeq X \otimes L_{M(\mathbb{Z}_{(p)})} \mathbb{S} \simeq X \otimes M(\mathbb{Z}_{(p)}).$$

If we instead consider the Moore spectrum $M(\mathbb{Z}_p)$ and we assume that X is connective, then the $M(\mathbb{Z}_p)$-localization is given by

$$X_p^\wedge := L_{M(\mathbb{Z}_p)} X = \lim\{\cdots \to X \otimes M(\mathbb{Z}_p^2) \to X \otimes M(\mathbb{Z}_p)\},$$

and is called the *p-completion functor*.

Now, consider $\mathbb{S}$ and note that an element $p \in \pi_0(\mathbb{S})$ corresponds to a map $p : \mathbb{S} \to \mathbb{S}$. Then we have that $\mathbb{S}_{(p)} = L_{M(\mathbb{Z}_{(p)})}\mathbb{S} \simeq \mathrm{cofib}(\mathbb{S} \xrightarrow{p} \mathbb{S})$. The same can be done for the Eilenberg-Maclane spectrum HM for M p-torsion free. In that case we get that $(HM)_{(p)} \simeq \mathrm{cofib}(HM \xrightarrow{p} HM) \simeq H(M/p)$.

Theorem 1.15. *Assume $X \in Sp$ is bounded below. Then the map*

$$\Delta_p : X \to (X^{\otimes p})^{tC_p}$$

exhibits $(X^{\otimes p})^{tC_p}$ as the p-completion.

We note that in the case $X = \mathbb{S}$ this is the Segal conjecture for the group C_p. To prove this theorem, we will need the following result from [NR12], which we will not prove:

Theorem 1.16. *$\Delta_p : H\mathbb{Z}_p \to T_p(H\mathbb{Z}_p)$ is an equivalence.*

Proof of theorem 1.15. First we note that $\Sigma : \mathrm{Sp} \to \mathrm{Sp}$ is an equivalence since Sp is stable, so by finitely many applications of Σ we may shift X such that we can assume that X is connective, i.e. homotopy groups in negative degrees vanish. We claim that

$$X \simeq \lim_n \tau_{\leq n} X, \quad (X^{\otimes p})^{tC_p} \simeq \lim_n \left(((\tau_{\leq n} X)^{\otimes p})^{tC_p} \right).$$

That $X \simeq \lim_n \tau_{\leq n} X$ is clear since $\pi_i(\tau_{\leq n} X) \cong \pi_i(X)$ for all $i \leq n$. To prove the second equivalence we have to show that both

$$\begin{aligned} (X^{\otimes p})_{hC_p} &\to \lim_n((\tau_{\leq n} X)^{\otimes p})_{hC_p} \\ (X^{\otimes p})^{hC_p} &\to \lim_n((\tau_{\leq n} X)^{\otimes p})^{hC_p} \end{aligned}$$

are equivalences, because we then will obtain an equivalence of cofibers induced by the norm map

$$\begin{array}{ccccc} (X^{\otimes p})_{hC_p} & \xrightarrow{Nm} & (X^{\otimes p})^{hC_p} & \longrightarrow & (X^{\otimes p})^{tC_p} \\ \downarrow \simeq & & \downarrow \simeq & & \vdots \simeq \\ \lim_n((\tau_{\leq n} X)^{\otimes p})_{hC_p} & \xrightarrow{Nm} & \lim_n((\tau_{\leq n} X)^{\otimes p})^{hC_p} & \longrightarrow & \lim_n((\tau_{\leq n} X)^{\otimes p})^{tC_p}. \end{array}$$

To prove this, we first want to prove $X^{\otimes p} \simeq \lim_n((\tau_{\leq n} X)^{\otimes p})$. We have $c_n : X \to \tau_{\leq n} X$ which is n-connected, so $\mathrm{cofib}(c_n)$ is n-connected. We will show that the composition

$$X \otimes X \xrightarrow{c_n \otimes id_X} \tau_{\leq n} X \otimes X \xrightarrow{id_{\tau_{\leq n} X} \otimes c_n} \tau_{\leq n} X \otimes \tau_{\leq n} X$$

is again n-connected. Since

$$\tau_{>n} X \to X \to \tau_{\leq n} X$$

is a distinguished triangle, and smashing preserves colimits, we have that

$$\tau_{\leq n} X \otimes \tau_{>n} X \to \tau_{\leq n} X \otimes X \xrightarrow{id_X \otimes c_n} \tau_{\leq n} X \otimes \tau_{\leq n} X$$

is a distinguished triangle as well. We have that $\mathrm{cofib}(id_X \otimes c_n) = \Sigma(\tau_{\leq n} X \otimes \tau_{>n} X)$. Using the Tor spectral sequence we get

$$E^2_{p,q} = \bigsqcup_{s+t=q} \mathrm{Tor}_p(\pi_s(\tau_{\leq n} X), \pi_t(\tau_{>n} X)) \Rightarrow \pi_{p+q}(\tau_{\leq n} X \otimes \tau_{>n} X).$$

For $q \leq n$ either $s < 0$ or $t \leq n$, so either $\pi_s(\tau_{\leq n}X)$ or $\pi_t(\tau_{>n})$ is 0. So $\Sigma(\tau_{\leq n}X \otimes \tau_{>n}X)$ is n-connected, so $id_X \otimes c_n$ is n-connected. With the same argument we get that $c_n \otimes id_{\tau_{\leq n}X}$ is n-connected, so this is also true for the composite. Repeated use of this result gives us that the comparison map

$$X^{\otimes p} \to (\tau_{\leq n}X)^{\otimes p}$$

is n-connected. This gives us isomorphisms

$$\pi_i(X^{\otimes p}) \xrightarrow{\sim} \pi_i((\tau_{\leq n}X)^{\otimes p})$$

for all $i \leq n$, which gives isomorphisms

$$\pi_i(X^{\otimes p}) \xrightarrow{\sim} \pi_i(\lim_n((\tau_{\leq n}X)^{\otimes p})$$

for all i, hence $X^{\otimes p} \simeq \lim_n((\tau_{\leq n}X)^{\otimes p})$. Since homotopy fixed points commutes with limits we get the equivalence

$$(X^{\otimes p})^{hC_p} \simeq \lim_n((\tau_{\leq n}X)^{\otimes p})^{hC_p}.$$

To get the equivalence when considering homotopy orbits instead of homotopy fixed points, we first note that we have shown that $X^{\otimes p} \simeq \lim_n((\tau_{\leq n}X)^{\otimes p})$. Since $\lim(-)$, when considered as a functor, commutes with finite limits, we get that it commutes with finite colimits on Sp, since Sp is stable. So in particular we get the equivalence

$$(X^{\otimes p})_{hC_p} \simeq \lim_n((\tau_{\leq n}X)^{\otimes p})_{hC_p}.$$

By the above proved claim, we can now assume that X is bounded. We want to reduce to the case where $X = HM$ is an Eilenberg-Maclane spectrum. We have already reduced to the case that X is bounded and connective, i.e. the case where X is concentrated in the range 0 to n for some $n \in \mathbb{Z}_{\geq 0}$. For $n = 0$ it is clear that we may assume that X is equivalent to an Eilenberg-Maclane spectrum. We wish to proceed by induction on n, so assume we have proved the statement for $n-1$. We know we have a map from the n-truncation $c_{n-1} : X \to \tau_{\leq n-1}X$ which gives us a fiber sequence

$$\mathrm{fib}(c_{n-1}) \to X \xrightarrow{c_{n-1}} \tau_{\leq n-1}X.$$

A model for this fiber is $\Sigma^n H\pi_n X$, which is an Eilenberg-Maclane spectrum. The claim then follows by exactness of T_p by proposition 1.8, and exactness of Bousfield localization, which in particular gives us that p-completion is exact. So assume $X = HM$ for some abelian group M. We want to further reduce to the case where M is finitely generated and p-torsion free. Write $M = \mathrm{colim}_i M_i$, where M_i are finitely generated subgroups of M. We have that

$$\begin{aligned}(\mathrm{colim}_i M_i)^{tC_p} &= \mathrm{cofib}((\mathrm{colim}_i M_i)_{hC_p} \xrightarrow{Nm} (\mathrm{colim}_i M_i)^{hC_p})) \\ &\simeq \mathrm{cofib}(\mathrm{colim}_i((M_i)_{hC_p}) \to \mathrm{colim}_i(M_i^{hC_p})) \\ &\simeq \mathrm{colim}_i(M_i^{tC_p}),\end{aligned}$$

where we have used that $(-)_{hC_p}$ is left adjoint and $H^p(C_p, M)$ preserves filtered colimits. This gives us that T_p preserves filtered colimits, so we may reduce to the case where M is finitely generated. So we have a short exact sequence

$$0 \longrightarrow \ker(\theta) \hookrightarrow \mathbb{Z}^{\otimes n} \xrightarrow{\theta} M \longrightarrow 0$$

where we note that $\ker(\theta)$ in particular is p-torsion free. Again using that T_p is exact by proposition 1.8 together with the fact that $H(-)$ takes short exact sequences to fiber sequences, we get fiber sequences

$$\begin{array}{ccccc} T_p(H(\ker(\theta))) & \hookrightarrow & T_p(H(\mathbb{Z}^{\otimes n})) & \xrightarrow{\theta} & T_p(H(M)) \\ {\scriptstyle T_p}\uparrow & & {\scriptstyle T_p}\uparrow & & \uparrow \\ H(\ker(\theta)) & \hookrightarrow & H(\mathbb{Z}^{\otimes n}) & \xrightarrow{\theta} & HM. \end{array}$$

Since equivalences on the left and center maps will induce an equivalence to the right, we may assume that $X = HM$ where M is a finitely generated p-torsion free abelian group.

Now, using [NS18, I.2.9] we have that $(X^{\otimes p})^{tC_p}$ is p-complete, so it is sufficient to show that Δ_p is a p-adic equivalence, i.e. an equivalence after smashing with $\mathbb{S}_{(p)}$. Using exactness we have

$$\begin{aligned} HM \otimes \mathbb{S}_{(p)} &\simeq HM \otimes \operatorname{cofib}(\mathbb{S} \xrightarrow{p} \mathbb{S}) \\ &\simeq \operatorname{cofib}(HM \otimes \mathbb{S} \xrightarrow{id \otimes p} HM \otimes \mathbb{S}) \\ &\simeq \operatorname{cofib}(HM \xrightarrow{p} HM) \\ &\simeq H(M/p). \end{aligned}$$

Note that $p : HM \to HM$ could be 0 if M was not p-torsion free, so it is necessary to reduce to this case. On the other side, again by exactness, we get

$$\begin{aligned} (HM^{\otimes p})^{tC_p} \otimes \mathbb{S}_{(p)} &\simeq ((HM \otimes \mathbb{S}_{(p)})^{\otimes p})^{tC_p} \\ &\simeq (H(M/p)^{\otimes p})^{tC_p}, \end{aligned}$$

so it is sufficient to prove that Δ_p is an equivalence for $X = H(M/p)$.

We have that M/p is a finite direct sum of copies of $\mathbb{Z}_p$, so using the exactness of T_p from proposition 1.8 and the equivalence from theorem 1.16 we are done.

□

2 Topological Hochschild homology

A very useful tool in algebraic K-theory is topological Hochschild homology which is an additive invariant. We wish to consider the general abstract construction as presented in [NS18, Chp.3] on higher algebraic analogues of algebras in symmetric monoidal ∞-categories. Before we will be able to do this, we will have to introduce cyclic objects and some properties related to these, since Toplogical Hochshild homology is a specific colimit of a functor, which is constructed by a specific cyclic object induced from some algebra object.

2.1 Cyclic objects and active parts

In this section we wish to introduce some of the preliminaries which are necessary for the next section where we will construct topological Hochschild homology. Therefore many of the proofs will be omitted, but references are given. We will mainly follow [NS18, App. B]. First we need to introduce several categories. Let Fin_* denote the category of finite pointed sets and for all $n \geq 0$ write $\langle n \rangle \in \mathrm{Fin}_*$ for the finite set $\{0, 1, \cdots, n\}$ pointed at 0. Then we define the *associative operad* $\mathrm{Ass}^{\otimes}$ as follows: The objects are finite pointed sets, which we denote by $\langle n \rangle_{\mathrm{Ass}}$ when considered as object in $\mathrm{Ass}^{\otimes}$. $\mathrm{Hom}_{\mathrm{Ass}^{\otimes}}(\langle n \rangle_{\mathrm{Ass}}, \langle m \rangle_{\mathrm{Ass}})$ is the set of all maps $f \in \mathrm{Fin}_*(\langle n \rangle, \langle m \rangle)$ together with a linear ordering on $f^{-1}(i) \subseteq \langle n \rangle$ for all $i \in \{1, 2, \cdots, m\} \subseteq \langle m \rangle$. To understand the composition let $f : S \to T$ be a map of finite sets, with a linear ordering on $t_1 < t_2 < \cdots < t_m$ and on each of the preimages $f^{-1}(t_1), f^{-1}(t_2), \cdots, f^{-1}(t_m)$. Then there is a natural linear ordering on S, by first having the ordering of $f^{-1}(t_1)$, then the ordering of $f^{-1}(t_2)$ and so on. Note that there is a natural functor

$$F : \mathrm{Ass}^{\otimes} \to \mathrm{Fin}_*$$

which forgets the linear ordering.

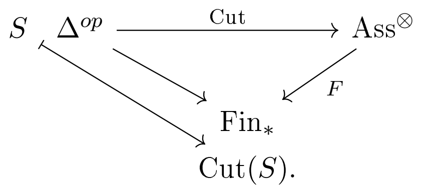
Let Δ denote the category of totally ordered non-empty finite sets, and for $n \geq 0$ let $[n] = \{0, 1, \cdots, n\} \in \Delta$. Then we have a natural functor

$$\begin{aligned} \Delta^{op} &\to \mathrm{Fin}_* \\ S &\mapsto \mathrm{Cut}(S), \end{aligned}$$

where $\mathrm{Cut}(S)$ denotes the set of disjoint decompositions $S = S_1 \sqcup S_2$ which satiesfies that for all $s_1 \in S_1, s_2 \in S_2$ we have $s_1 < s_2$ and we identify $\emptyset \sqcup S \simeq S \sqcup \emptyset$. We point $\mathrm{Cut}(S)$ at $\emptyset \sqcup S$. In particular we get that $[n] \mapsto \langle n \rangle$. We wish to construct a compatible functor $\mathrm{Cut} : \Delta^{op} \to \mathrm{Ass}^{\otimes}$ over Fin_*. On objects we simply define it as $\mathrm{Cut}([n]) = \langle n \rangle$. For $\alpha \in \Delta([n], [m])$ we define $\mathrm{Cut}(\alpha)$ by

$$\mathrm{Cut}(\alpha)(i) = \begin{cases} j & \text{If there exists } j \text{ such that } \alpha(j-1) < i < \alpha(j) \\ * & \text{otherwise} \end{cases}$$

where we endow each $\mathrm{Cut}(\alpha)^{-1}(\{j\})$ with the linear ordering induced by its inclusion into $\langle n \rangle^{\circ} = \{1, 2, \cdots, n\} \in \Delta^{op}$. So we have a commutative triangle



Definition 2.1. Let PoSet denote the category of partially ordered sets and non-decreasing maps, and let $\mathbb{Z}$PoSet be the category of objects in PoSet equipped with a $\mathbb{Z}$-action. Then we define the *Paracyclic category* Λ_∞ as the full subcategory of $\mathbb{Z}$PoSet, consisting of all objects isomorphic to $(1/n)\mathbb{Z}$ for $n \geq 1$.

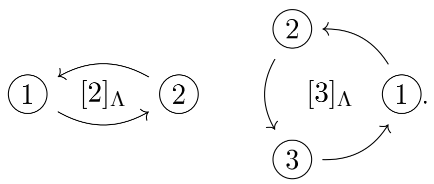
In this definition we consider $(1/n)\mathbb{Z}$ as an object of $\mathbb{Z}$PoSet by using its natural ordering and the $\mathbb{Z}$-action induced by addition. We write $[n]_{\Lambda_\infty}$ for $(1/n)\mathbb{Z}$ when considered as an object of Λ_∞. In general, when we have a G-action on each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of an ∞-category $\mathcal{C}$, which is compatible with the composition, we say that $\mathcal{C}$ is equipped with a BG-action. We equip Λ_∞ with a $B\mathbb{Z}$-action by sending morphisms $f : (1/n)\mathbb{Z} \to (1/m)\mathbb{Z}$ to $\sigma(f) = f + 1$, where σ is a generator of $\mathbb{Z}$. For $p \geq 1$ we can also divide the morphism space by the action of σ^p, which lets us define the category

$$\Lambda_p := \Lambda_\infty / B(p\mathbb{Z}),$$

which has the same objects of Λ_∞, but different morphism spaces. That means the objects of Λ_p still has the form $(1/n)\mathbb{Z}$, up to isomorphism, but we denote the corresponding object by $[n]_{\Lambda_p}$ when considered in Λ_p. We have a remaining $BC_p = B\mathbb{Z}/B(p\mathbb{Z})$-action on the category Λ_p, so we get

$$\Lambda_1 = \Lambda_p / B(p\mathbb{Z})$$

for all $p \geq 1$. For abbreviation we write $\Lambda := \Lambda_1$. This category should be thought of as consisting of cyclic graphs



It can be shown that Λ_∞ is self-dual, and that the self duality $\Lambda_\infty \simeq \Lambda_\infty^{op}$ is $B\mathbb{Z}$-equivariant. That means Λ_p is self-dual for all p. In particular $\Lambda \simeq \Lambda^{op}$ which will be used often without comments.

When we have a symmetric monoidal ∞-category we can form a new ∞-category

$$\mathcal{C}^{\otimes}_{\mathrm{act}} := \mathcal{C}^{\otimes} \times_{N(\mathrm{Fin}_*)} N(\mathrm{Fin}).$$

We have that the fiber of $\mathcal{C}^{\otimes}_{\mathrm{act}}$ over a finite set $I \in \mathrm{Fin}$ is $\mathcal{C}^I$. Using this, we see that this refines to a symmetric monoidal ∞-category $(\mathcal{C}^{\otimes}_{\mathrm{act}})^{\otimes}$ using disjoint unions sending

$(X_i)_{i\in I} \in \mathcal{C}^I$ and $(X_j)_{j\in J} \in \mathcal{C}^J$ to $(X_k)_{k\in I\sqcup J} \in \mathcal{C}^{I\sqcup J}$. $\mathcal{C}^{\otimes}_{\text{act}}$ is called the *active part* of $\mathcal{C}$, since it equivalently can be defined as the subcategory of $\mathcal{C}^{\otimes}$ spanned by all objects and active morphisms. We get a natural lax symmetric monoidal functor

$$\mathcal{C} \to \mathcal{C}^{\otimes}_{\text{act}}$$

whose underlying functor is the inclusion of $\mathcal{C}$ into the fiber of $\mathcal{C}^{\otimes}_{\text{act}}$ over the 1-element set. In particular we are interested in $\text{Ass}^{\otimes}_{\text{act}}$ and $\text{Sp}^{\otimes}_{\text{act}}$. Using this category we can formulate the following result, which is proved in [NS18, B.1]

Proposition 2.2. *The functor*

$$\begin{aligned} V : \Lambda \to & Fin \\ [n]_{\Lambda_\infty} \mapsto & \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{(n-1)}{n}\right\} \end{aligned}$$

refines to a functor $\Lambda \to Ass^{\otimes}_{act}$*, which we still denote* V*.*

We will mainly consider this functor from the opposite category, and in that case we will write

$$V^{\circ} : \Lambda^{op} \simeq \Lambda \xrightarrow{V} \text{Ass}^{\otimes}_{\text{act}}.$$

Definition 2.3. Let $\mathcal{C}$ be some ∞-category, then a *cyclic object* (resp. *paracyclic object*) in $\mathcal{C}$ is a $\mathcal{C}$ valued presheaf on Λ (resp. Λ_∞), i.e. a functor $\Lambda^{op} \to \mathcal{C}$ (resp. $\Lambda_\infty^{op} \to \mathcal{C}$). Cyclic objects (resp. paracyclic objects) in Sp are called *cyclic spectra* (resp. *paracyclic spectra*).

So both V and V° are cyclic objects in $\text{Ass}^{\otimes}_{\text{act}}$. Recall that an object of $\text{Fun}(\Delta^{op}, \mathcal{C})$ is called a *simplicial object* in $\mathcal{C}$. Then we have the following result by [NS18, B.5]:

Proposition 2.4. *For any ∞-category $\mathcal{C}$ admitting geometric realization of simplicial objects, there is a natural functor*

$$Fun(N(\Lambda^{op}), \mathcal{C}) \to \mathcal{C}^{B\mathbb{T}}$$

from cyclic objects in $\mathcal{C}$ to the $\mathbb{T}$-equivariant objects in $\mathcal{C}$. The underlying objects is given by

$$(F : N(\Delta^{op}) \to \mathcal{C}) \mapsto colim_{N(\Delta^{op})} j^* F,$$

where $j : \Delta^{op} \to \Lambda^{op}$ sends $[n]$ to $[n+1]_\Lambda$.

It is worth noting that j factors over a functor $j_\infty : \Delta^{op} \to \Lambda_\infty^{op}$. When we talk about the geometric realization of a cyclic object, we refer to the image of the object under the above functor.

The last thing we wish to consider in this section is a functor $\text{sd}_p : \Lambda_p \to \Lambda$ which induces a homotopy equivalence of geometric realization. This is done by first defining an endofunctor $\Lambda_\infty \to \Lambda_\infty$ by

$$
\begin{aligned}
[n]_{\Lambda_\infty} &\mapsto [pn]_{\Lambda_\infty} \\
\Big(f : (1/n)\mathbb{Z} \to (1/m)\mathbb{Z})\Big) &\mapsto \Big((1/p)f(p \cdot (-)) : (1/pn)\mathbb{Z} \to (1/pm)\mathbb{Z}\Big).
\end{aligned}
$$

In general, for any $T \in \Lambda_\infty$ we let $\mathrm{sd}_p(T) = (1/p)T$ denote the same underlying set as T, but with action of $n \in \mathbb{Z}$ given by multiplication by pn. By composing with the quotient by $B\mathbb{Z}$ we get the desired functor

$$\mathrm{sd}_p : \Lambda_p \to \Lambda,$$

which sends $[n]_{\Lambda_p}$ to $[pn]_\Lambda$. This map is called the *simplicial subdivision*.

An important property of the simplicial subdivision is that for every cyclic object X we get a subdivided Λ_p-object $\mathrm{sd}_p^* X$ which has the same geometric realization. For the case which we will need, this is made specific in the following proposition from [NS18, B.19]:

Proposition 2.5. *Let X be a paracyclic spectrum, then there is a natural $\mathbb{T}$-equivariant equivalence $|X| \cong |sd_p^* X|$.*

There is also a nice compatibility between simplicial subdivision and the functor $V : \Lambda \to \mathrm{Ass}_{act}^{\otimes}$ which will play a crucial role in the next section. To see this, we first let Free_{C_p} denote the category of finite sets with a free C_p-action and note that we have a natural functor from Free_{C_p} to Fin defined by $S \mapsto \overline{S} = S/C_p$. That means we can consider the pull-back $\mathrm{Free}_{C_p} \times_{\mathrm{Fin}} \mathrm{Ass}_{act}^{\otimes}$. Then we define a functor $V_p : \Lambda_p \to \mathrm{Free}_{C_p} \times_{\mathrm{Fin}} \mathrm{Ass}_{act}^{\otimes}$ which sends $[n]_{\Lambda_p}$ to $(V(\mathrm{sd}_p([n]_{\Lambda_p})), V([n]_\Lambda))$. Here we note that $V(\mathrm{sd}_p([n]_{\Lambda_p}))$ has a C_p-action which is induced from the C_p-action on $[n]_{\Lambda_p}$. This gives us a natural commutative diagram

$$
\begin{array}{ccccc}
\Lambda^{op} & \xrightarrow{\simeq} & \Lambda & \xrightarrow{\quad V \quad} & \mathrm{Ass}_{act}^{\otimes} \\
{\scriptstyle q}\uparrow & & {\scriptstyle q}\uparrow & & \uparrow \\
\Lambda_p^{op} & \xrightarrow{\simeq} & \Lambda_p & \xrightarrow{V_p} & \mathrm{Free}_{C_p} \times_{\mathrm{Fin}} \mathrm{Ass}_{act}^{\otimes} \\
{\scriptstyle \mathrm{sd}_p}\downarrow & & {\scriptstyle \mathrm{sd}_p}\downarrow & & \downarrow \\
\Lambda^{op} & \xrightarrow{\simeq} & \Lambda & \xrightarrow{\quad V \quad} & \mathrm{Ass}_{act}^{\otimes},
\end{array}
$$

where q is the quotient. It is worth commenting on the two vertical arrows to the right, since these do not denote the same map. The upper arrow denotes the projection to the second factor. The lower arrow is given by projection to the first factor Free_{C_p}, which is given a total ordering on the preimages induced by the second factor.

Another functor which will be necessary for us comes from some more general considerations about symmetric monoidal functors from the active parts of symmetric monoidal ∞-categories, which we will discuss a bit now.

Proposition 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and let $Fun_{\otimes}(\mathcal{C}^{\otimes}_{act}, \mathcal{D}) \subseteq Fun(\mathcal{C}^{\otimes}_{act}, \mathcal{D})$ denote the full subcategory of symmetric monoidal functors. Then restriction along the inclusion $\mathcal{C} \subseteq \mathcal{C}^{\otimes}_{act}$ gives an equivalence of categories*

$$Fun_{\otimes}(\mathcal{C}^{\otimes}_{act}, \mathcal{D}) \simeq Fun_{lax}(\mathcal{C}, \mathcal{D}).$$

Proof. Using that the inclusion is the fiber of $\{1\}$ this follows directly from [Lur17, 2.2.4.9], where we note that $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}) = \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ and that $\mathrm{Env}(\mathcal{C})$ can be identified with $\mathcal{C}^{\otimes}_{\mathrm{act}}$ (see [Lur17, 2.2.4.3]). □

This in particular gives us that $id_{\mathcal{C}} \in \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})$ corresponds to some functor $\otimes : \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$. Since an object $X \in \mathcal{C}^{\otimes}_{\mathrm{act}}$ can be thought of as a list $X_1, ..., X_n$ of objects in $\mathcal{C}$, we can describe $\otimes$ as sending $X_1, ..., X_n$ to $X_1 \otimes \cdots \otimes X_n$. This functor can also be characterized as the left adjoint to the inclusion $\mathcal{C} \subseteq \mathcal{C}^{\otimes}_{\mathrm{act}}$, which will follow from the following proposition.

Proposition 2.7. *For a symmetric monoidal ∞-category $\mathcal{C}$, the full, lax symmetric monoidal inclusion $\mathcal{C} \subseteq \mathcal{C}^{\otimes}_{act}$ admits a symmetric monoidal left adjoint L.*

Proof. Let $\overline{X} \in \mathcal{C}^{\otimes}_{\mathrm{act}} \subseteq \mathcal{C}^{\otimes}$, then $\overline{X}$ lies over some $\langle n \rangle \in \mathrm{Fin}_*$. We have a unique active morphism $\langle n \rangle \to \langle 1 \rangle$, and using the symmetric monoidal structure of $\mathcal{C}^{\otimes}$ we get a coCartesian lift $f : \overline{X} \to X$ in $\mathcal{C}^{\otimes}$. We see that f is a morphism in $\mathcal{C}^{\otimes}_{\mathrm{act}}$ and $X \in \mathcal{C} \subset \mathcal{C}^{\otimes}_{\mathrm{act}}$. Since every morphism from $\overline{X}$ to some other object of $\mathcal{C}$ has to cover the active morphism $\langle n \rangle \to \langle 1 \rangle$ it follows from the property of coCartesian lifts that f is initial among such morphisms. This means that for every $\overline{X} \in \mathcal{C}^{\otimes}$ there is some $X \in \mathcal{C}$ which is the reflection of $\overline{X}$ into $\mathcal{C} \subseteq \mathcal{C}^{\otimes}_{\mathrm{act}}$. So by [Lur09, 5.2.7.8 & 5.2.7.9] this inclusion admits a left adjoint $L : \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$, which we can consider as sending a list $X_1, ..., X_n$ of objects in $\mathcal{C}$ to $X_1 \otimes \cdots \otimes X_n$.

To show that L is a symmetric monoidal localization we wish to prove the assumption of [Lur17, 2.2.1.9]. In our setting this means we have to prove that for any $f \in \mathcal{C}^{\otimes}(\overline{X}, \overline{Y})$ satisfying that $L(f)$ is an equivalence, and for every $\overline{Z} \in \mathcal{C}^{\otimes}$, the morphism $L(f \otimes \overline{Z})$ is again an equivalence. Here $\otimes$ denotes the tensor product in $\mathcal{C}^{\otimes}_{\mathrm{act}}$. We know that we can consider $\overline{X}, \overline{Y}$ and $\overline{Z}$ as list of objects in $\mathcal{C}$;

$$\overline{X} = X_1, \cdots, X_n, \ \ \overline{Y} = Y_1, \cdots, Y_m, \ \ \overline{Z} = Z_1, \cdots, Z_r.$$

So if

$$L(f) : X_1 \otimes \cdots \otimes X_n \mapsto Y_1 \otimes \cdots \otimes Y_m$$

is an equivalence, it is clear that the same is true for

$$L(f \otimes \overline{Z}) : X_1 \otimes \cdots X_n \otimes Z_1 \otimes \cdots \otimes Z_r \mapsto Y_1 \otimes \cdots \otimes Y_m \otimes Z_1 \otimes \cdots \otimes Z_r,$$

which lets us conclude that L is a symmetric monoidal left adjoint to the inclusion. □

2.2 THH and cyclotomic spectra

We are finally ready to introduce the main topics of these notes. First we will define the algebra objects which will induce a specific cyclic spectrum, using our results from last section. Taking the geometric realization of this cyclic spectrum then gives us the topological Hochschild homology of the algebra object. We will finish off by considering the cyclotomic structure maps on these spectra.

Definition 2.8. The ∞-category $\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$ of *associative algebras*, or $\mathbb{E}_1$-*algebras*, in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is given by the ∞-category of operad maps

$$A^{\otimes} : N(\mathrm{Ass}^{\otimes}) \to \mathcal{C}^{\otimes}.$$

Equivalently, it is the ∞-category of functors $N(\mathrm{Ass}^{\otimes}) \to \mathcal{C}^{\otimes}$ over $N(\mathrm{Fin}_*)$ that carries inert maps to inert maps.

When we consider the associative algebras in Sp we will often call them $\mathbb{E}_1$-*ring spectrum.* That it carries inert to inert basically gives for $n \geq 1$ that

$$\begin{array}{ccccc} & \langle n\rangle_{\mathrm{Ass}} & \longmapsto & (A, \cdots, A) & \\ \langle n\rangle_{\mathrm{Ass}} & \mathrm{Ass}^{\otimes} & \dashrightarrow & \mathrm{Sp} & \\ \downarrow & & \searrow \quad \swarrow & & \\ \downarrow & & \mathrm{Fin}_* & & \\ & \searrow & \langle n\rangle & & \end{array}$$

where $A \in \mathrm{Sp}$ is the image of $\langle 1\rangle_{\mathrm{Ass}}$.

There is an equivalent way to describe associative algebras, which uses the condition that $A : N(\Delta^{op}) \to \mathcal{C}$ sends inert morphisms to p-coCartesian morphisms in $\mathcal{C}^{\otimes}$. This is called the *Segal condition.* This condition can equivalently be described in the following, more concrete, way: Consider the diagram

$$\begin{array}{ccccc} N(\Delta^{op}) & \xrightarrow{\mathrm{Cut}} & N(\mathrm{Ass}^{\otimes}) & \xrightarrow{A^{\otimes}} & \mathcal{C}^{\otimes} \\ & \searrow^{p} & & & \downarrow q \\ & & & & N(\mathrm{Fin}_*). \end{array}$$

We will also use the notation $A^{\otimes}$ for the composition of the vertical arrows. Let $A = A^{\otimes}([1]) \in \mathcal{C}^{\otimes}_{\langle 1\rangle} = \mathcal{C}$ and consider $A^{\otimes}([n]) \in \mathcal{C}^{\otimes}_{\langle n\rangle} \simeq \mathcal{C}^n$, which we may identify with a sequence $A_1, ..., A_n \in \mathcal{C}$. We have n maps

$$\begin{aligned} \rho_i &: [1] \to [n] \\ \{0, 1\} &\mapsto \{i-1, 1\}, \end{aligned}$$

for $i = 1, ..., n$. Considering this as morphisms in Δ^{op} and using p, we get maps

$$\begin{aligned} \rho^i : \langle n \rangle &\to \langle 1 \rangle \\ i &\mapsto 1 \\ j \neq i &\mapsto 0 \end{aligned}$$

in Fin_*. Using that q is a coCartesian fibration, this induces the functor

$$\begin{aligned} \rho^i_! : \mathcal{C}^n \simeq \mathcal{C}^{\otimes}_{\langle n \rangle} &\to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C} \\ (A_1, ..., A_n) &\mapsto A_i. \end{aligned}$$

In particular we get that the maps $A^{\otimes}([n]) \to A^{\otimes}([1]) = A$ induces a map

$$A_1 = \rho^i_! A^{\otimes}([n]) \to A.$$

If this map is always an equivalence, then $A^{\otimes}$ satiesfies the Segal condition. Using this condition, we have the following equivalent definition of $\mathbb{E}_1$-ring spectra:

Proposition 2.9. *Restricting along the natural functor* $Cut : \Delta^{op} \to Ass^{\otimes}$ *defines an equivalence between* $Alg_{\mathbb{E}_1}(Sp)$ *and the* ∞*-category of functors* $A^{\otimes} : N(\Delta^{op}) \to Sp^{\otimes}$ *which satiesfies the Segal condition and makes the diagram*

$$\begin{array}{ccc} N(\Delta^{op}) & \xrightarrow{A^{\otimes}} & Sp^{\otimes} \\ & \searrow & \downarrow \\ & & N(Fin_*) \end{array}$$

commute.

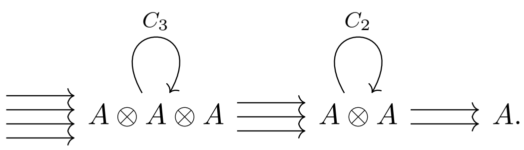
The proof of this result can be found in [NS18, III.2]. Now, let $A^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. Using the symmetric monoidal functor

$$\begin{aligned} \otimes : \mathrm{Sp}^{\otimes}_{\mathrm{act}} &\to \mathrm{Sp} \\ (X_1, \cdots, X_n) &\mapsto X_1 \otimes \cdots \otimes X_n, \end{aligned}$$

from the discussion after proposition 2.6, together with the earlier described functor V°, we get a cyclic spectrum by the composition

$$N(\Lambda^{op}) \xrightarrow{V^{\circ}} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A^{\otimes}} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathrm{Sp}.$$

This cyclic spectrum formally describes the diagram



Definition 2.10. Let $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. Then we define the *topological Hochschild homology* $\mathrm{THH}(A) \in \mathrm{Sp}^{B\mathbb{T}}$ of A as the geometric realization of the cyclic spectrum

$$N(\Lambda^{op}) \xrightarrow{V^{\circ}} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A^{\otimes}} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathrm{Sp}.$$

It can be shown that for $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$, $\mathrm{THH}(A)$ can be given the structure of a *cyclotomic spectrum*, i.e. $\mathbb{T}/C_p \cong \mathbb{T}$-equivariant maps $\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ for all primes p. We will sketch the idea of how to construct these maps, but for a throughout explanation of this, see [NS18, III.3-4].

The idea is to extend the Tate diagonal to a map of cyclic spectra

$$\begin{array}{ccccc}
\cdots \; A \otimes A \otimes A \; (\circlearrowright C_3) & \rightrightarrows & A \otimes A \; (\circlearrowright C_2) & \rightrightarrows & A \\
\downarrow \Delta_p & & \downarrow \Delta_p & & \downarrow \Delta_p \\
\cdots \; (A^{\otimes 3p})^{tC_p} \; (\circlearrowright C_3) & \rightrightarrows & (A^{\otimes 2p})^{tC_p} \; (\circlearrowright C_2) & \rightrightarrows & (A^{\otimes p})^{tC_p}.
\end{array}$$

First we need to formally consider the lower cyclic spectrum. We have the following composition

$$\begin{aligned}
G : N(\Lambda_p^{op}) \simeq N(\Lambda_p) &\xrightarrow{V_p} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \\
&\xrightarrow{A^{\otimes}} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \\
&\xrightarrow{H} (\mathrm{Sp}^{\otimes}_{\mathrm{act}})^{BC_p} \\
&\xrightarrow{\otimes} \mathrm{Sp}^{BC_p} \\
&\xrightarrow{(-)^{tC_p}} \mathrm{Sp},
\end{aligned}$$

where H is the symmetric monoidal functor

$$\begin{aligned}
H : N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} &\to (\mathrm{Sp}_{\mathrm{act}})^{BC_p} \\
(S, (X_{\bar{s}})_{\bar{s} \in \bar{S}}) &\mapsto (S, (X_{\bar{s}})_{s \in S})
\end{aligned}$$

given by [NS18, III.3.6]. So this composition is BC_p-equivariant, hence it factors over a functor

$$N(\Lambda^{op}) \simeq N(\Lambda_p^{op})/BC_p \to \mathrm{Sp},$$

which we will also denote G. This cyclic spectrum is exactly

$$\cdots \; (A^{\otimes 3p})^{tC_p} \circlearrowleft C_3 \quad \rightrightarrows \quad (A^{\otimes 2p})^{tC_p} \circlearrowleft C_2 \quad \rightrightarrows \quad (A^{\otimes p})^{tC_p}.$$

On page 395 of [NS18] it is shown that the geometric realization of a cyclic object and $(-)^{tC_p}$ commutes, so the geometric realization of the above cyclic spectrum maps $\mathbb{T}$-equivariant to $\mathrm{THH}(A)^{tC_p}$. So it is sufficient to construct a natural transformation of BC_p-equivariant functors from

$$F' : N(\Lambda^{op}) \xrightarrow{V^\circ} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A^\otimes} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathrm{Sp}$$

to G. We claim that the geometric realization of the composition

$$\begin{aligned} F : N(\Lambda_p^{op}) \simeq N(\Lambda_p) &\xrightarrow{V_p} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \\ &\xrightarrow{A^\otimes} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \\ &\xrightarrow{\pi_2} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \\ &\xrightarrow{\otimes} \mathrm{Sp}, \end{aligned}$$

where π_2 denotes the projection to the second factor, is equivalent to the geometric realization of F'. To show this write $q_p : \Lambda_p \to \Lambda = \Lambda_p/B(p\mathbb{Z})$ for the projection, and consider the diagram

$$\begin{array}{ccccccccc} N(\Lambda_p) & \xrightarrow{V_p} & N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) & \xrightarrow{A^\otimes} & N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} & \xrightarrow{\pi_2} & \mathrm{Sp}^{\otimes}_{\mathrm{act}} & \xrightarrow{\otimes} & \mathrm{Sp}. \\ \downarrow{\scriptstyle q_p} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & & \nearrow{\scriptstyle \otimes} & \\ N(\Lambda) & \xrightarrow[V]{} & N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) & \xrightarrow[A^\otimes]{} & \mathrm{Sp}^{\otimes}_{\mathrm{act}} & & & & \end{array}$$

The middle square and the right triangle clearly commutes, and the left square commutes by the construction of V_p. This gives us that $F = F' \circ q_p$. We know by proposition 2.4 that the geometric realization of a cyclic object only depends on the geometric realization of the paracyclic object obtained by composing with the quotient. So since F' and $F' \circ q_p$ have the same underlying paracyclic objects, we get that their geometric realizations are equivalent. So this gives us that $|F'| \simeq |F' \circ q_p| \simeq |F|$. So it is sufficient to construct a natural transformation from F to G.

From [NS18, III.3.8] we have an essentially unique BC_p-equivariant lax symmetric monoidal natural transformation from

$$\begin{array}{rcl} I : N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{\pi_2} \mathrm{Sp}_{\mathrm{act}} & \xrightarrow{\otimes} & \mathrm{Sp} \\ (S, (X_{\bar{s}})_{\bar{s} \in \bar{S} = S/C_p}) & \longmapsto & \bigotimes_{\bar{s} \in \bar{S}} X_{\bar{s}} \end{array}$$

to

$$\begin{array}{rl}\tilde{T}_p : N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{H} (\mathrm{Sp}^{\otimes}_{\mathrm{act}})^{BC_p} \xrightarrow{\otimes} \mathrm{Sp}^{BC_p} \xrightarrow{(-)^{tC_p}} \mathrm{Sp} \\ (S, (X_{\bar{s}})_{\bar{s}\in\bar{S}}) \longmapsto \left(\bigotimes_{s\in S} X_{\bar{s}}\right)^{tC_p}.\end{array}$$

Composing this natural transformation $I \to \tilde{T}_p$ with the functor

$$N(\Lambda_p^{op}) \xrightarrow{V_p} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} N(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A^{\otimes}} N(\mathrm{Free}_{C_p}) \times_{N(\mathrm{Fin})} \mathrm{Sp}^{\otimes}_{\mathrm{act}},$$

we get a natural transformation $F \to G$, which gives the cyclotomic structure on THH.

We wish to give a more concrete description of this structure, when considering THH on an easier class of algebras, the so called $\mathbb{E}_\infty$-algebras.

Definition 2.11. Let $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category. An $\mathbb{E}_\infty$-*algebra object* of $\mathcal{C}$ is a section of p, i.e. a map

$$A : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}, \text{ such that } p \circ A \simeq \mathrm{id}_{N(\mathrm{Fin}_*)},$$

which sends inert maps to p-coCartesian maps. Let $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ denote the full subcategory of $\mathrm{Map}_{N(\mathrm{Fin}_*)}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes})$, spanned by the $\mathrm{Alg}_{\mathbb{E}_\infty}$-aglebra objects of $\mathcal{C}$. These objects are sometimes called *commutative algebra objects*.

An object of $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ is called an $\mathbb{E}_\infty$*-ring spectrum*.

Note that an $\mathbb{E}_\infty$-ring in particular is an $\mathbb{E}_1$-ring, so we may consider THH(A) for $A \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$. We wish to discuss the cyclotomic spectra structure on THH(A) in the case where $A \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$. First we want to show that this structure can be given by $\mathbb{E}_\infty$-maps, so we wish to prove the following theorem

Theorem 2.12. *Assume that $A \in Alg_{\mathbb{E}_\infty}(Sp)$. Then THH$(A)$ can be equipped with cyclotomic structur maps φ_p : THH$(A) \to$ THH$(A)^{tC_p}$, where φ_p is an $\mathbb{E}_\infty$-map for every prime p.*

To prove this we first need to consider the category of cyclotomic spectra and endow it with the structure of a symmetric monoidal ∞-category. Let $\mathbb{P}$ denote all primes.

Definition 2.13. The ∞-category of *cyclotomic spectra* is defined as the pullback

$$\begin{array}{ccc}\mathrm{CycSp} & \longrightarrow & (\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}})^{\Delta^1} \\ \downarrow & & \downarrow (\mathrm{ev}_0,\mathrm{ev}_1) \\ \mathrm{Sp}^{B\mathbb{T}} & \xrightarrow{((\mathrm{id}_{\mathrm{Sp}^{B\mathbb{T}}})_{p\in\mathbb{P}},((-)^{tC_p})_{p\in\mathbb{P}})} & (\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}}) \times (\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}}).\end{array}$$

Here we have used that $(-)^{tC_p} : \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{B(\mathbb{T}/C_p)} \simeq \mathrm{Sp}^{B\mathbb{T}}$. So we have that objects in CycSp are pairs (X, F) with $X \in \mathrm{Sp}^{B\mathbb{T}}$ and $F : X \to X^{tC_p}$. That means THH$(A) \in$ CycSp for $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. We wish to show that CycSp is a symmetric monoidal ∞-category. First recall that $\mathrm{Sp}^{B\mathbb{T}}$ is a symmetric monoidal ∞-category and the same is true for $\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}}$. Then we can define the total space of CycSp as the pullback

$$\begin{array}{ccc} \mathrm{CycSp}^{\otimes} & \xrightarrow{\quad\quad\quad\quad\quad\quad} & ((\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}})^{\otimes})^{\Delta^1}_{\mathrm{id}} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ (\mathrm{Sp}^{B\mathbb{T}})^{\otimes} & \xrightarrow{((\mathrm{id}_{\mathrm{Sp}^{B\mathbb{T}}})_{p\in\mathbb{P}},((-)^{tC_p})_{p\in\mathbb{P}})} & (\prod_{p\in\mathbb{P}}(\mathrm{Sp}^{B\mathbb{T}})^{\otimes}) \times (\prod_{p\in\mathbb{P}}(\mathrm{Sp}^{B\mathbb{T}})^{\otimes}). \end{array}$$

where we have used that $\mathrm{id}_{\mathrm{Sp}^{B\mathbb{T}}}$ is symmetric monoidal and $(-)^{tC_p}$ is lax symmetric monoidal by theorem 1.1. Here $((\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}})^{\otimes})^{\Delta^1}_{\mathrm{id}} \subseteq ((\prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}})^{\otimes})^{\Delta^1}$ is the full subcategory consisting of those morphisms which are send to identities in $N(\mathrm{Fin}_*)$. We get from the above, that giving a lax symmetric monoidal functor $\mathcal{E} \to \mathrm{CycSp}$, where $\mathcal{E}$ is some symmetric monoidal ∞-category, is equivalent to giving a lax symmetric monoidal functor

$$H : \mathcal{E} \to \mathrm{Sp}^{B\mathbb{T}},$$

together with a lax symmetric monoidal transformation

$$\mathrm{id}_{\mathrm{Sp}^{B\mathbb{T}}} \circ H \Rightarrow (-)^{tC_p} \circ H,$$

for every $p \in \mathbb{P}$. Using this we are now ready to prove the theorem.

Proof of theorem 2.12. First we wish to show that $\mathrm{THH}(A)$ of an $\mathbb{E}_\infty$-ring spectrum is again an $\mathbb{E}_\infty$-ring spectrum. This is done by first proving that THH gives a lax symmetric monoidal functor $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \to \mathrm{CycSp}$, which we do by using the above description of lax symmetric monoidal functors from symmetric monoidal ∞-categories to CycSp.

Recall that $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ is a symmetric monoidal ∞-category (see [Lur17, 3.2.4.4]) and note that for $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ there is a natural multiplication $m : A^{\otimes p} \to A$. We want to show that $\mathrm{THH} : \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \to \mathrm{Sp}^{B\mathbb{T}}$ is lax symmetric monoidal. Since THH is a composition of first sending A to a cyclic object and then taking geometric realization, it is sufficient to show that both of these are lax symmetric monoidal. Using that geometric realization is a colimit by proposition 2.4 we get that this is indeed lax symmetric monoidal. Since the other part is post-composing with the functor $\otimes$, which is symmetric monoidal by proposition 2.7, and pre-composing with some functor, in this case V°, we have that this is also lax symmetric monoidal, hence the desired holds for THH. We know that m is lax symmetric monoidal and by theorem 1.1 and lemma 1.12 we get that the same is true for both $(-)^{tC_p}$ and Δ_p. Hence

$$(m)^{tC_p} \circ \Delta_p : \mathrm{THH} \Rightarrow (-)^{tC_p} \circ \mathrm{THH}$$

is naturally lax symmetric monoidal. So we get that $\mathrm{THH} : \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \to \mathrm{CycSp}$ is lax symmetric monoidal. This gives us

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}) \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})) \xrightarrow{\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{THH})} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{CycSp}),$$

so if $A \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ then $\mathrm{THH}(A) \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{CycSp})$. Using the forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}$ we get that $\mathrm{THH}(A)$ is a $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-ring spectrum with $\mathbb{T}/C_p \cong \mathbb{T}$-equivariant $\mathbb{E}_\infty$-maps $\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$, for each $p \in \mathbb{P}$. □

Next we wish to give a more concrete description of these φ_p. By proposition 2.4 we know that THH(A) is the colimit of the cyclic spectrum in definition 2.10, so inclusion of the bottom cells into THH(A) gives a map

$$i : A \to \mathrm{THH}(A).$$

Theorem 2.14. *For $A \in Alg_{\mathbb{E}_\infty}(Sp)$ the map $i : A \to THH(A)$ is initial among all maps from A to an $\mathbb{E}_\infty$-ring spectrum equipped with a $\mathbb{T}$-action through $\mathbb{E}_\infty$-maps.*

Proof. So what we want to show is, that if A' is an $\mathbb{E}_\infty$-ring spectrum with $\mathbb{T}$-action and we have a map $f : A \to A'$, then there exist an unique $\mathbb{T}$-equivariant map $\overline{f} : \mathrm{THH}(A) \to A'$, such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i} & \mathrm{THH}(A) \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & A'. \end{array}$$

It is sufficient to prove that $\mathrm{THH}(A) \simeq A \otimes S^1$. To prove this we use the simplicial model for S^1 which is given by $S_\bullet = \Delta^1/\partial\Delta^1$. We know that $S_\bullet$ have $(n+1)$ different , perhaps degenerate, n-vertices S_n. So in $\mathscr{S}$ we have $\mathrm{Colim}_{\Delta^{op}} S_n \simeq S^1$, hence

$$\begin{aligned} A \otimes S^1 &\simeq A \otimes \mathrm{Colim}_{\Delta^{op}} S_n \\ &\simeq \mathrm{Colim}_{\Delta^{op}}(A \otimes S_n) \\ &\simeq \mathrm{Colim}_{\Delta^{op}} A^{\otimes(n+1)} \\ &\simeq \mathrm{THH}(A). \end{aligned}$$

Here we have used that by [Lur17, 3.2.4.7], $A \otimes S_n \simeq A^{\otimes(n+1)}$ is the $(n+1)$-fold coproduct in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$. □

Still using that $\otimes$ is the coproduct in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ we get that $i : A \to \mathrm{THH}(A)$ induces a C_p-equivariant $\mathbb{E}_\infty$-map

$$m' : A^{\otimes p} \to \mathrm{THH}(A).$$

Using theorem 2.14 we get that there is a unique $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-map $\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ which makes the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{i} & \mathrm{THH}(A) \\ {\scriptstyle \Delta_p}\downarrow & & \downarrow{\scriptstyle \varphi_p} \\ (A^{\otimes p})^{tC_p} & \xrightarrow[(m')^{tC_p}]{} & \mathrm{THH}(A)^{tC_p}. \end{array}$$

Since

$$\begin{array}{ccccc} A & \xrightarrow{\quad i \quad} & \mathrm{THH}(A) & & \\ & & & \searrow^{\Delta_p} & \\ \Big\downarrow{\scriptstyle \Delta_p} & & & & (\mathrm{THH}(A)^{\otimes p})^{tC_p} \\ & & & \swarrow_{(m')^{tC_p}} & \\ (A^{\otimes p})^{tC_p} & \xrightarrow[(m')^{tC_p}]{} & \mathrm{THH}(A)^{tCp} & & \end{array}$$

commutes, we get that $\varphi_p = (m')^{tC_p} \circ \Delta_p$, which is called the *Frobenius map*. There is also the following version of the Frobenious map:

Definition 2.15. Let A be an $\mathbb{E}_\infty$-ring spectrum and $p \in \mathbb{P}$. Then the $\mathbb{E}_\infty$-map defined as the composition

$$\varphi_A : A \xrightarrow{\Delta_p} (A^{\otimes p})^{tC_p} \xrightarrow{m^{tC_p}} A^{tC_p}$$

is called the *Tate-valued Frobenius* of A. Here $m : A^{\otimes p} \to A$ is the multiplication map, which is a C_p-equivariant $\mathbb{E}_\infty$-map when A is equipped with the trivial C_p-action.

Using that i is initial by theorem 2.14 we get that there exists a $\mathbb{T}$-equivariant retract $\pi : \mathrm{THH}(A) \to A$, when A is an $\mathbb{E}_\infty$-ring spectrum equipped with trivial $\mathbb{T}$-action.

Corollary 2.16. *Let A be an $\mathbb{E}_\infty$-ring spectrum, then*

$$A \to THH(A) \xrightarrow{\varphi_p} THH(A) \xrightarrow{\pi^{tC_p}} A^{tC_p}$$

is equivalent to the Tate-valued Frobenious of A.

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\qquad i \qquad} & \mathrm{THH}(A) & & \\ {\scriptstyle \Delta_p}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_p} & & \\ (A^{\otimes p})^{tC_p} & \xrightarrow[(m')^{tC_p}]{} & \mathrm{THH}(A)^{tC_p} & \xrightarrow[\pi^{tC_p}]{} & A^{tC_p}, \end{array}$$

so it is sufficient to prove that

$$A^{\otimes p} \xrightarrow{m'} \mathrm{THH}(A) \xrightarrow{\pi} A$$

as a C_p-equivariant map is equivalent to the multiplication m of A. We have that $A^{\otimes p} \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})^{BC_p}$ is induced from A, so since m' was induced from i we get that it is sufficient to show that

$$A \xrightarrow{i} \mathrm{THH}(A) \xrightarrow{\pi} A$$

is equivalent to the identity, which we know is true by the definition. $\square$

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