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Master Thesis

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The Ring Spectrum of Stable Power Operations

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Abstract

For a commutative ring spectrum E, we recall the description of the associative ring spectrum $\mathrm{Pow}(E)$ of stable power operations due to Lurie and Glasman-Lawson, which naturally acts on the underlying spectrum of any commutative E-algebra in a way compatible with stable operations [Lur07, Lec.24], [GL20]. The homotopy of this ring describes the stable power operations on the generalized cohomology theory $E^*(-)$ associated to E. It further fits into a string of associative ring spectra $E \to \mathrm{Pow}(E) \to \mathrm{End}(E)$, which describes the restriction of stable power operations to stable operations.

We show that $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ is isomorphic to a natural completion of the big Steenrod algebra, which shows that it captures the structure of the Steenrod operations and all sums hereof, hence it describes all the stable power operations on ordinary mod-p cohomology.

We further consider the case of Morava E-theory $E(\mathbb{F}_{p^n}, \Gamma)$, where Γ is the Honda formal group law of height n. This will be done by constructing an associative ring spectrum $\widehat{\mathrm{Pow}}(E)$, which is the $K(n)$-local version of $\mathrm{Pow}(E)$, where $K(n)$ denotes the Morava K-theory at height n. We end this thesis by sketching the calculation of $\pi_*\widehat{\mathrm{Pow}}(E)$ at height 1 by using results of Rezk [Rez09].

Contents

1 Introduction

The Steenrod operations $P^i : H^n(X, \mathbb{F}_p) \to H^{n+2i(p-1)}(X, \mathbb{F}_p)$, acting on ordinary mod-p cohomology, are the fundemental example of stable power operations. They are usually constructed as power operations, but it can be shown that they are stable operations as well, and it is from this point of view they are most often considered. The sums and composition of such operations can be formed into a non-commutative $\mathbb{F}_p$-algebra $\mathfrak{B}_p$ called the big Steenrod algebra, which describes the stable power operations on $H^*(-, \mathbb{F}_p)$. We know that $H^n(X; \mathbb{F}_p) \cong \pi_{-n}(H\mathbb{F}_p)^X$, where $(H\mathbb{F}_p)^X$ denotes the function spectrum, which has a natural structure of a commutative $H\mathbb{F}_p$-algebra. Therefore it makes sense to try and understand the stable power operations from a spectral point of view.

To do so, we describe an associative ring spectrum $\mathrm{Pow}(H\mathbb{F}_p)$ of stable power operations on $H^*(-, \mathbb{F}_p)$. This ring acts on the underlying spectrum of any commutative $H\mathbb{F}_p$-algebra, in a way which is compatible with the stable cohomology operations. In particular we get that it acts on the function spectrum $H\mathbb{F}_p^X$, which then induces an action of $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ on $H^*(X, \mathbb{F}_p)$. Therefore, it makes sense to try and understand the algebraic structure of the stable power operations on $H^*(-, \mathbb{F}_p)$ by understanding $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$. We will show that this is the correct approach, by showing that the latter is isomorphic to a suitable completion of the big Steenrod algebra.

Theorem (2.1.9)**.** *The homotopy groups $\pi_*Pow(H\mathbb{F}_p)$ of the ring of stable power operations on commutative $H\mathbb{F}_p$-algebras is isomorphic to the completion $(\mathfrak{B}_p)^\wedge$ of the big Steenrod algebra with respect to the excess filtration.*

We wish to generalize this theory to any commutative ring spectrum E. For any such E we have an associated generalized cohomology theory $E^*(-)$, which satisfies $\pi_*E^X \cong E^{-*}(X)$ for $X \in \mathscr{S}$ a space. A stable cohomology operation on $E^*(-)$ is then an operation $E^k(-) \to E^{k+n}(-)$ for all k, which commutes with suspension. These are in bijective correspondence with the homotopy classes of maps of spectra $E \to \Sigma^n E$. Since we can consider the function spectrum E^X as an $\mathrm{End}(E)$-module, where $\mathrm{End}(E) := \mathrm{map}_{\mathrm{Sp}}(E, E)$, we get that $E^*(-)$ has stable operations.

To understand stable power operations we proceed as follows. Let $F : \mathrm{CAlg}_E \to \mathrm{Sp}$ denote the forgetful functor, and consider the mapping spectrum

$$\mathrm{End}(F) := \mathrm{map}_{\mathrm{Fun}(\mathrm{CAlg}_E, \mathrm{Sp})}(F, F)$$

which has a natural structure of an associative ring spectrum induced by composition. Note that $\mathrm{End}(F)$ acts on the underlying spectrum $F(M)$, of any commutative E-algebra M.

Since $\pi_n\mathrm{End}(F)$ corresponds to homotopy classes of natural transformations $F \to \Sigma^n F$, we get that every element of this ring gives a map $M \to \Sigma^n M$ and therefore a map

$\pi_k M \to \pi_{k+n} M$. This is functorial with respect to the commutative E-algebra M, so we can consider the elements of $\pi_* \mathrm{End}(F)$ as giving rise to operations which acts on the homotopy groups of the underlying spectrum of any commutative E-algebra. These operations are what we call stable power operations, and therefore we define the ring of stable power operations as $\mathrm{Pow}(E) := \mathrm{End}(F)$.

As mentioned above we have $\pi_* E^X \cong E^{-*}(X)$ for $X \in \mathscr{S}$ and $E \in \mathrm{CAlg}$, so using that $E^X \in \mathrm{CAlg}_E$ we see that $E^*(-)$ has power operations. The way that the function spectrum $E^{(-)}$ captures both the stable operations and the stable power operations, is captured in the following diagram, which commutes up to natural isomorphism

$$\begin{tikzcd} \mathscr{S}^{op} \arrow[r, "E^{(-)}"] \arrow[d, "E^{(-)}"'] & \mathrm{LMod}_{\mathrm{End}(E)} \arrow[d] \\ \mathrm{CAlg}_E \arrow[r] & \mathrm{LMod}_E. \end{tikzcd} \tag{1}$$

Here the map $\mathrm{LMod}_{\mathrm{End}(E)} \to \mathrm{LMod}_E$ comes from the fact that if $\mathrm{End}(E)$ acts on a spectrum M, then the map $E \to \mathrm{End}(E)$ induces an action of E on M.

We will give an explicit description of $\mathrm{Pow}(E)$ in terms of endomorphisms of certain cospectrum objects, following [GL20] and [Lur07, Lec.24]. Using this description we will prove in the following theorem, that $\mathrm{Pow}(E)$ acts on the underlying spectrum of any commutative E-algebra in a way, which is compatible with the stable operations.

Theorem (2.0.19)**.** *For any commutative ring spectrum E, the ring of stable power operations fits into a diagram of associative ring spectra*

$$E \to Pow(E) \to End(E).$$

The associative ring spectrum $Pow(E)$ has a natural action on the underlying spectrum of any commutative E-algebra, in a matter compatible with stable cohomology operations, in the sense that there is a canonical lift in the diagram

$$\begin{tikzcd} \mathscr{S}^{op} \arrow[r, "E^{(-)}"] \arrow[dd, "E^{(-)}"'] & LMod_{End(E)} \arrow[d] \\ & LMod_{Pow(E)} \arrow[d] \\ CAlg_E \arrow[r] \arrow[ur, dashed] & LMod_E. \end{tikzcd}$$

The first half of this thesis begins with describing this associative ring spectrum $\mathrm{Pow}(E)$, proving that it is equivalent to an endomorphism ring of a specific cospectrum object and proving the above theorem. We then calculate the fundamental example $\pi_*\mathrm{Pow}(H\mathbb{F}_p) \cong (\mathfrak{B}_p)^\wedge$ as mentioned above. The key elements to this will be the fact that the underlying spectrum of $\mathrm{Pow}(E)$ is $\lim_m \Sigma^m \widetilde{\mathrm{Free}}_E(\Omega^m E)$, where $\widetilde{\mathrm{Free}}_E$ denotes the free non-unital commutative E-algebra, together with the free functor which equips an $\mathbb{F}_p$-vector space with a generalized version of the Steenrod operations.

The second half of this thesis is devoted to understanding stable power operations on the p-adic K-theory $K_p^\wedge$. It can be shown that $K_p^\wedge$ is equivalent to the first in a family of cohomology theories called Morava E-theory, where each one is a 2-periodic commutative ring spectrum $E(\mathbb{F}_{p^n}, \Gamma)$, for fixed prime p and integer n. Most of the abstract theory we need to consider before we can give a description of the stable power operations on $K_p^\wedge$, can be done for a general height n, so we will begin in this generality, since $E(\mathbb{F}_{p^n}, \Gamma)$ is an interesting commutative ring spectrum in it's own right, as it is a central part of chromatic homotopy theory. For brevity, let us now fix a height and write $E := E(\mathbb{F}_{p^n}, \Gamma)$.

The spectrum E is $K(n)$-local, where $K(n)$ is Morava K-theory at height n, and so is the function spectrum E^X for any space X, so it is natural to restrict ourself to this setting. We recall the construction of the so called algebraic approximation functor $\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E*}$, which satisfies $(\mathbb{T}(\pi_* M))^\wedge_{\mathfrak{m}} \cong \pi_* L_{K(n)}\mathrm{Free}_E(M)$ for M a flat E-module and $\mathfrak{m}$ the maximal ideal of $\pi_0 E$, following [Rez09]. This indicates that when we wish to try and understand the homotopy of the underlying spectrum of $\mathrm{Pow}(E)$, which is given by $\lim_m \Sigma^m \widetilde{\mathrm{Free}}_E(\Omega^m E)$, it will indeed be necessary to restrict ourself to $K(n)$-local commutative E-algebras.

To take $K(n)$-locality into account, we adjust the construction of $\mathrm{Pow}(E)$. Most of the necessary adjustments are rather straightforward and we will construct an associative ring spectrum $\widehat{\mathrm{Pow}}(E)$ of $K(n)$-local stable power operations, with underlying spectrum $\lim_m \Sigma^m \widetilde{\mathrm{LFree}}_E(\Omega^m E)$. Here $\widetilde{\mathrm{LFree}}_E$ is a $K(n)$-local version of $\widetilde{\mathrm{Free}}_E$, satisfying $\widetilde{\mathrm{LFree}}_E \simeq L_{K(n)}\widetilde{\mathrm{Free}}_E$.

One would like to have an explicit description of $\mathrm{Pow}(E)$, but stable power operations in Morava E-theory at height $n > 1$ are understood only partially, and so we restrict to the height $n = 1$ case, where $E \simeq K_p^\wedge$ and a lot more is known. Due to McClure [BMMS86, IX.3] we have an operation

$$Q : \pi_*(K/p^r \otimes_{\mathbb{S}} X) \to \pi_*(K/p^{r-1} \otimes_{\mathbb{S}} X),$$

for $X \in \mathrm{CAlg}$, which we will extend to an operation

$$Q : \pi_* L_{K(1)}(K_p^\wedge \otimes_{\mathbb{S}} X) \to \pi_* L_{K(1)}(K_p^\wedge \otimes_{\mathbb{S}} X).$$

These operations gives us the following useful isomorphism

$$\mathbb{T}(\Omega^n \pi_* K_p^\wedge) \cong \begin{cases} \mathbb{Z}_p[x, Qx, Q^2x, ...][u^{\pm 1}], & n \text{ even} \\ \Lambda_{\mathbb{Z}_p}[x, Qx, Q^2x, ...][u^{\pm 1}], & n \text{ odd}, \end{cases}$$

where $|u| = 2$, from [BF15, 6.14]. This is a key element when we consider $\pi_*\widehat{\mathrm{Pow}}(K_p^\wedge)$. We will not be able to give a full calculation, but we will prove the following description of $\pi_*\widehat{\mathrm{Pow}}(K_p^\wedge)$

Theorem (3.2.10)**.** *The homotopy groups $\pi_*\widehat{Pow}(K_p^\wedge)$ of the ring of stable power operations on $K(1)$-local commutative $K_p^\wedge$-algebras, is equivalent to the limit of the following inverse system*

$$\cdots \xrightarrow{\pi_*\theta_{2n+1}} (\mathbb{Z}_p\{x, Qx, Q^2x, ...\}[u^{\pm 1}])_p^\wedge \xrightarrow{\pi_*\theta_{2n}} \Sigma(\Lambda_{\mathbb{Z}_p}\{x, Qx, Q^2x, ...\}[u^{\pm 1}])_p^\wedge$$
$$\xrightarrow{\pi_*\theta_{2n-1}} (\mathbb{Z}_p\{x, Qx, Q^2x, ...\}[u^{\pm 1}])_p^\wedge \xrightarrow{\pi_*\theta_{2n-2}} \cdots$$

with $|x| = |Q^i x| = 0$ and $|u| = 2$.

In the end we will give a sketch of how one could proceed to calculate this limit. This relies on a better understanding of the maps in the system above, but these are very complicated to determine. Surprisingly the expectation is that they are complicated enough so that there is nothing left in the limit except the identities, which would give us that $\pi_*\widehat{\mathrm{Pow}}(K_p^\wedge) \cong \mathbb{Z}_p\{x\}[u^{\pm 1}]$ with $|u| = 2$.

1.1 Notation

In these notes we will work in the setting of ∞-categories as developed by Joyal and Lurie, the standard reference being [Lur09]. Given a symmetric monoidal ∞-category $\mathcal{C}$ we will in general work with the ∞-category of *associative algebra objects* and the ∞-category of *commutative algebra objects of* $\mathcal{C}$, which we denote by $\mathrm{Alg}(\mathcal{C})$ and $\mathrm{CAlg}(\mathcal{C})$ respectively. Note that if E is a commutative algebra object in $\mathcal{C}$ then it is in particular an associative algebra object in $\mathcal{C}$.

If $E \in \mathrm{Alg}(\mathcal{C})$ we get a new ∞-category $\mathrm{LMod}_E(\mathcal{C})$ which we will call the ∞-category of *E-modules*. One can show that if E is a commutative algebra object in $\mathcal{C}$, then the relative tensor product equips $\mathrm{LMod}_E(\mathcal{C})$ with a structure of a symmetric monoidal ∞-cateogry. Objects in this category will be called *commutative E-algebras*. When denoting the direct sum and tensor product in these ∞-categories we will use $\oplus$ and $\otimes$ respectively. We will further use subscripts on the tensor product to emphasise which one, e.g. $\otimes_{\mathbb{S}}$, $\otimes_E$. We will further denote the terminal object in these ∞-categories by 0.

We will often consider algebra objects in the symmetric monoidal ∞-category of spectra Sp, with the sphere spectrum $\mathbb{S}$ as the monoidal unit. In this case we will omit the underlying category from the notation of the above categories, i.e. we will denote these by Alg, CAlg, LMod_E, CAlg_E. We will call objects in Alg *associative ring spectra* and objects in CAlg will be called *commutative ring spectra*. We will often need to consider the underlying spectrum or module of objects in these ∞-categories, and we will denote the most used forgetful functors as follows:

$$\begin{tikzcd} \mathrm{CAlg}_E \arrow[d, "G"'] \arrow[dr, "F"] & \\ \mathrm{LMod}_E \arrow[r, "H"'] & \mathrm{Sp}. \end{tikzcd}$$

Let $\mathscr{S}$ denote the ∞-category of spaces and $\mathcal{C}$ some ∞-category with $X, Y \in \mathcal{C}$. Then we can construct the *mapping space* $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathscr{S}$. If we further assume that $\mathcal{C}$ is stable, then this mapping space has a canonical structure of an infinite loop space and can be extended to a *mapping spectrum* $\mathrm{map}_{\mathcal{C}}(X, Y) \in \mathrm{Sp}$, such that $\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \Omega^\infty \mathrm{map}_{\mathcal{C}}(X, Y)$.

When we talk about (co)limits, we will always mean their ∞-categorical versions. In the more classical language of model categories, these would correspond to what is know as homotopy (co)limits.

1.2 Acknowledgements

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2 The Ring of Stable Power operations

The goal of this section is to describe the associative ring spectrum of stable power operations $\mathrm{Pow}(E)$, for any commutative ring spectrum $E \in \mathrm{CAlg}$. We further want to understand how $\mathrm{Pow}(E)$ acts on the underlying spectrum of any commutative E-algebra, and how this is compatible with the stable operations.

From [GH19, 7.4.14] we know that any stable ∞-category $\mathcal{C}$ is enriched over spectra, so we can consider $\mathrm{map}_{\mathcal{C}}(X, Y)$ as a spectrum for any $X, Y \in \mathcal{C}$. If we consider the case $X = Y$, we can further equip $\mathrm{map}_{\mathcal{C}}(X, X)$ with the structure of an associative ring spectrum by composition. When considered as such we call this the *endomorphism ring* of X and denote it by $\mathrm{End}_{\mathcal{C}}(X)$. When $\mathcal{C}$ is understood we will omit this from the notation, and simply write $\mathrm{End}(X)$.

In a similar way we can define $\mathrm{End}(F)$ for a functor $F : \mathcal{C} \to \mathrm{Sp}$. We we have from [Lur17, 1.1.3.1] that $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ is stable for any ∞-category $\mathcal{C}$, since Sp is stable by [Lur17, 1.4.3.6], so we can consider the mapping spectrum $\mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Sp})}(F, G)$. For $F = G$ we can again turn $\mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Sp})}(F, F)$ into an associative ring spectrum by composition, and when we consider $\mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Sp})}(F, F)$ with this structure, we denote it by $\mathrm{End}(F)$ and call it the *endomorphism ring* of the functor F. We are interested in this construction when applied to the forgetful functor $F : \mathrm{CAlg}_E \to \mathrm{Sp}$, for $E \in \mathrm{CAlg}$.

Definition 2.0.1. Let E be a commutative ring spectrum and $F : \mathrm{CAlg}_E \to \mathrm{Sp}$ the forgetful functor. Then $\mathrm{End}(F)$ is called the *ring of stable power operations* and we denote it by $\mathrm{Pow}(E)$.

Remark 2.0.2. We wish to justify this definition, so let $E \in \mathrm{CAlg}$, $M \in \mathrm{CAlg}_E$ and let $F : \mathrm{CAlg}_E \to \mathrm{Sp}$ denote the forgetful functor. An element of $\pi_0\mathrm{Pow}(E)$ corresponds to an equivalence class of natural transformation $F \Rightarrow F$, so the underlying spectrum of M admits an action of every object in $\mathrm{Pow}(E)$, given by the evaluation map from the natural transformation. In particular we note that $\mathrm{Pow}(E)$ acts on the underlying spectrum of E itself. We will later show in **Theorem 2.0.19** that this action of $\mathrm{Pow}(E)$ is compatible with the stable cohomology operations.

Using this action, we see that every element of $\pi_n\mathrm{Pow}(E)$ gives rise to a map $F(M) \to \Sigma^n F(M)$, which then further gives a map $\pi_k F(M) \to \pi_{k+n} F(M)$. This is functorical with respect to the commutative E-algebra M, hence we get that we can consider the elements of $\pi_*\mathrm{Pow}(E)$ as giving rise to operations on the homotopy groups of the underlying spectrum of any commutative E-algebra. These operations are exactly what we call the stable power operations. A throughout modern introduction to these operations can be found in [Law20].

Let $E^{(-)} : \mathscr{S}^{op} \to \mathrm{CAlg}_E$ denote the function spectrum. Then E^X is a commutative E-algebra for any space X, so using that $\pi_* E^X \cong E^{-*}(X)$ we get by the above that

$\pi_*\mathrm{Pow}(E)$ describes the stable power operations on the generalized cohomology theory $E^*(-)$.

There are two different descriptions of this associative ring spectrum in the literature. The above definition is how $\mathrm{Pow}(E)$ is defined by Lurie [Lur07, Lec.24], where it is denoted by R. Our first goal of this section is to show that this definition is equivalent to the definition given by Glasman-Lawson in [GL20, 7.3], which we will do in **Theorem 2.0.9**. An important tool, which we will need throughout the entire thesis, is functors represented by so called spectrum objects.

Definition 2.0.3. Let $\mathcal{C}$ be an ∞-category which admits finite limits, let $\mathcal{C}_*$ denote the ∞-category of pointed objects and $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ the loop functor. Then we define the ∞-category $\mathrm{Sp}(\mathcal{C})$ of *spectrum objects* of $\mathcal{C}$ as the limit ∞-category

$$\mathrm{Sp}(\mathcal{C}) := \lim(\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \cdots).$$

More specifically this means that a sequence $\{X_n\}$ of objects in a suitable ∞-category $\mathcal{C}$, is a spectrum object of $\mathcal{C}$ if $X_n \simeq \Omega X_{n+1}$ for each n. In the same way we get that $\{X_n\}$ is a cospectrum object of $\mathcal{C}$, i.e. is in $\mathrm{Sp}(\mathcal{C}^{op})$, if $X_n \in \mathcal{C}$ for each n and if they satisfy $X_n \simeq \Sigma X_{n+1}$.

The definition of spectrum objects of a general ∞-category $\mathcal{C}$ is a bit more involved, but in the cases relevant for us it follows from [Lur17, 1.4.2.25] that we can use the above definition. From [Lur17, 4.8.1.23] we get that $\mathrm{Sp}(\mathscr{S}) \simeq \mathrm{Sp}$, which is the main example which we are interested in. We further get from [Lur17, 1.4.2.17] that $\mathrm{Sp}(\mathcal{C})$ is a stable ∞-category for any suitable ∞-category $\mathcal{C}$, which means that it makes sense to consider $\mathrm{End}_{\mathrm{Sp}(\mathcal{C})}(X)$ for $X \in \mathrm{Sp}(\mathcal{C})$.

Remark 2.0.4. We want to make a connection between specific functors and spectrum objects. To do so, we need to understand a variant of the Yoneda embedding involving the latter. Let $\mathcal{C}$ be a locally small ∞-category which admits finite limits. By [Lur09, 5.1.3.1] we then get that the Yoneda embedding

$$\begin{aligned} j : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{op}, \mathscr{S}) \\ X &\mapsto \mathrm{Map}_{\mathcal{C}}(-, X), \end{aligned}$$

is fully faithful. By further applying [Lur09, 5.1.3.2] we get that j preserves finite limits in $\mathcal{C}$, hence it induces the so called *spectral Yoneda embedding*

$$\begin{aligned} \Upsilon : \mathrm{Sp}(\mathcal{C}) &\to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}) \\ c = \{c_n\} &\mapsto \{\mathrm{Map}_{\mathcal{C}}(-, c_n)\} \simeq \mathrm{map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_+(-), c), \end{aligned}$$

which is still fully faithful.

Definition 2.0.5. Let $\mathcal{C}$ be a locally small ∞-category with finite limits. Then we say that a functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ is *represented* by a spectrum object $c \in \mathrm{Sp}(\mathcal{C})$ if it is the image of c under the spectral Yoneda embedding.

Remark 2.0.6. To be able to talk about a functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ being represented, we need $\mathcal{C}$ to be locally small and admit finite limits, and when we will later need to consider functors represented by cospectrum objects, this has to hold for $\mathcal{C}^{op}$ as well. If $\mathcal{C}$ is a presentable ∞-category, then it is in particular locally small and admits all small limits and colimits. Hence both $\mathcal{C}$ and $\mathcal{C}^{op}$ admits finite limits in this case, and is locally small. Therefore it is sufficient for us to show that either $\mathcal{C}$ or $\mathcal{C}^{op}$ is presentable, for the ∞-categories which we are interested in.

By [Lur09, 5.5.1.8] we have that $\mathscr{S}$ is presentable, so since [Lur17, 1.4.4.4 (1)] tells us that $\mathrm{Sp}(\mathcal{C})$ is presentable for any presentable ∞-category $\mathcal{C}$, we get that $\mathrm{Sp} \simeq \mathrm{Sp}(\mathscr{S})$ is also presentable. [Lur17, 4.2.3.7 (1)] tells us that $\mathrm{LMod}_E(\mathcal{C})$ is presentable for all such $\mathcal{C}$ and $E \in \mathrm{Alg}(\mathcal{C})$, so we get that this also holds for LMod_E. Lastly we get from [Lur17, 3.2.3.5] that both CAlg_E and CAlg are presentable since both LMod_E and Sp are presentable ∞-categories with sufficiently nice symmetric monoidal structures.

The spectral Yoneda embedding is in general very well-behaved when we are working in the setting of presentable ∞-categories.

There exists an ∞-category Pr^L whose objects are presentable ∞-categories and morphisms are colimit-preserving functors, which we by [Lur17, 4.8.1.15] can equip with a symmetric monoidal structure. From [Lur17, 4.8.1.16] we get that for $\mathcal{C} \in \mathrm{Pr}^L$, the full subcategory $\mathrm{Fun}^R(\mathcal{C}^{op}, \mathrm{Sp}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})$ consisting of all limit-preserving and accesible functors, is presentable. Note that in the case where $\mathcal{C}^{op}$ is a presentable ∞-category, this assumption on the functors is equivalent to being a right adjoint by the Adjoint functor theorem [Lur09, 5.5.2.9], which explains the R in the notation. We then get from [Lur17, 4.8.1.23] that whenever $\mathcal{C}$ is a presentable ∞-category there is an equivalence in Pr^L

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Fun}^R(\mathcal{C}^{op}, \mathrm{Sp}).$$

This equivalence is exactly the spectral Yoneda embedding, so if $\mathcal{C}$ is a presentable ∞-category, this gives us that any limit-preserving accessible functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ is represented by some spectrum object in $\mathcal{C}$.

Using this understanding of functors represented by spectrum objects, we are ready to turn our focus to proving that the two different ways of defining $\mathrm{Pow}(E)$, are equivalent. Therefore we will now define this associative ring spectrum, as done by Glasman-Lawson [GL20]. For this we need the following left adjoint.

Proposition 2.0.7 (Prop. 7.1 [GL20])**.** *Let E be a commutative ring spectrum. Then the forgetful functor $G : CAlg_E \to LMod_E$ admits a left adjoint $Free_E$ which is given by*

$$Free_E(M) = \bigoplus_{n\geq 0} (M^{\otimes_E n})_{h\Sigma_n} \simeq E \oplus \left(\bigoplus_{n>0} (M^{\otimes_E n})_{h\Sigma_n} \right).$$

Definition 2.0.8. Let $E \in \mathrm{CAlg}$ be any commutative ring spectrum and $\mathrm{Free}_E : \mathrm{LMod}_E \to \mathrm{CAlg}_E$ the left adjoint to the forgetful functor as above. Then we call $\mathrm{Free}_E(M)$ the *free commutative E-algebra* on the E-module M.

We can now consider two examples of cospectrum objects, which will be necessary for us. The first example is $\{\Omega^n E\} \in \mathrm{Sp}(\mathrm{LMod}_E^{op})$, which we see holds since LMod_E is stable by [Lur17, 7.1.1.5], so $\Omega^n E \simeq \Sigma\Omega^{n+1}E$ for all $n \in \mathbb{Z}$. The second example is $\{\mathrm{Free}_E(\Omega^n E)\} \in \mathrm{Sp}(\mathrm{CAlg}_E^{op})$. To see that this holds, we use that Free_E is a left adjoint by **Proposition 2.0.7**, and therefore commutes with colimits, which gives us that

$$\mathrm{Free}_E(\Omega^n E) \simeq \mathrm{Free}_E(\Sigma\Omega^{n+1}E) \simeq \Sigma\mathrm{Free}_E(\Omega^{n+1}E)$$

as desired.

Glasman-Lawson defines $\mathrm{Pow}(E)$ as the endomorphism ring

$$\mathrm{Pow}(E) := \mathrm{End}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\{\mathrm{Free}_E(\Omega^n E)\})$$

of this cospectrum object, in [GL20].

We will in the following theorem show that this definition agrees with Luries definition in [Lur07, Lec.24].

Theorem 2.0.9. *Let E be a commutative ring spectrum. Then*

$$Pow(E) \simeq End_{Sp(CAlg_E^{op})}(\{Free_E(\Omega^n E)\}).$$

To prove this, we need to show that

$$\mathrm{End}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\{\mathrm{Free}_E(\Omega^n E)\}) \simeq \mathrm{End}(F : \mathrm{CAlg}_E \to \mathrm{Sp}),$$

which requires a better understanding of endomorphism rings of functors represented by spectrum objects.

Proposition 2.0.10. *Let $\mathcal{C}$ be any locally small ∞-category and $F : \mathcal{C}^{op} \to Sp$ some functor. Then F has a natural lift to $LMod_{End(F)}$. If we further assume that $\mathcal{C}$ has finite limits and that F is represented by a spectrum object $Y \in Sp(\mathcal{C})$, then there is a canonical identification*

$$End(F) \simeq End_{Sp(\mathcal{C})}(Y).$$

Proof. First, it is clear that F naturally is a left $\mathrm{End}(F)$-module, since $\mathrm{End}(F)$ has a natural action on $F(X)$ for each $X \in \mathcal{C}^{op}$, given by the evaluation map associated to each object in $\mathrm{End}(F)$. So we have the desired lift in the diagram

$$\begin{tikzcd} & \mathrm{LMod}_{\mathrm{End}(F)} \arrow[d, "H"] \\ \mathcal{C}^{op} \arrow[ur, dotted] \arrow[r, "F"'] & \mathrm{Sp}, \end{tikzcd}$$

where H denotes the forgetful functor. The second part relies on the spectral Yoneda embedding

$$\Upsilon : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}).$$

Using that F is assumed to be represented by Y and the fact that Υ is fully faithful, we get the desired equivalence

$$\mathrm{End}_{\mathrm{Sp}(\mathcal{C})}(Y) = \mathrm{map}_{sp(\mathcal{C})}(Y, Y) \simeq \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp})}(F, F) = \mathrm{End}(F).$$

$\square$

This means that if we have a functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ which is represented by $Y \in \mathrm{Sp}(\mathcal{C})$, then it is equivalent to $\mathrm{map}_{\mathrm{Sp}(\mathcal{C})}(\Sigma^\infty_+(-), Y)$ and it has a natural lift $\mathcal{C}^{op} \to \mathrm{LMod}_{\mathrm{End}(F)} \simeq \mathrm{LMod}_{\mathrm{End}_{\mathrm{Sp}(\mathcal{C})}(Y)}$.

Lemma 2.0.11. *Let E be a commutative ring spectrum. Then the forgetful functor $H : LMod_E \to Sp$ is represented by the cospectrum object $\{\Omega^n E\} \in Sp(LMod_E^{op})$.*

Proof. Recall the following adjunction

$$\begin{tikzcd} \mathrm{Sp}(\mathrm{LMod}_E^{op}) \arrow[r, bend right, "\Omega^\infty"'] \arrow[r, phantom, "\perp"] & \mathrm{LMod}_E^{op}, \arrow[l, bend right, "\Sigma^\infty_+"'] \end{tikzcd}$$

from [Lur17, 1.4.4.4]. Using this we get the following calculation for any $M \in \mathrm{LMod}_E$

$$\begin{aligned} \mathrm{map}_{\mathrm{Sp}(\mathrm{LMod}_E^{op})}(\Sigma^\infty_+(M), \{\Omega^n E\}) &\simeq \mathrm{map}_{\mathrm{LMod}_E^{op}}(M, \Omega^\infty(\{\Omega^n E\})) \\ &\simeq \mathrm{map}_{\mathrm{LMod}_E}(\Omega^\infty(\{\Omega^n E\}), M) \\ &\simeq \mathrm{map}_{\mathrm{LMod}_E}(E, M) \\ &\simeq H(M). \end{aligned}$$

$\square$

We are now ready to prove **Theorem 2.0.9**.

Proof of theorem 2.0.9. By **Proposition 2.0.10** we get that it is sufficient to prove that the cospectrum object $\{\mathrm{Free}_E(\Omega^n E)\}$ represents $F : \mathrm{CAlg}_E \to \mathrm{Sp}$. Using the adjunction between Free_E and the forgetful functor $G : \mathrm{CAlg}_E \to \mathrm{LMod}_E$ given in **Proposition 2.0.7**, followed by **Lemma 2.0.11** which says that the forgetful functor $H : \mathrm{LMod}_E \to \mathrm{Sp}$ is represented by $\{\Omega^n E\} \in \mathrm{Sp}(\mathrm{LMod}_E^{op})$, we get that for any $M \in \mathrm{Sp}(\mathrm{CAlg}_E^{op})$

$$\begin{aligned}\mathrm{map}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\Sigma^\infty_+(M), \{\mathrm{Free}_E(\Omega^n E)\}) &\simeq \mathrm{map}_{\mathrm{Sp}(\mathrm{LMod}_E^{op})}(\Sigma^\infty_+(G(M)), \{\Omega^n E\})\\ &\simeq H \circ G(M)\\ &\simeq F(M).\end{aligned}$$

So we get that

$$Pow(E) = \mathrm{End}(F : \mathrm{CAlg}_E \to \mathrm{Sp}) \simeq \mathrm{End}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\{\mathrm{Free}_E(\Omega^n E)\}),$$

hence the two definitions are equivalent. □

Next, we wish to understand the underlying spectrum of $\mathrm{Pow}(E)$. To do so, we need the following definition.

Definition 2.0.12. Let $E \in \mathrm{CAlg}$ and $M \in \mathrm{LMod}_E$. Then we define the *free non-unital commutative E-algebra* on M as $\widetilde{\mathrm{Free}}_E(M) := \mathrm{fib}(\mathrm{Free}_E(M) \to E)$.

Remark 2.0.13. First we note that this map exists since $E \simeq \oplus_{n\geq 0}(0^{\otimes_E n})_{h\Sigma_n} \simeq \mathrm{Free}_E(0)$, so $M \to 0$ induces a natural transformation $\mathrm{Free}_E(M) \to \mathrm{Free}_E(0)$. Using **Proposition 2.0.7** we get that

$$\begin{aligned}\widetilde{\mathrm{Free}}_E(M) &= \mathrm{fib}(\mathrm{Free}_E(M) \to E)\\ &\simeq \mathrm{fib}\left(E \oplus \left(\bigoplus_{n>0}(M^{\otimes_E n})_{h\Sigma_n}\right) \to E\right)\\ &\simeq \bigoplus_{n>0}(M^{\otimes_E n})_{h\Sigma_n}.\end{aligned}$$

There is a general notion of non-unital commutative ring spectra, and we wish to prove that the above definition of $\widetilde{\mathrm{Free}}_E$ is in fact left adjoint to the forgetful functor from non-unital commutative E-algebras to E-modules. First we will give an idea of the general notion of non-unital algebras. The general abstract definition is given in [Lur17, 5.4.4.1], but we will only go through the idea of this construction in our case. Note that it will not be necessary to understand the technical details containing ∞-operads to understand the rest of this project.

Define Surj to be the subcategory of Fin_* consisting of all objects of Fin_* such that $\alpha : \langle m\rangle \to \langle n\rangle$ belongs to Surj exactly when it is surjective. Further, for $\mathcal{O}^\otimes$ an ∞-operad we define $\mathcal{O}^\otimes_{nu} := \mathcal{O}^\otimes \times_{N(\mathrm{Fin}_*)} N(\mathrm{Surj})$. Then, given a fibration of ∞-operads

$\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ and a map of ∞-operads $\mathcal{O}^\otimes \to \mathcal{D}^\otimes$ we define the *∞-category of non-unital $\mathcal{O}$-algebra objects in $\mathcal{C}$* as $\mathrm{Alg}^{nu}_{\mathcal{O}/\mathcal{D}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}_{nu}/\mathcal{D}}(\mathcal{C})$.

We simplify $\mathrm{Alg}^{nu}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})$ to the case $\mathcal{D}^\otimes \simeq \mathcal{O}^\otimes \simeq N(\mathrm{Fin}_*)$. In this case $N(\mathrm{Fin}_*)_{nu} \simeq N(\mathrm{Surj})$ and we write $\mathrm{CAlg}^{nu}(\mathcal{C}) := \mathrm{Alg}^{nu}_{N(\mathrm{Fin}_*)/N(\mathrm{Fin}_*)}(\mathcal{C})$ for the ∞-category of *non-unital commutative algebra objects of $\mathcal{C}$*. We can see that this is indeed the non-unital version of $\mathrm{CAlg}(\mathcal{C})$, by considering how the structure arises in $\mathrm{CAlg}(\mathcal{C})$. The unit structure comes from the map $\langle 0\rangle \mapsto \langle 1\rangle$ in $N(\mathrm{Fin}_*)$, which is not surjective and hence not present in $\mathrm{CAlg}^{nu}(\mathcal{C})$. On the other hand, the multiplicative structure is still present in $\mathrm{CAlg}^{nu}(\mathcal{C})$ since this comes from the map $\langle 2\rangle \mapsto \langle 1\rangle$, which is surjective.

By [Lur17, 5.4.4.8] we get that the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}^{nu}(\mathcal{C})$ admits a left adjoint, which is given by

$$\begin{aligned} \mathrm{Free}^{nu} : \mathrm{CAlg}^{nu}(\mathcal{C}) &\to \mathrm{CAlg}(\mathcal{C}) \\ M &\mapsto M \oplus I_{\mathcal{C}}, \end{aligned}$$

where $I_{\mathcal{C}}$ denotes the unit in $\mathcal{C}$. We will now turn our focus to the case where $\mathcal{C} = \mathrm{LMod}_E$ for $E \in \mathrm{CAlg}$. In this case we write $\mathrm{CAlg}^{nu}_E := \mathrm{CAlg}^{nu}(\mathrm{LMod}_E)$ and Free^{nu}_E for the above mentioned left adjoint, which in this case maps $X \in \mathrm{CAlg}^{nu}_E$ to $X \oplus E$.

Proposition 2.0.14. *The functor $\widetilde{\mathrm{Free}}_E : \mathrm{LMod}_E \to \mathrm{CAlg}^{nu}_E$ is left adjoint to the forgetful functor $\mathrm{CAlg}^{nu}_E \to \mathrm{LMod}_E$.*

Proof. From [Lur17, 3.1.3.13] we get that the forgetful functor $\mathrm{CAlg}^{nu}_E \to \mathrm{LMod}_E$ admits a left adjoint which we denote by Free'_E. Consider the following diagram

$$\begin{tikzcd} \mathrm{LMod}_E \arrow[d, "\mathrm{Free}'_E"'] \arrow[dr, "\mathrm{Free}_E"] & \\ \mathrm{CAlg}^{nu}_E \arrow[r, "\mathrm{Free}^{nu}_E"'] & \mathrm{CAlg}_E, \end{tikzcd}$$

which commutes since adjoints compose. By the above discussion we know that $\mathrm{Free}^{nu}_E(M) \simeq M \oplus E$ and from **Proposition 2.0.7** we know that $\mathrm{Free}_E(M) \simeq \oplus_{n\geq 0}(M^{\otimes_E n})_{h\Sigma_n}$, hence

$$\mathrm{Free}'_E(M) \simeq \oplus_{n>0}(M^{\otimes_E n})_{h\Sigma_n}$$

which is exactly $\widetilde{\mathrm{Free}}_E$ by **Remark 2.0.13**. □

Proposition 2.0.15 (p.15 [GL20])**.** *Let E be a commutative ring spectrum. Then the underlying spectrum of $Pow(E)$ is $\lim_n \Sigma^n \widetilde{\mathrm{Free}}_E(\Omega^n E)$.*

Remark 2.0.16. The limit in the above proposition exists for any $M \in \mathrm{LMod}_E$, and we wish to clarify how the maps $\Sigma^n \widetilde{\mathrm{Free}}_E(\Omega^n M) \to \Sigma^{n-1}\widetilde{\mathrm{Free}}_E(\Omega^{n-1}M)$ are obtained. We first note that for any $M \in \mathrm{LMod}_E$ there is a map $M \hookrightarrow \widetilde{\mathrm{Free}}_E(M)$ which is just the inclusion into the $n = 1$ term of the direct sum $\widetilde{\mathrm{Free}}_E(M) \simeq \oplus_{n>0}(M^{\otimes_E n})_{h\Sigma_n}$. Using that $\widetilde{\mathrm{Free}}_E$ is reduced, since $\widetilde{\mathrm{Free}}_E(0) \simeq \oplus_{n>0}(0^{\otimes_E n})_{h\Sigma_n} \simeq 0$, we get that the pullback diagram

$$
\begin{tikzcd}
\Omega M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
0 \arrow[r] & M
\end{tikzcd}
$$

induces the commutative cube

$$
\begin{tikzcd}
& \widetilde{\mathrm{Free}}_E(\Omega M) \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
\Omega M \arrow[ur, hook] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
& 0 \arrow[rr] & & \widetilde{\mathrm{Free}}_E(M). \\
0 \arrow[ur] \arrow[rr] & & M \arrow[ur, hook] &
\end{tikzcd}
$$

Considering the front and back squares of this commutative cube, together with the fact that $\Sigma(-)$ is a pushout, we get the following two unique maps

$$
\begin{tikzcd}
\Omega M \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma\Omega M \arrow[dr, dotted, "\exists!"] & \\
& & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathrm{Free}}(\Omega M) \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma\widetilde{\mathrm{Free}}_E(\Omega M) \arrow[dr, dotted, "\exists!"] & \\
& & \widetilde{\mathrm{Free}}_E(M).
\end{tikzcd}
$$

If we use the spectrum $\Omega^{n-1}M$ and repeat the above, we get the following two unique maps

$$
\Sigma^n\Omega^n M \xrightarrow{\simeq} \Sigma^{n-1}\Omega^{n-1}M,
$$
$$
\Sigma^n\widetilde{\mathrm{Free}}_E(\Omega^n M) \to \Sigma^{n-1}\widetilde{\mathrm{Free}}_E(\Omega^{n-1}M),
$$

where the first map is known to be an equivalence since LMod_E is stable, and where the second map is the one which are used in $\lim_n(\Sigma^n\widetilde{\mathrm{Free}}_E(\Omega^n M))$.

Later when we need to understand this limit in greater details for specific spectra, it will be useful to note that the commutativity of the cube above, together with the construction of the two unique maps by pushouts, gives us the commutative diagram

$$\begin{tikzcd}
\Sigma\Omega M \arrow[r, "\simeq"] \arrow[d] & M \arrow[d] \\
\Sigma\widetilde{\mathrm{Free}}_E(\Omega M) \arrow[r] & \widetilde{\mathrm{Free}}_E(M),
\end{tikzcd}$$

which further induces the following string of commutative squares

$$\begin{tikzcd}
\cdots \arrow[r, "\simeq"] & \Sigma^2\Omega^2 M \arrow[r, "\simeq"] \arrow[d] & \Sigma\Omega M \arrow[r, "\simeq"] \arrow[d] & M \arrow[d] \\
\cdots \arrow[r] & \Sigma^2\widetilde{\mathrm{Free}}_E(\Omega^2 M) \arrow[r] & \Sigma\widetilde{\mathrm{Free}}_E(\Omega M) \arrow[r] & \widetilde{\mathrm{Free}}_E(M),
\end{tikzcd}$$

where the lower string of horizontal maps is the inverse system in $\lim_n\Sigma^n\widetilde{\mathrm{Free}}_E(\Omega^n M)$.

Before we move on, we want to get a further understanding of the maps in this limit.

Proposition 2.0.17. *Let $E \in CAlg$ and $M \in LMod_E$. Then the maps in $lim_n(\Sigma^n\widetilde{Free}_E(\Omega^n M))$ takes products in homotopy to zero.*

Later when we calculate examples of the homotopy of this limit, we will get that each $\Sigma^n\widetilde{\mathrm{Free}}_E(\Omega^n E)$ will be some specific algebra, where knowing that these maps annihilates products lets us simplify the system greatly.

Before we can prove this proposition, we need a better understanding of the multiplicative structure on non-unital commutative ring spectra.

Lemma 2.0.18. *Let $R \in CAlg^{nu}$ be a non-unital commutative ring spectrum. Then we can equip ΩR with a canonical multiplication which is trivial.*

Proof. Let $R \in \mathrm{CAlg}^{nu}$ and X a pointed space. Then we can equip the mapping spectrum $\mathrm{map}_{\mathrm{Sp}}(X, R) := \mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty X, R)$ with the following canonical multiplication:

$$\begin{aligned}
\mathrm{map}_{\mathrm{Sp}}(X, R) \otimes_{\mathbb{S}} \mathrm{map}_{\mathrm{Sp}}(X, R) &\to \mathrm{map}_{\mathrm{Sp}}(X \wedge X, R \otimes_{\mathbb{S}} R) \\
&\to \mathrm{map}_{\mathrm{Sp}}(X, R),
\end{aligned}$$

where the first map is the external tensor product, and the second map is induced by precomposing with the diagonal map $X \to X \wedge X$ and then postcomposing with the multiplication $R \otimes_{\mathbb{S}} R \to R$. Hence we can consider $\mathrm{map}_{\mathrm{Sp}}(X, R) \in \mathrm{CAlg}^{nu}$.

Now, consider the association $X \mapsto \mathrm{map}_{\mathrm{Sp}}(X, R)$, which gives us the functor composed by the left adjoint $\Sigma^\infty(-)$ and the right adjoint $\mathrm{map}_{\mathrm{Sp}}(-, R)$. This means that it takes colimits in $\mathscr{S}_*$ to limits in Sp. By [Lur17, 3.2.2.5] we know that limits in non-unital commutative ring spectra coincides with the limits in spectra, hence we get that this functor takes colimits in $\mathscr{S}_*$ to limits in CAlg^{nu}. Consider the following pushout diagram in $\mathscr{S}_*$

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{pt.} \arrow[d] \\ \mathrm{pt.} \arrow[r] & S^1. \end{tikzcd}$$

By applying the above mentioned functor to this diagram, and noting that $\mathrm{map}_{\mathrm{Sp}}(-, R)$ is a reduced contravariant functor, we get the following pullback diagram in CAlg^{nu}

$$\begin{tikzcd} \mathrm{map}_{\mathrm{Sp}}(S^1, R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{map}_{\mathrm{Sp}}(S^0, R). \end{tikzcd}$$

This implies that $\mathrm{map}_{\mathrm{Sp}}(S^1, R) \simeq 0 \times_{\mathrm{map}_{\mathrm{Sp}}(S^0,R)} 0$, so since $\mathrm{map}_{\mathrm{Sp}}(S^0, R) \simeq R$ we get

$$\mathrm{map}_{\mathrm{Sp}}(S^1, R) \simeq 0 \times_R 0 \simeq \Omega R.$$

This means that ΩR is a non-unital commutative ring spectrum with multiplication induced by the diagonal map $S^1 \to S^1 \wedge S^1 \simeq S^2$, which is null-homotopic. Hence multiplication on ΩR is trivial no matter the multiplication on R. $\square$

Proof of 2.0.17. We first note that ΩM is the pullback of the diagram $0 \to M \leftarrow 0$ in spectra, which by applying $\widetilde{\mathrm{Free}}_E(-)$ becomes the following commutative square of non-unital commutative ring spectra

$$\begin{tikzcd} \widetilde{\mathrm{Free}}_E(\Omega M) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \widetilde{\mathrm{Free}}_E(M), \end{tikzcd}$$

where we have used that $\widetilde{\mathrm{Free}}_E$ is reduced. So using the universal property for the pullback $\Omega\widetilde{\mathrm{Free}}_E(M)$, we get a unique map which makes the diagram commutes

$$\begin{tikzcd}
\widetilde{\mathrm{Free}}_E(\Omega M) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega\widetilde{\mathrm{Free}}_E(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
& 0 \arrow[r] & \widetilde{\mathrm{Free}}_E(M).
\end{tikzcd}$$

By **Lemma 2.0.18** we know that $\Omega\widetilde{\mathrm{Free}}_E(M)$ is a non-unital commutative ring spectrum with trivial multiplication, so the map $\widetilde{\mathrm{Free}}_E(\Omega M) \to \Omega\widetilde{\mathrm{Free}}_E(M)$ takes products in homotopy to zero. Hence the same is true for the functor which we are interested in, just with a degree shift. □

Now that we got a basic understanding of $\mathrm{Pow}(E)$ and its underlying spectrum, we will turn our focus to our first main theorem.

Theorem 2.0.19. *For any commutative ring spectrum E, the ring of stable power operations fits into a diagram of associative ring spectra*

$$E \to Pow(E) \to End(E).$$

The ring $Pow(E)$ has a natural action on the underlying spectrum of any commutative E-algebra in a matter compatible with stable cohomology operations, in the sense that there is a canonical lift in the diagram

$$\begin{tikzcd}
\mathscr{S}^{op} \arrow[rr, "E^{(-)}"] \arrow[dd, "E^{(-)}"'] & & LMod_{End(E)} \arrow[d] \\
& & LMod_{Pow(E)} \arrow[d] \\
CAlg_E \arrow[rru, dashed] \arrow[rr] & & LMod_E.
\end{tikzcd}$$

We note that the diagram in the theorem above, fits into diagram **(1)** in the introduction, which describes how $E^{(-)}$ can be used to describe both the stable operations and the stable power operations on the underlying spectrum of any commutative E-algebra.

To prove this theorem we need the following result.

Proposition 2.0.20 (Prop. 5.7 [GL20]). *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories with finite limits, and let $G : \mathcal{C} \to \mathcal{D}$ be a functor with a left adjoint H. Further assume that $F : \mathcal{C}^{op} \to Sp$ is a functor represented by a spectrum object $Y \in Sp(\mathcal{C})$. Then there is a map of associative ring spectra $End_{Sp(\mathcal{C})}(Y) \to End_{Sp(\mathcal{D})}(GY)$ and a commutative diagram*

$$\begin{tikzcd}
\mathcal{D}^{op} \arrow[r, "map_{Sp(\mathcal{D})}(\Sigma^\infty_+(-), GY)"] \arrow[d, "H^{op}"'] & LMod_{End_{Sp(\mathcal{D})}(GY)} \arrow[d] \\
\mathcal{C}^{op} \arrow[r, "map_{Sp(\mathcal{C})}(\Sigma^\infty_+(-), Y)"'] & LMod_{End_{Sp(\mathcal{C})}(Y)}.
\end{tikzcd}$$

We know that $\mathrm{Pow}(E)$ is the endomorphism ring of a functor, so it makes sense to describe E and $\mathrm{End}(E)$ as endomorphism rings of functors as well.

Lemma 2.0.21. *Let E be a commutative ring spectrum. Then*

$$End(E) \simeq End(E^{(-)} : \mathscr{S}^{op} \to Sp), \quad E \simeq End(H : LMod_E \to Sp),$$

where $E^{(-)}$ denotes the function spectrum and H the forgetful functor.

Proof. To show the first part we note that $E^{(-)} \simeq \mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty_+(-), E)$, so using that $\mathrm{Sp} \simeq \mathrm{Sp}(\mathscr{S})$, we get that the function spectrum $E^{(-)} : \mathscr{S}^{op} \to \mathrm{Sp}$ is represented by E. So by applying **Proposition 2.0.10** we get that $\mathrm{End}(E) \simeq \mathrm{End}(E^{(-)} : \mathscr{S}^{op} \to \mathrm{Sp})$.

To prove the second part, we note that by **Lemma 2.0.11** the forgetful functor $H : \mathrm{LMod}_E \to \mathrm{Sp}$ is represented by the cospectrum object $\{\Omega^n E\}_{n\in\mathbb{Z}}$, which implies that it is represented by $\{\Sigma^n E\}_{n\in\mathbb{Z}} \in \mathrm{Sp}(\mathrm{LMod}_E^{op})$ since LMod_E is stable. So using **Proposition 2.0.10** again, we get

$$\begin{aligned}
\mathrm{End}(H) &\simeq \mathrm{End}_{\mathrm{Sp}(\mathrm{LMod}_E^{op})}(\{\Sigma^n E\}_{n\in\mathbb{Z}}) \\
&= \mathrm{map}_{\mathrm{Sp}(\mathrm{LMod}_E^{op})}(\{\Sigma^n E\}_{n\in\mathbb{Z}}, \{\Sigma^n E\}_{n\in\mathbb{Z}}) \\
&\simeq H(\{\Sigma^n E\}_{n\in\mathbb{Z}}) \\
&\simeq E.
\end{aligned}$$

□

We are now ready to prove **Theorem 2.0.19**.

Proof of theorem 2.0.19. We wish to apply **Proposition 2.0.20** to $\mathcal{C} := \mathrm{CAlg}_E^{op}$, $\mathcal{D} := \mathscr{S}$ and $G := \mathrm{Map}_{\mathrm{CAlg}_E}(-, E)^{op}$. By [GL20, 7.1 (2)] we get that $E^{(-),op}$ is left adjoint to $\mathrm{Map}_{\mathrm{CAlg}_E}(-, E)^{op}$. We further have from the proof of **Theorem 2.0.9** that the forgetful functor $F : \mathrm{CAlg}_E \to \mathrm{Sp}$ is represented by the cospectrum object $Y := \{\mathrm{Free}_E(\Omega^n E)\}$. So by **Proposition 2.0.20** we get the following map of associative ring spectra

$$\mathrm{End}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\{\mathrm{Free}_E(\Omega^n E)\}) \to \mathrm{End}_{\mathrm{Sp}(\mathscr{S})}(\mathrm{Map}_{\mathrm{CAlg}_E}(\{\mathrm{Free}_E(\Omega^n E)\}, E)^{op}).$$

Using **Proposition 2.0.10** we first get that

$$\mathrm{End}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\{\mathrm{Free}_E(\Omega^n E)\}) \simeq \mathrm{End}(F : \mathrm{CAlg}_E \to \mathrm{Sp}) = \mathrm{Pow}(E).$$

Let $G : \mathrm{CAlg}_E \to \mathrm{LMod}_E$ denote the forgetful functor and recall the adjunction $G \vdash \mathrm{Free}_E$ from **Proposition 2.0.7**. Using that LMod_E is stable we then get

$$\begin{aligned}\{\mathrm{Map}_{\mathrm{CAlg}_E}(\mathrm{Free}_E(\Omega^n E), E)^{op}\} &\simeq \{\mathrm{Map}_{\mathrm{LMod}_E}(\Omega^n E, G(E))^{op}\}\\ &\simeq \{\mathrm{Map}_{\mathrm{LMod}_E}(G(E), \Sigma^n G(E))^{op}\}\\ &\simeq \{\Omega^\infty \Sigma^n F(E)\}\\ &\simeq F(E).\end{aligned}$$

This gives us that the above map is equivalent to $\mathrm{Pow}(E) \to \mathrm{End}(E)$.

By **Lemma 2.0.21** we know that $E \simeq \mathrm{End}(H : \mathrm{LMod}_E \to \mathrm{Sp})$, with H the forgetful functor, so using the following commutative diagram of forgetful functors

$$\begin{tikzcd} \mathrm{CAlg}_E \arrow[d, "G"'] \arrow[dr, "F"] & \\ \mathrm{LMod}_E \arrow[r, "H"'] & \mathrm{Sp}, \end{tikzcd}$$

we get a natural map $E \to \mathrm{Pow}(E)$ by precomposing with G. Hence we got the desired string of associative ring spectra

$$E \to \mathrm{Pow}(E) \to \mathrm{End}(E).$$

Using that we know from the proof of **Lemma 2.0.21** that $E^{(-)}$ is represented by $E \in \mathrm{Sp}(\mathscr{S})$, and hence by **Proposition 2.0.10** can be considered as the functor $\mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty_+(-), E) : \mathscr{S}^{op} \to \mathrm{LMod}_{\mathrm{End}(E)}$, we get the following commutative diagram by **Proposition 2.0.20**

$$\begin{tikzcd} \mathscr{S}^{op} \arrow[r, "E^{(-)}"] \arrow[d, "E^{(-)}"] & \mathrm{LMod}_{\mathrm{End}(E)} \arrow[d] \\ \mathrm{CAlg}_E \arrow[r, "\mathrm{map}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\Sigma^\infty_+(-), \{\mathrm{Free}_E(\Omega^n E)\})"'] & \mathrm{LMod}_{\mathrm{Pow}(E)}. \end{tikzcd}$$

We see that this map $\mathrm{CAlg}_E \to \mathrm{LMod}_{\mathrm{Pow}(E)}$ is the desired lift since the following triangle commutes

$$\begin{tikzcd}
\mathrm{CAlg}_E \arrow[rr, "\mathrm{map}_{\mathrm{Sp}(\mathrm{CAlg}_E^{op})}(\Sigma^\infty_+(-),\{\mathrm{Free}_E(\Omega^n E)\})"] \arrow[drr, "G"'] & & \mathrm{LMod}_{\mathrm{Pow}(E)} \arrow[d] \\
& & \mathrm{LMod}_E.
\end{tikzcd}$$

□

2.1 Steenrod Operations

Now that we have a basic understanding of the ring of stable power operations $\mathrm{Pow}(E)$, we would like to consider it in the fundamental case, namely on the commutative ring spectrum $H\mathbb{F}_p$ for p a prime. This will let us show that the stable power operations described by $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ on ordinary mod-p cohomology $H^*(-,\mathbb{F}_p)$, agrees with the well-known theory of Steenrod operations. Hence the goal of this section is to calculate $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$.

To be able to calculate $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$, we will recall the Steenrod operations $P^i : H^n(X,\mathbb{F}_p) \to H^{n+2i(p-1)}(X,\mathbb{F}_p)$ together with their basic properties and then give them algebraic structure by defining the big Steenrod algebra $\mathfrak{B}_p$ using their sums and composition. These operations are usually constructed as power operations, but it can be shown that they are stable cohomology operations as well, hence it makes sense to expect $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ to capture these.

There are two different ways to define the Steenrod operations: Through a construction or by an axiomatic characterisation. We will define them axiomatically for a fixed odd prime p. We will not consider the case $p = 2$, which is similar, if slightly easier.

Definition 2.1.1. The *ith Steenrod operation* is a homomorphism

$$P^i : H^n(X;\mathbb{F}_p) \to H^{n+2i(p-1)}(X;\mathbb{F}_p),$$

which satisfy the following axioms for $x, y \in H^n(X;\mathbb{F}_p)$ and X, Y topological spaces

- Additivity: $P^i(x + y) = P^i(x) + P^i(y)$
- Natural: $P^i(f^*(x)) = f^*(P^i(x))$ for $f : X \to Y$
- Cartan formula: $P^i(x \smile y) = \sum_{j+l=i}(P^j(x) \smile P^l(y))$
- Squaring: $P^i(x) = x^p$ if $2i = |x|$

- Instability: $P^i(x) = 0$ if $2i > |x|$
- Unitality: $P^0 = 1$ the identity.

We note that the squaring property explains why P^i raises the degree by $2i(p-1)$. In [Ste62, VI.(1)] it is shown that these operations exists uniquely and that they are stable, i.e. commutes with the boundary map in the long exact sequence of cohomology. It is further shown that composition of these operations satisfy the *Adem relations*

$$\begin{aligned}
P^aP^b &= \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj}P^{a+b-j}P^j \quad \text{if } a < pb \\
P^a\beta P^b &= \sum_j (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta P^{a+b-j}P^j \\
&+ \sum_j (-1)^{a+j+1}\binom{(p-1)(b-j)-1}{a-pj-1}P^{a+b-j}\beta P^j \quad \text{if } a \leq pb,
\end{aligned}$$

where β denotes the Bockstein homomorphism $\beta : H^n(X;\mathbb{F}_p) \to H^{n+1}(X;\mathbb{F}_p)$.

Definition 2.1.2. The *big Steenrod algebra* $\mathfrak{B}_p$ is the quotient of the free non-commutative $\mathbb{F}_p$-algebra generated by $\beta, P^0, P^1, P^2, ...,$ by the Adem relations.

We equip $\mathfrak{B}_p$ with the grading

$$\deg(P^i) = 2i(p-1), \ \ \deg(\beta) = 1.$$

We can consider $H^*(X;\mathbb{F}_p)$ as a $\mathfrak{B}_p$-module for any space X, but it has more structure than that, due to the cup product giving it a multiplicative structure, which is compatible with the action of $\mathfrak{B}_p$. This extra structure is captured in the following definition.

Definition 2.1.3. An *algebra with Dyer-Lashof operations* is a graded-commutative $\mathbb{F}_p$-algebra A equipped with operations

$$\begin{aligned}
P^i_A &: A_n \to A_{n+2i(p-1)}, \\
\beta P^i_A &: A_n \to A_{n+2i(p-1)-1},
\end{aligned}$$

written as $\beta^\varepsilon P^i_A$ for $\varepsilon = 0$ or 1, which satisfy the following properties

- Additivity: $\beta^\varepsilon P^i_A(x+y) = \beta^\varepsilon P^i_A(x) + \beta^\varepsilon P^i_A(y)$
- Squaring: $P^i_A(x) = x^p$ if $2i = |x|$
- Instability: $\beta^\varepsilon P^i_A(x) = 0$ if $2i + \varepsilon < |x|$

- Unitality: $P^i_A(1) = 0$ if $i > 0$
- Cartan formula:

$$P^s_A(xy) = \sum_{i+j=s} P^i_A(x)P^j_A(y)$$
$$\beta P^s_A(xy) = \sum_{i+j=s} \beta P^i_A(x)P^j_A(y) + \sum_{i+j=s} P^i_A(x)\beta P^j_A(y).$$

In addition they are assumed to satisfy the Adem relations. These relations still holds after formally applying β on the left and eliminating terms involving $\beta\beta$.

It is clear that $\mathfrak{B}_p$ makes $H^*(X;\mathbb{F}_p)$ into an algebra with Dyer-Lashof operations, where $\beta^\varepsilon P^i_{H^*(X;\mathbb{F}_p)} = \beta^\varepsilon P^{-i}$ is the composition of the Steenrod operations P^{-i} and the Bockstein homomorphism β, which has degree $-\varepsilon$. Note the difference in the grading, which occurs due to the change from homological grading to cohomological grading. We want a more general way to describe the structure of these operations.

Definition 2.1.4. Let A be an algebra with Dyer-Lashof operations and consider a sequence $I = (\varepsilon_1, s_1, \cdots, \varepsilon_k, s_k)$ with $\varepsilon_i = 0$ or 1 and $s_i \in \mathbb{Z}$. Then I determines an operation

$$P^I_A = \beta^{\varepsilon_1} P^{s_1}_A \cdots \beta^{\varepsilon_k} P^{s_k}_A,$$

where $\beta^0 P^{s_i}_A = P^{s_i}_A$ and $\beta^1 P^{s_i}_A = \beta P^{s_i}_A$. We say that I has *length* k, and we call it *admissible* if $s_i \geq ps_{i+1} + \varepsilon_{i+1}$. Then we define the *excess* of I as

$$e(I) = 2s_k + \varepsilon_1 + \sum_{i=1}^{k-1}(2s_i - 2ps_{i+1} - \varepsilon_{i+1}) = 2s_1 + \varepsilon_1 - \sum_{i=2}^{k}(2s_i(p-1) + \varepsilon_i).$$

By convention, the empty set determines the identity operation, has length 0, is admissible and has excess $e(\emptyset) = -\infty$. The grading difference mentioned above is encoded in this sequence I, where the s_i changes sign whether we consider it as a composite of Dyer-Lashof operations or of Steenrod operations. Just to clarify the signs, consider the following composition of Steenrod operations $\beta^{\varepsilon_1} P^{s_1} \cdots \beta^{\varepsilon_k} P^{s_k}$ on $H^*(X;\mathbb{F}_p)$. If we wish to consider this in terms of Dyer-Lashof operations, it corresponds to $P^I_{H^*(X;\mathbb{F}_p)}$ where $I = (\varepsilon_1, -s_1, \cdots, \varepsilon_k, -s_k)$. So a Steenrod operation determined by this I, is admissible if $s_j \leq ps_{j+1}\varepsilon_{j+1}$ and has excess given by $e(I) = \varepsilon_1 - 2s_1 + \sum_{j=2}^{k}(2s_j(p-1) - \varepsilon_j)$.

From the following result we get that it is sufficient for us to consider those compositions of Steenrod operations P^I, which are determined by admissible sequences I.

Theorem 2.1.5 (Prop.5.2 [Man01])**.** *The set $\{P^I | I$ admissible $\}$ is a basis for the underlying $\mathbb{F}_p$-module $\mathfrak{B}_p$.*

Using this, we can describe the grading on $\mathfrak{B}_p$ as follows: Given a sequence I of length k as above, we have

$$\deg(P^I) = \sum_{i=1}^{k} \varepsilon_i + 2s_i(p-1).$$

Using the instability we get that $P^{s_i}(x) = 0$ for $2s_i > |x|$, so in any P^I there is only finitely many of the $P^{s_i}(x)$ which is non-zero. So we get that $\mathfrak{B}_p$ acts naturally on $H^*(X;\mathbb{Z}_p)$ in a way which preserves the grading.

This instability further gives us that any sum $\sum_{i\geq 0} P^i(x)$ is finite. In $\mathfrak{B}_p$ we are only allowed to take finite sums, but due to the instability we see that we can also make sense to considering an infinite sum of Steenrod operations, as an operation. Since $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ describes all the stable power operations on ordinary mod-p cohomology, we get that we would expect this to also capture these infinite sums. Therefore we will need the following definition.

Definition 2.1.6. Let $F_i\mathfrak{B}_p := \{P^I | I \text{ admissible }, e(I) \geq i\}$. This gives a decreasing filtration $\mathcal{F} = (F_i\mathfrak{B}_p)_{i\in\mathbb{Z}}$ of $\mathfrak{B}_p$, which we call the *excess filtration.*

We see that the completion $(\mathfrak{B}_p)^\wedge := \lim \mathfrak{B}_p/F_i\mathfrak{B}_p$ of $\mathfrak{B}_p$ with respect to the excess filtration, describes both all the Steenrod operations as well as the finite and infinite sums therefore. Note here that we are considering the limit as an ordinary limit in the category of graded $\mathbb{F}_p$-algebras. We wish to show that $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ is isomorphic to this completion.

To do so we will need a way to equip graded $\mathbb{F}_p$-vector speces with Dyer-Lashof operations. Let $\mathbb{Q}$ denote the free functor from the category of graded $\mathbb{F}_p$-vector spaces to the category of graded $\mathbb{F}_p$-algebras equipped with Dyer-Lashof operations, which is left adjoint to the inclusion. We then have the following concrete description.

Theorem 2.1.7 (Thm. 10.2 [GL20])**.** *Let V be a graded $\mathbb{F}_p$-vector space and $\{e_i\}$ a basis. Then $\mathbb{Q}(V)$ is a free graded-commutative algebra on admissible monomials of excess $e(I) \geq |e_i|$.*

Assuming that the monomials are admissible implies that it is those where we cannot apply the Adem relations. The next theorem then gives us a connection between the algebraic description above and topology.

Theorem 2.1.8 (Thm.10.2 [GL20])**.** *Let X be a spectrum. Then we have a canonical isomorphism between the homotopy of the free commutative $H\mathbb{F}_p$-algebra $\mathrm{Free}_{H\mathbb{F}_p}(H\mathbb{F}_p \otimes_{\mathbb{S}} X)$ and the free algebra $\mathbb{Q}(H^*(X;\mathbb{F}_p))$ in the category of $\mathbb{F}_p$-algebras with Dyer-Lashof operations.*

Note that here $\mathrm{Free}_{H\mathbb{F}_p}$ denotes the functor constructed in **Proposition 2.0.7**. We are now ready to calculate $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$.

Theorem 2.1.9. *The homotopy groups $\pi_* Pow(H\mathbb{F}_p)$ of the ring of stable power operations on commutative $H\mathbb{F}_p$-algebras is isomorphic to the completion $(\mathfrak{B}_p)^\wedge$ of the big Steenrod algebra with respect to the excess filtration.*

Proof. We start by recalling that by **Proposition 2.0.15** we know that the underlying spectrum of $\mathrm{Pow}(H\mathbb{F}_p)$ is $\lim_n(\Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p))$, hence we are interested in understanding $\pi_* \lim_n(\widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p))$. Consider the Milnor short exact sequence for $\Sigma^n \widetilde{\mathrm{LFree}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)$ which is given by

$$\begin{aligned} 0 &\longrightarrow \lim\nolimits^1_n \pi_{q+1}(\Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)) \\ &\longrightarrow \pi_q(\lim\nolimits_n \Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)) \\ &\longrightarrow \lim\nolimits_n(\pi_q \Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)) \longrightarrow 0. \end{aligned}$$

We will later show that the $\lim^1$-term vanishes, so we will consider $\lim_n \pi_q(\Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p))$. First we note that

$$\begin{aligned} \pi_* \Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p) &\cong \pi_{*-n} \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p) \\ &\cong \Sigma^n \pi_* \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p). \end{aligned}$$

Using **Theorem 2.1.8** with $X = \Omega^n \mathbb{S}$, we get that

$$\begin{aligned} \pi_*(\mathrm{Free}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)) &\cong \pi_*(\mathrm{Free}_{H\mathbb{F}_p}(\Omega^n \mathbb{S} \otimes H\mathbb{F}_p)) \\ &\cong \mathbb{Q}(H^*(\Omega^n \mathbb{S}; \mathbb{F}_p)) \\ &\cong \mathbb{Q}(\Omega^n \mathbb{F}_p), \end{aligned}$$

so using that π_* commutes with finite limits, we get

$$\begin{aligned} \widetilde{\mathbb{Q}}(\Omega^n \mathbb{F}_p) &:= \ker(\mathbb{Q}(\Omega^n \mathbb{F}_p) \to \mathbb{F}_p) \\ &\cong \ker(\pi_*(\mathrm{Free}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p)) \to \mathbb{F}_p) \\ &\cong \pi_* \mathrm{fib}(\mathrm{Free}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p) \to H\mathbb{F}_p) \\ &\cong \pi_* \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p). \end{aligned}$$

This gives us that $\lim_n \pi_*(\Sigma^n \widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p))$ is equivalent to the inverse system

$$\cdots \xrightarrow{\theta_{n+1}} \Sigma^n \widetilde{\mathbb{Q}}(\Omega^n \mathbb{F}_p) \xrightarrow{\theta_n} \Sigma^{n-1} \widetilde{\mathbb{Q}}(\Omega^{n-1} \mathbb{F}_p) \xrightarrow{\theta_{n-1}} \cdots .$$

Using **Theorem 2.1.7** we know that $\mathbb{Q}(\Omega^n\mathbb{F}_p)$ is a free graded-commutative $\mathbb{F}_p$-algebra on admissible monomials $P^I e_n$ of excess $e(I) \geq |e_n|$. We then obtain $\widetilde{\mathbb{Q}}(\Omega^n\mathbb{F}_p)$ by removing the unit from this algebra. Note that $\Omega^n\mathbb{F}_p$ is a graded vector space in degree $-n$ on e_n, i.e. $|e_n| = -n$.

Next we wish to get a better understanding of the maps θ_n. By **Remark 2.0.16** we have the following commutative square

$$\begin{tikzcd}
\Sigma^n\Omega^n H\mathbb{F}_p \arrow[r, "\simeq"] \arrow[d] & \Sigma^{n-1}\Omega^{n-1}H\mathbb{F}_p \arrow[d] \\
\Sigma^n\widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^n H\mathbb{F}_p) \arrow[r] & \Sigma^{n-1}\widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(\Omega^{n-1}).
\end{tikzcd}$$

By applying $\pi_*(-)$ and using the above calculation, we get the following commutative diagram containing the maps θ_n

$$\begin{tikzcd}
\cdots \arrow[r, "\cong"] & \pi_*(\Sigma^2\Omega^2 H\mathbb{F}_p) \arrow[r, "\cong"] \arrow[d] & \pi_*(\Sigma\Omega H\mathbb{F}_p) \arrow[r, "\cong"] \arrow[d] & \pi_*(H\mathbb{F}_p) \cong \mathbb{F}_p \arrow[d] \\
\cdots \arrow[r, "\theta_3"] & \Sigma^2\widetilde{\mathbb{Q}}(\Omega^2\mathbb{F}_p) \arrow[r, "\theta_2"] & \Sigma\widetilde{\mathbb{Q}}(\Omega\mathbb{F}_p) \arrow[r, "\theta_1"] & \widetilde{\mathbb{Q}}(\mathbb{F}_p).
\end{tikzcd}$$

Let 1 denote the generating element of $\mathbb{F}_p$, and note that the isomorphism $\pi_* H\mathbb{F}_p \cong \mathbb{F}_p$ maps some generator $e_0 \in \pi_* H\mathbb{F}_p$ to 1. Further, the isomorphism $\widetilde{\mathbb{Q}}(\mathbb{F}_p) \xrightarrow{\cong} \pi_*(\widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(H\mathbb{F}_p))$ from **Theorem 2.1.8** maps the formal $P^I e_0$ to $P^I(e_0)$, and similar for products. In particular this means that e_0 is mapped to e_0. So from the commutative square

$$\begin{tikzcd}
\pi_* H\mathbb{F}_p \arrow[r, "\cong"] \arrow[d] & \mathbb{F}_p \arrow[d] \\
\pi_*(\widetilde{\mathrm{Free}}_{H\mathbb{F}_p}(H\mathbb{F}_p)) & \widetilde{\mathbb{Q}}(\mathbb{F}_p) \arrow[l, "\cong"']
\end{tikzcd}$$

we get that the right vertical arrow needs to map 1 to e_0. This gives us that $\theta_1(e_1) = e_0$. The top horizontal string of isomorphisms $\pi_*(\Sigma^n\Omega^n H\mathbb{F}_p) \cong \pi_*(\Sigma^{n-1}\Omega^{n-1}H\mathbb{F}_p)$ maps e_n to e_{n-1}, so since the above string of squares commutes, we get that $\theta_n(e_n) = e_{n-1}$. Further using that the Dyer-Lashof operations are stable [Law20, 5.13], we get that $\theta_n(P^I e_n) \cong P^I\theta_n(e_n) \cong P^I e_{n-1}$ for $P^I \in \Sigma^n\tilde{\mathbb{Q}}(\Omega^n\mathbb{F}_p)$. By **Proposition 2.0.17** we know that θ_n annihilates products, hence the above system is equivalent to the inverse system

$$\cdots \longrightarrow \{P^I e_n | e(I) \geq -n\} \longrightarrow \{P^I e_{n-1} | e(I) \geq -(n-1)\} \longrightarrow \cdots,$$

with I running through all admissible sequences. These maps are surjective, hence the system satisfies the Mittage-Leffler condition, so the $\lim^1$-term in the first short exact sequence disappears. This gives us that

$$\pi_q(\lim_n \Sigma^n \widetilde{\text{Free}}(\Omega^n H\mathbb{F}_p)) \cong \lim_n(\pi_q \Sigma^n \widetilde{\text{Free}}(\Omega^n H\mathbb{F}_p)),$$

so the homotopy group of $\lim_n \Sigma^n(\widetilde{\text{Free}}\Omega^n H\mathbb{F}_p)$ is equivalent to the above inverse system. Using that the underlying spectrum of $\text{Pow}(H\mathbb{F}_p)$ is exactly $\lim_n \Sigma^n(\widetilde{\text{Free}}\Omega^n H\mathbb{F}_p)$, we get that $\pi_*\text{Pow}(H\mathbb{F}_p)$ is the completion of the group with basis $\{P^I e_n \mid I \text{ admissible }\}$, with respect to the excess filtration. Using that this is the basis for the big Steenrod algebra $\mathfrak{B}_p$ by **Theorem 2.1.5** we get that $\pi_*\text{Pow}(H\mathbb{F}_p) \cong (\mathfrak{B}_p)^\wedge$ as desired. □

3 Morava E-theory

Let K denote the complex K-theory and recall that this is a 2-periodic spectrum which satisfy $K_* := \pi_* K \cong \mathbb{Z}[u^{\pm 1}]$, with $|u| = 2$. We then define the *p-adic K-theory* as $K_p^\wedge := \lim_r K/p^r$, for some fixed prime p. The goal for the rest of this thesis is to prove a result similar to **Theorem 2.1.9** for $K_p^\wedge$ instead of $H\mathbb{F}_p$. It will not be possible for us to calculate $\pi_*\text{Pow}(K_p^\wedge)$ in general, but if we restrict ourself to a localized version, we get a useful notion of commutative $K_p^\wedge$-algebras, for which we can sketch the calculation of the stable power operations. Therefore we will first try and understand this localization together with its connection to completion.

It is show in [Lur10, Lec.24, prop.12] that $K_p^\wedge$ is in fact equivalent to the first in a family of cohomology theories called *Morava E-theory*. These are parametrized by a choice of prime p and an integer n, and each one of these is a 2-periodic commutative ring spectrum $E(\mathbb{F}_{p^n}, \Gamma)$ called *Morava E-theory at height n*. To get deeper into the construction of these would take us too far afield, but it is sufficient to say that $E(\mathbb{F}_{p^n}, \Gamma)$ is obtained from the universal deformation of a height n formal group law Γ by applying the Landweber exact theorem.

The abstract theory which we need to develop for $K_p^\wedge$ holds for $E(\mathbb{F}_{p^n}, \Gamma)$ at any height n, and we will therefore do this part of the set-up in this generality.

We will fix an integer n and a prime p throughout. We will therefore omit these from the notation and simply write $E := E(\mathbb{F}_{p^n}, \Gamma)$. We further write E_* for $\pi_* E$, so $E_*(X) = \pi_*(E \otimes_{\mathbb{S}} X)$ for $X \in \text{Sp}$. It can be shown that $E_* \cong W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$, where $W(\mathbb{F}_{p^n})$ denotes the ring of p-typical Witt vectors on $\mathbb{F}_{p^n}$ and where $|u_i| = 0$, $|u| = 2$. It can be shown that $E_0 \cong W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]]$, which is a complete Noetherian regular ring with maximal ideal $\mathfrak{m} = (p, u_1, ..., u_{n-1})$, where we write $u_0 = p$ to ease notation.

This implies that $E_* \cong E_0[u^{\pm 1}]$. Even though E is 2-periodic we will consider E_*-modules, including E_* itself, as $\mathbb{Z}$-graded. For a good introduction to the construction of this cohomology theory see the lecture notes [Lur10], in particular lectures 16, 21 and 22.

Let $K(n)$ denote the *Morava K-theory* at height n, which is the unique E-module satisfying $\pi_* K(n) \cong E_*/\mathfrak{m}$ and plays the role of a residue field for E. Consider the Bousfield localization of spectra with respect to Morava K-theory

$$L_{K(n)} : \mathrm{Mod}_E \to \mathrm{Mod}_E.$$

We say that an E-module M is $K(n)$-*local* if the $K(n)$-localization map $M \to L_{K(n)}M$ is an equivalence.

Definition 3.0.1. Let X be some spectrum. Then we define the *completed E-homology* of X as

$$E_*^\wedge(X) := \pi_* L_{K(n)}(E \otimes_{\mathbb{S}} X).$$

We will later discuss how $K(n)$-localization is related to the algebraic process of completion which will explain the name of $E_*^\wedge(-)$.

Definition 3.0.2. Let M be any E_*-module and $\mathfrak{m} \subset E_0$ the maximal ideal. Then we define the *completion with respect to* $\mathfrak{m}$ as $M_\mathfrak{m}^\wedge := \lim M/\mathfrak{m}^k$, and we say that M is $\mathfrak{m}$-*complete* if the canonical map $M \to M_\mathfrak{m}^\wedge$ is an isomorphism.

Further, let $L_s : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ denote the sth left derived functor of $M \mapsto M_\mathfrak{m}^\wedge$. Then the natural map $M \to M_\mathfrak{m}^\wedge$ factors canonically as $M \xrightarrow{\eta_M} L_0 M \to M_\mathfrak{m}^\wedge$, and we call M *L-complete* if η_M is an isomorphism.

Note that Mod_{E_*} denotes the abelian category of graded E_*-modules, while Mod_E denotes the symmetric monoidal ∞-category of E-modules.

Remark 3.0.3. In the above we chose to define our completion with respect to the maximal ideal of E_0, but it would seem more natural to define it using an ideal of E_*. The reason behind our choice is that both options gives the same completion, but our definition is a bit easier to work with in practice. As mentioned in the beginning of this section we know that $E_* \cong E_0[u^{\pm 1}]$, $|u| = 2$, so since E_0 is a regular local ring we have a unique maximal ideal $\mathfrak{m} \subset E_0$. Consider the homogenous graded ideal $I := \mathfrak{m}E_* \subset E_*$ and let $M_* \in \mathrm{Mod}_{E_*}$. Then we have

$$(M_*)_I^\wedge = \lim M_*/I^k M_*, \quad (M_*)_\mathfrak{m}^\wedge = \lim M_*/\mathfrak{m}^k M_*,$$

so to prove that the completions agree, it is sufficient to show that $IM_* \cong \mathfrak{m}M_*$. For $x \in \mathfrak{m}$ and $m \in M_*$ we get the following isomorphism

$$\begin{aligned} IM_* &\cong \mathfrak{m}M_* \\ (x \cdot u^k) \cdot m &\mapsto x \cdot (u^k \cdot m) \\ (x \cdot u^0) \cdot m &\leftarrow\!\shortmid x \cdot m. \end{aligned}$$

Our first goal of this section is to construct an endofunctor $\mathbb{T}$ on the abelian category Mod_{E_*} of graded E_*-modules, which will satisfy $(\mathbb{T}(\pi_* M))^{\wedge}_{\mathfrak{m}} \cong \pi_* L_{K(n)}\mathrm{Free}_E(M)$ for M a so called flat E-module. This functor can be considered as the E-theoretic analogue of the functor $\mathbb{Q}$, which we used in the calculation of $\pi_*\mathrm{Pow}(H\mathbb{F}_p)$ in **Theorem 2.1.9**. This part follows [Rez09, Section 4].

Definition 3.0.4. We say that an E-module M is *finite free* if $\pi_* M$ is a finitely generated free E_*-module. We let $\mathrm{Mod}_E^{\mathrm{ff}}$ denote the full subcategory of Mod_E consisting of the finite free modules. Further let $h\mathrm{Mod}_E^{\mathrm{ff}}$ denote the full subcategory of $h\mathrm{Mod}_E$ which are spanned by the finite free E-modules.

Note that every finite free E-module are equivalent to a module of the form $\oplus_{i=1}^{k}\Sigma^{d_i}E$, where Σ^{d_i} denotes the suspension. So in particular we have that $E \in \mathrm{Mod}_E^{\mathrm{ff}}$.

Proposition 3.0.5 (Prop. 3.12 [Rez09])**.** *The functor $\pi_* : hMod_E^{ff} \to Mod_{E_*}^{ff}$, which takes any finite free E-module M to its homotopy groups $\pi_* M$, is an equivalence of categories.*

It is a strong assumption to assume that a module is finite free, so we will often only need the following weaker definition.

Definition 3.0.6. We say that an E-module M is *flat*, if $\pi_* M$ is flat when considered as a graded E_*-module.

Note that if M is a finite free E-module, then M is a flat E-module as well, since finite free E_*-modules in particular are flat. When we wish to work with this notion, we will need the following higher algebraic version of Lazard's theorem for flat modules over a ring, here stated with the "ring" being Morava E-theory.

Proposition 3.0.7 (Prop. 3.14 [Rez09])**.** *An object $M \in Mod_E$ is flat if and only if it is weakly equivalent to the colimit of some functor $F : \mathcal{C} \to Mod_E^{ff}$ which preserves all small limits in $\mathcal{C}$, for some filtered category $\mathcal{C}$ enriched over topological spaces.*

This means that a filtered colimit $\mathrm{colim}_{j\in J} M_j$ over a collection $\{M_j\}_J$ of finite free E-modules, is a flat E-module.

We wish to try and understand the correspondence between the different completions mentioned above and $K(n)$-localizations, in particular when considering flat E-modules. To do so, we will need the following two results.

Proposition 3.0.8 (Prop. 3.6 [Rez09])**.** *The functors $L_s : Mod_{E_*} \to Mod_{E_*}$ vanishes for $s > n$. If M_* is a flat E_*-module, then $L_0(M_*) \cong (M_*)^{\wedge}_{\mathfrak{m}}$ and $L_s(M_*) = 0$ for $s > 0$.*

Proposition 3.0.9 (Prop 3.7 [Rez09])**.** *There exists a conditionally and strongly convergent spectral sequence of E_*-modules*

$$E_2^{s,t} = L_s \pi_t M \Rightarrow \pi_{s+t} L_{K(n)} M,$$

which vanishes for $s > n$.

Proposition 3.0.10. *For any flat E-module M we have a string of isomorphisms*

$$(\pi_* M)^\wedge_{\mathfrak{m}} \cong L_0(\pi_* M) \cong \pi_*(L_{K(n)} M).$$

Proof. Let M be a flat E-module. Then we know that $L_s(\pi_* M) = 0$ for $s > 0$ by **Proposition 3.0.8**, hence the only non-trivial entries on the E_2-page of the spectral sequence from **Proposition 3.0.9** are $E_2^{0,t} = L_0 \pi_t M$. This gives us that all the differentials are trivial, so the spectral sequence degenerates at the E_2-page, i.e. $E_2^{s,t} \cong E_\infty^{s,t}$. The convergence part of **Proposition 3.0.9** implies that there exists some sequence of abelian groups

$$0 \subseteq F^{s+t}_{s+t} \subseteq F^{s+t}_{s+t-1} \subseteq \cdots \subseteq F^{s+t}_0 \cong \pi_{s+t} L_{K(n)} M,$$

such that $E_\infty^{s,t} \cong F^{s+t}_t / F^{s+t}_{t+1}$. First we have that

$$L_0 \pi_{s+t} M \cong E_\infty^{0,s+t} \cong F^{s+t}_{s+t} / F^{s+t}_{s+t+1} \cong F^{s+t}_{s+t}.$$

Using that $E_\infty^{k,s+t-k} \cong 0$ for all $k \neq 0$, we get that

$$F^{s+t}_{s+t-1} / F^{s+t}_{s+t} \cong E_\infty^{1,s+t-1} \cong 0,$$

which implies that $F^{s+t}_{s+t-1} \cong F^{s+t}_{s+t}$. By repeating this argument we get a string of isomorphisms

$$L_0 \pi_{s+t} M \cong F^{s+t}_{s+t} \cong F^{s+t}_{s+t-1} \cong \cdots \cong F^{s+t}_0 \cong \pi_{s+t} L_{K(n)} M.$$

Hence $L_0 \pi_*(M) \cong \pi_* L_{K(n)} M$ for any flat E-module M, so by applying **Proposition 3.0.8** once more we get

$$(\pi_* M)^\wedge_{\mathfrak{m}} \cong L_0(\pi_* M) \cong \pi_*(L_{K(n)} M).$$

□

Example 3.0.11. If we consider Morava E-theory at height 1, then we know that $\pi_* E(\mathbb{F}_p, \Gamma) \cong \mathbb{Z}_p[u^{\pm 1}]$ with $|u| = 2$ and $\mathbb{Z}_p$ the p-adic integers, and then the maximal ideal is $\mathfrak{m} = (p) \subset \pi_0 E(\mathbb{F}_p, \Gamma)$. Let $M_* \in \mathrm{Mod}^{\mathrm{ff}}_{\pi_* E(\mathbb{F}_p,\Gamma)}$ and $M \in h\mathrm{Mod}^{\mathrm{ff}}_{E(\mathbb{F}_p,\Gamma)}$ such that $\pi_*(M) \cong M_*$. Then **Proposition 3.0.10** gives us that M_* is L-complete if and only if it is p-complete, which further holds if and only if M is $K(1)$-local.

Definition 3.0.12. Let M be an E-module. Then we define the *mth extended power* as $\mathbb{P}_m(M) := (M^{\otimes_E m})_{h\Sigma_m}$, which gives a functor $\mathbb{P}_m : \mathrm{Mod}_E \to \mathrm{Mod}_E$. This definition also passes to an endofunctor on $h\mathrm{Mod}_E$, which we also denote by $\mathbb{P}_m$.

First we will need these functors $\mathbb{P}_m$ to define the above mentioned functor $\mathbb{T}$, but later it will also be important that $\mathrm{Free}_E M \simeq \oplus_{m \geq 0} \mathbb{P}_m(M)$. Before we can use the above definition to construct the desired functor $\mathbb{T}$, we first need to recall the notion of a Left Kan extension.

Let $F : \mathcal{E} \to \mathcal{D}$ and $H : \mathcal{E} \to \mathcal{C}$ be two functors between ∞-categories. Then a *Left Kan extension* of F along H is a functor $\mathrm{Lan}_H F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\eta : F \to \mathrm{Lan}_H F \circ H$, which is the initial such pair

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "H"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\eta"] & & \mathcal{D}. \\ & \mathcal{C} \arrow[ur, dashed, "\mathrm{Lan}_H F"'] & \end{tikzcd}$$

That it is the initial pair means that if $(G : \mathcal{C} \to \mathcal{D},\ \theta : F \to G \circ H)$ is another pair as above, then there exists a natural transformation $\omega : \mathrm{Lan}_H F \to G$ such that $\theta = \omega H \circ \eta$. This universality is equivalent to the following: If $\mathrm{Lan}_H F$ is as above, then the universality gives a bijection

$$\mathcal{D}^{\mathcal{C}}(\mathrm{Lan}_H F, G) \simeq \mathcal{D}^{\mathcal{E}}(F, GH)$$

for any $G \in \mathcal{D}^{\mathcal{C}}$, where $\mathcal{D}^{(-)} = \mathrm{Fun}(-, \mathcal{D})$.

Now, consider the following diagram

$$\begin{tikzcd} \mathrm{hMod}_E^{\mathrm{ff}} \arrow[rrr, bend left, "\pi_* L_{K(n)} \mathbb{P}_m i"] \arrow[rr, hook, "i"] \arrow[dr, "\pi_*", "\simeq"'] & & \mathrm{hMod}_E \arrow[dr, "\pi_*"] & \mathrm{Mod}_{E_*} \\ & \mathrm{Mod}_{E_*}^{\mathrm{ff}} \arrow[rr, hook, "j"] & & \mathrm{Mod}_{E_*} \arrow[u, dashed, "\mathrm{Lan}_{\pi_* i} \pi_* L_{K(n)} \mathbb{P}_m i"'] \end{tikzcd}$$

where i and j are the respective inclusions of the full subcategories, and where $\pi_* : \mathrm{hMod}_E^{\mathrm{ff}} \to \mathrm{Mod}_{E_*}^{\mathrm{ff}}$ is an equivalence by **Proposition 3.0.5**. Using that $h\mathrm{Mod}_E^{\mathrm{ff}}$ is essentially small we get from [Rez09, 4.3] that there exists a left Kan extension of the composed functor $\pi_* L_{K(n)} \mathbb{P}_m i$ along $\pi_* i \cong j\pi_* : h\mathrm{Mod}_E^{\mathrm{ff}} \to \mathrm{Mod}_{E_*}$. Therefore we can make the following definition.

Definition 3.0.13. With i and j as above, we define the *mth algebraic approximation functor* $\mathbb{T}_m$ as the left Kan extension of the composed functor $\pi_* L_{K(n)} \mathbb{P}_m i$ along $\pi_* i \cong j\pi_*$, i.e.

$$\mathbb{T}_m := \mathrm{Lan}_{\pi_* i} \pi_* L_{K(n)} \mathbb{P}_m i : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}.$$

We further define the functor $\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ as $\mathbb{T}(M) := \oplus_{m\geq 0}\mathbb{T}_m(M)$.

Remark 3.0.14. Since $\pi_* : \mathrm{hMod}_E^{\mathrm{ff}} \to \mathrm{Mod}_{E_*}^{\mathrm{ff}}$ is an equivalence of categories by **Proposition 3.0.5** and j is an inclusion, we get that the composite $\pi_* i \cong j\pi_*$ is fully faithful. This gives us, that the diagram used to describe the left Kan extension $\mathbb{T}_m$, commutes by [Lur09, 4.3.3.5]. So if M is a finite free E-module, we get that $\mathbb{T}_m(\pi_* M) \cong \pi_*(L_{K(n)}\mathbb{P}_m(M))$.

This will be useful when we wish to consider $\mathbb{T}_m(E_*)$, and in particular it gives us the following result.

Proposition 3.0.15. *There exists an isomorphism* $\mathbb{T}(E_*) \cong \oplus_{m\geq 0} E_*^\wedge(B\Sigma_m)$.

Proof. Using **Remark 3.0.14** we get the following calculation

$$\begin{aligned}
\mathbb{T}(E_*) &= \bigoplus_{m\geq 0} \mathbb{T}_m(E_*) \\
&\cong \bigoplus_{m\geq 0} \pi_* L_{K(n)}\mathbb{P}_m(E) \\
&= \bigoplus_{m\geq 0} \pi_* L_{K(n)} \left(E^{\otimes_E m}\right)_{h\Sigma_m} \\
&\cong \bigoplus_{m\geq 0} \pi_* L_{K(n)} \left(E \otimes_{\mathbb{S}} B\Sigma_m\right) \\
&= \bigoplus_{m\geq 0} E_*^\wedge(B\Sigma_m).
\end{aligned}$$

□

Another property which will come in handy later on, but which we will not prove, is the following proposition.

Proposition 3.0.16 (Prop. 4.12 [Rez09])**.** *The functors $\mathbb{T}_m : Mod_{E_*} \to Mod_{E_*}$ commutes with filtered colimits.*

Our next goal is to prove the following property of $\mathbb{T}$, which will later be used as an analogue of how [GL20, 10.2] was used in the proof of **Theorem 2.1.9**, and which is one of our motivations to consider this functor.

Theorem 3.0.17. *If M is a flat E-module, then there exist isomorphisms $L_0\mathbb{T}(\pi_* M) \to (\mathbb{T}(\pi_* M))^\wedge_{\mathfrak{m}}$ and $L_0\mathbb{T}(\pi_* M) \to \pi_* L_{K(n)} Free_E(M)$.*

To prove this theorem, we will need the following two lemmas.

Lemma 3.0.18. *If $\{M_i\}_{i\in I}$ is a collection of flat E-modules, then $\oplus_{i\in I} M_i$ is again a flat E-module.*

Proof. We have assumed that each M_i is flat, so $\pi_* M_i$ is a flat E_*-module. We recall that this means that for any injective map $K \to L$ of E_*-modules, the induced map $K \otimes_{E_*} M_i \to L \otimes_{E_*} M_i$ is again injective. So assuming that we have such a map $K \to L$, we wish to show that the map

$$K \otimes_{E_*} \left(\bigoplus_{i\in I} \pi_* M_i \right) \to L \otimes_{E_*} \left(\bigoplus_{i\in I} \pi_* M_i \right)$$

is again injective. Using that tensor product distributes over direct sum, we get that this map is isomorphic to

$$\bigoplus_{i\in I} (K \otimes_{E_*} \pi_* M_i) \to \bigoplus_{i\in I} (L \otimes_{E_*} \pi_* M_i).$$

Since each map $K \otimes_{E_*} \pi_* M_i \to L \otimes_{E_*} \pi_* M_i$ is injective, the desired follows. □

Lemma 3.0.19. *Let $L : \mathcal{C} \to \mathcal{C}$ be a localization of an ∞-category $\mathcal{C}$, and X_n a diagram in $\mathcal{C}$. Then $L(colim_n(LX_n)) \simeq L(colim_n(X_n))$ in $\mathcal{C}$.*

Proof. When we consider L as a functor $\mathcal{C} \to L\mathcal{C}$ we know that it is left adjoint to the inclusion $i : L\mathcal{C} \hookrightarrow \mathcal{C}$, and hence preserves colimits. This gives us that

$$L(\operatorname{colim}_n(X_n)) \simeq \operatorname{colim}_n(LX_n),$$

where the colimit on the left is computed in $\mathcal{C}$ and the one on the right is computed in $L\mathcal{C}$. We further have that

$$\operatorname{colim}_n(LX_n) \simeq \operatorname{colim}_n(L(iLX_n)) \simeq L(\operatorname{colim}_n(iLX_n)),$$

hence $\operatorname{colim}_n(LX_n)$ computed in $L\mathcal{C}$ is equivalent to $L(\operatorname{colim}_n(LX_n))$ in $\mathcal{C}$. This gives us that

$$L(\operatorname{colim}_n(X_n)) \simeq L(\operatorname{colim}(LX_n))$$

in $\mathcal{C}$. □

Proof of Theorem 3.0.17. First we wish to prove that there exists an isomorphism $L_0(\mathbb{T}(\pi_* M)) \xrightarrow{\cong} (\mathbb{T}(\pi_* M))^\wedge_{\mathfrak{m}}$. Since we have assumed that M is a flat E-module, we know that $\pi_* M$ is flat as an E_*-module. From **Proposition 3.0.16** we know that $\mathbb{T}_n$ commutes with filtered colimits, hence the same is true for $\mathbb{T}$. Applying **Proposition 3.0.7** it follows that also $\mathbb{T}(\pi_* M)$ is flat, so by **Proposition 3.0.8** we get that

$$L_0(\mathbb{T}(\pi_* M)) \cong (\mathbb{T}(\pi_* M))^\wedge_{\mathfrak{m}}.$$

To show the second isomorphism we first note that since M is assumed to be a flat E-module, we know from **Proposition 3.0.7** that $M \simeq \mathrm{colim}_{j\in J} M_j$ for J some filtered ∞-category and $M_j \in \mathrm{Mod}_E^{\mathrm{ff}}$ for each $j \in J$. Now, consider the E-module

$$N = \bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} L_{K(n)} \mathbb{P}_i M_j.$$

Since each M_j is finite free, we get from [Rez09, 3.17] that each $L_{K(n)}\mathbb{P}_i M_j$ is finite free as well. By **Proposition 3.0.7** we then get that $\mathrm{colim}_{j\in J} L_{K(n)}\mathbb{P}_i M_j$ is flat for every i, so by **Lemma 3.0.18** we conclude that N is a flat E-module.

Using [Rez09, 3.8] this gives us that the map $\pi_* N \to \pi_* L_{K(n)} N$ factors through an isomorphism $L_0 \pi_* N \xrightarrow{\cong} \pi_* L_{K(n)} N$.

First we consider $\pi_* N$. Using that π_* commutes with both direct sums and filtered colimits we get that

$$\begin{aligned} \pi_* N &= \pi_* \left(\bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} (L_{K(n)}\mathbb{P}_i M_j) \right) \\ &\cong \bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} (\pi_*(L_{K(n)}\mathbb{P}_i M_j)). \end{aligned}$$

As mentioned above we have that each $L_{K(n)}\mathbb{P}_i M_j$ is finite free, so using **Remark 3.0.14** followed by **Proposition 3.0.16** we get that

$$\begin{aligned} \bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} (\pi_*(L_{K(n)}\mathbb{P}_i M_j)) &\cong \bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} (\mathbb{T}_i(\pi_* M_j)) \\ &\cong \bigoplus_{i\geq 0} \mathbb{T}_i(\mathrm{colim}_{j\in J}(\pi_* M_j)) \\ &\cong \bigoplus_{i\geq 0} \mathbb{T}_i \pi_*(\mathrm{colim}_{j\in J}(M_j)) \\ &\cong \mathbb{T}(\pi_* M), \end{aligned}$$

hence $\pi_* N \cong \mathbb{T}(\pi_* M)$.

Next we consider $\pi_* L_{K(n)} N$. Noting that the direct sum is in fact a colimit, we can apply **Lemma 3.0.19**, which gives us

$$\begin{aligned} \pi_* L_{K(n)} N &= \pi_* L_{K(n)} \left(\bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} L_{K(n)} \mathbb{P}_i M_j \right) \\ &\cong \pi_* L_{K(n)} \left(\bigoplus_{i\geq 0} \mathrm{colim}_{j\in J} \mathbb{P}_i M_j \right) \\ &\cong \pi_* L_{K(n)} \mathrm{Free}_E(M). \end{aligned}$$

So we conclude that the map

$$\mathbb{T}(\pi_* M) \cong \pi_* N \to \pi_* L_{K(n)} N \cong \pi_* L_{K(n)}(\mathrm{Free}_E(M))$$

induces the desired isomorphism

$$L_0 \mathbb{T}(\pi_* M) \to \pi_* L_{K(n)}(\mathrm{Free}_E(M)).$$

$\square$

It is the composed isomorphism which will be interesting for us later, i.e. the isomorphism

$$(\mathbb{T}(\pi_* M))^{\wedge}_{\mathfrak{m}} \cong \pi_* L_{K(n)} \mathrm{Free}_E(M).$$

This result indicates that the $K(n)$-local commutative E-algebras are well-behaved, and therefore we will now introduce a $K(n)$-local version of the associative ring spectrum $\mathrm{Pow}(E)$.

3.1 $K(n)$-local E-modules

In the section above we showed that $(\mathbb{T}(\pi_* M))^{\wedge}_{\mathfrak{m}} \cong \pi_* L_{K(n)} \mathrm{Free}_E(M)$ for any flat E-module M. This indicates that the $K(n)$-local commutative E-algebras are nicely behaved, and we would like to restrict our attention to this case. We will therefore adjust the construction of the ring of stable power operations on E, and make an analogue which only takes this restricted case into account. It turns out that going through the constructions from section 2 in this setting is relatively straight forward, and we will be able to define a $K(n)$-local associative ring spectrum $\widehat{\mathrm{Pow}}(E)$ of stable power operations on $K(n)$-local commutative E-algebras. It is in this local setting that we in the end will give an outline of the calculation of $\pi_* \widehat{\mathrm{Pow}}(E(\mathbb{F}_p, \Gamma))$.

Definition 3.1.1. Let $\widehat{\mathrm{Mod}}_E \subseteq \mathrm{Mod}_E$ denote the full subcategory consisting of all $K(n)$-local E-modules.

Example 3.1.2. We know from [Lur10, Lec.35, Lem.1] that E is $K(n)$-local, hence E is naturally in $\widehat{\mathrm{Mod}}_E$. Since Mod_E is a stable ∞-category we get that both tensor and limits commutes with finite sums, so by [Rez09, 3.4 (3)] we get that $L_{K(n)}$ commutes with the latter. Using that every finite free E-module is a finite sum of the form $\oplus_{i=1}^{k} \Sigma^{d_i} E$, we get that

$$\mathrm{Mod}_E^{\mathrm{ff}} \subseteq \widehat{\mathrm{Mod}}_E \subseteq \mathrm{Mod}_E.$$

We want to also define a local version of CAlg_E, but before we can do so, we need to show that $\widehat{\mathrm{Mod}}_E$ admits a symmetric monoidal structure which is compatible with the structure on Mod_E.

Definition 3.1.3. Let $\mathcal{C}^{\otimes}$ denote a symmetric monoidal ∞-category and $L : \mathcal{C} \to \mathcal{C}$ a localization of the underlying ∞-category $\mathcal{C}$. Then L is said to be *compatible with the symmetric monoidal structure* on $\mathcal{C}$, if for every L-equivalence $X \to Y$ in $\mathcal{C}$ and every $Z \in \mathcal{C}$, the induced map $X \otimes Z \to Y \otimes Z$ is again an L-equivalence.

Proposition 3.1.4. *The localization $L_{K(n)} : Mod_E \to Mod_E$ is compatible with the symmetric monoidal structure on Mod_E.*

Proof. Using [Rez09, 3.4.(1)] we first note that $L_{K(n)}$ is equivalent to the localization functor with respect to the homology theory on E-modules, defined by tensoring with $E \otimes_{\mathbb{S}} K(n)$. This means that a morphism $f : M \to N$ in Mod_E, is an $L_{K(n)}$-equivalence exactly when

$$f \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n) : M \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n) \to N \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n)$$

is an equivalence. Assume that this holds for f, and consider $f \otimes_{\mathbb{S}} W : M \otimes_{\mathbb{S}} W \to N \otimes_{\mathbb{S}} W$ for W some E-module. We know that $f \otimes_{\mathbb{S}} W \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n) \cong f \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n) \otimes_{\mathbb{S}} W$, so since $f \otimes_{\mathbb{S}} E \otimes_{\mathbb{S}} K(n)$ is an equivalence by assumption and W is the identity as a morphism, we get that $f \otimes_{\mathbb{S}} W$ is an $L_{K(n)}$-equivalence as desired. □

Remark 3.1.5. If we consider $L_{K(n)}$ as a localization functor $\mathrm{Sp} \to \mathrm{Sp}$, then an argument analogue to the one given in the proof above, will show that $L_{K(n)}$ is compatible with the symmetric monoidal structure on Sp as well.

Corollary 3.1.6. *For a commutative ring spectrum E, the associative ring spectrum $\widehat{Mod}_E$ carries an essentially unique symmetric monoidal structure for which the localization functor $L_{K(n)}$ is symmetric monoidal.*

Proof. This follows from **Proposition 3.1.4** together with [Lur17, 2.2.1.9] □

We can now define the ∞-category of $K(n)$*-local commutative E-algebras*

$$\widehat{\mathrm{CAlg}}_E := \mathrm{CAlg}(\widehat{\mathrm{Mod}}_E) \subseteq \mathrm{CAlg}_E.$$

We then have a forgetful functor $\widehat{F} : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$, and we use this to define the $K(n)$-*local ring of stable power operations* as the endomorphism ring of this functor

$$\widehat{\mathrm{Pow}}(E) := \mathrm{End}(\widehat{F} : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}).$$

Proposition 3.1.7. *The associative ring spectrum $\widehat{Pow}(E)$ is $K(n)$-local.*

Proof. Since every object in $\widehat{\mathrm{CAlg}}_E$ is $K(n)$-local, the same holds for their underlying spectra, hence the forgetful functor $\widehat{F} : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$ is $K(n)$-local, in the sense that it takes values in $K(n)$-local spectra.

Recall that a spectrum X is $K(n)$-local if and only if, for every $L_{K(n)}$-equivalence $\alpha : W \to Z$ in spectra, the map $\mathrm{Map}_{\mathrm{Sp}}(Z, X) \to \mathrm{Map}_{\mathrm{Sp}}(W, X)$ induced by precomposing with α, is an equivalence of spaces. Assuming that we are given such a map α, we wish to prove that the map

$$\mathrm{Map}_{\mathrm{Sp}}(Z, \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(\widehat{F}, \widehat{F})) \to \mathrm{Map}_{\mathrm{Sp}}(W, \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(\widehat{F}, \widehat{F}))$$

is an equivalence. Using the universal property of mapping spectra, we get the following diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Sp}}(Z, \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(\widehat{F}, \widehat{F})) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Sp}}(W, \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(\widehat{F}, \widehat{F})) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(Z \otimes_{\mathbb{S}} \widehat{F}, \widehat{F}) \arrow[r] & \mathrm{Map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E, \mathrm{Sp})}(W \otimes_{\mathbb{S}} \widehat{F}, \widehat{F}),
\end{tikzcd}$$

where the lower horizontal arrow is induced by precomposition with $\alpha \otimes_{\mathbb{S}} \widehat{F}$. This diagram commutes, since the vertical equivalences comes from the universal property of mapping spectra, which is natural.

We claim that $\alpha \otimes_{\mathbb{S}} \widehat{F}$ is a $K(n)$-equivalence. First we note that $(W \otimes_{\mathbb{S}} \widehat{F})(A) = W \otimes_{\mathbb{S}} \widehat{F}(A)$, so we get that the map $W \otimes_{\mathbb{S}} \widehat{F} \to Z \otimes_{\mathbb{S}} \widehat{F}$ is given componentwise by

$$\alpha \otimes_{\mathbb{S}} \widehat{F}(A) : W \otimes_{\mathbb{S}} \widehat{F}(A) \to Z \otimes_{\mathbb{S}} \widehat{F}(A).$$

Using that $\alpha : W \to Z$ is a $K(n)$-equivalence and $\widehat{F}(A)$ is $K(n)$-local, we get that this is an equivalence, which further implies that

$$(\alpha \otimes_{\mathbb{S}} \widehat{F}(A)) \otimes_{\mathbb{S}} (E \otimes_{\mathbb{S}} K(n))$$

is an equivalence. Using the equivalent description of $K(n)$-localization, given in [Rez09, 3.4.(1)], we see that $\alpha \otimes_{\mathbb{S}} \widehat{F}$ is indeed a $K(n)$-equivalence. Using that $\widehat{F}$ is $K(n)$-local, and that we have just showed that the lower horizontal map in the diagram above is induced by precomposing with a $K(n)$-equivalence, we get that the map is an equivalence due to the if and only if statement characterizing $K(n)$-local spectra mentioned above. Hence the same holds for the upper horizontal map as desired. □

To get a better understanding of $\widehat{\mathrm{Pow}}(E)$, we need to understand the functor $\mathrm{CAlg}_E \to \widehat{\mathrm{CAlg}}_E$ induced by $L_{K(n)}$, more precisely we want to show that this induced functor is left adjoint to the inclusion $\widehat{\mathrm{CAlg}}_E \hookrightarrow \mathrm{CAlg}_E$. This requires a finer analysis of localization functors on symmetric monoidal ∞-categories. We start by recalling the following result.

Proposition 3.1.8 (Prop. 5.2.7.4 [Lur09]). *Let $\mathcal{C}$ be an ∞-category and $L : \mathcal{C} \to \mathcal{C}$ an endofunctor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. Then the following is equivalent:*

1) There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ and an equivalence between $G \circ F$ and L,

2) When L is considered as a functor $\mathcal{C} \to L\mathcal{C}$, it is left adjoint to the inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$,

3) There exists a natural transformation $\eta : \mathcal{C} \times \Delta^1 \to \mathcal{C}$ from $id_{\mathcal{C}}$ to L, such that for every object $c \in \mathcal{C}$, the morphisms

$$L(\eta(c)),\ \eta(Lc) : Lc \to LLc,$$

in $\mathcal{C}$, are equivalences.

In part two we see that $L : \mathcal{C} \to L\mathcal{C}$ clearly is a localization. If we conversely have a localization $L : \mathcal{C} \to \mathcal{D}$, we can identify $\mathcal{D}$ with a full subcategory of $\mathcal{C}$, by using the right adjoint from part one, hence we are in the case of part two again. This means that this proposition gives us equivalent ways to describe localization functors. Note that using part three, we get that a localization L is equipped with a natural transformation $\eta : id_{\mathcal{C}} \to L$.

Proposition 3.1.9. *Let $\mathcal{C}^{\otimes}$ denote a symmetric monoidal ∞-category and $L : \mathcal{C} \to \mathcal{C}$ a compatible localization. Then L induces a functor $CAlg(\mathcal{C}) \to CAlg(L\mathcal{C})$ which is left adjoint to the inclusion $CAlg(L\mathcal{C}) \hookrightarrow CAlg(\mathcal{C})$.*

Proof. Let $\eta : id_{\mathcal{C}} \to L$ denote the natural transformation which exhibits L as a localization by **Proposition 3.1.8**. Since L is assumed to be compatible with the symmetric monoidal structure on $\mathcal{C}$, we get from [Lur17, 2.2.1.9 (3)] that it induces a functor $L^{\otimes} : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(L\mathcal{C})$, which we then compose with the inclusion to obtain the functor

$$\mathrm{CAlg}(\mathcal{C}) \xrightarrow{L^{\otimes}} \mathrm{CAlg}(L\mathcal{C}) \hookrightarrow \mathrm{CAlg}(\mathcal{C}).$$

We wish to show that this is a localization, since then it is a left adjoint to the inclusion of the essential image, which we will argue is exactly $\mathrm{CAlg}(L\mathcal{C})$. The above composed functor can be identified with the functor

$$L_* : \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes}) \to \mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes}),$$

given by post composing with $L^{\otimes}$, when restricted to the full subcategory spanned by the commutative algebra objects of $\mathcal{C}$. First we note that the image of L_* only contains functors with essential image contained in $L\mathcal{C}$. Conversely, if $F : N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ has essential image contained in $L\mathcal{C}$, then $\eta_F : F \to LF$ is an equivalence at each object, hence a natural equivalence. This means that the essential image of L_* is exactly those functors with image in $L\mathcal{C}$.

Next we wish to show that L_* is a localization. To do so, we construct a natural transformation $\alpha : id_{\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes})} \to L_*$ in the category

$$\mathrm{Fun}(\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes}),\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes})).$$

A natural transformation in this category is equivalent to a map $\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes}) \times \Delta^1 \times N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$. Composing the map $\eta_{id} : \mathcal{C}^{\otimes} \times \Delta^1 \to \mathcal{C}^{\otimes}$ with the evaluation map $ev : \mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes}) \times N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ we get a map

$$\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes}) \times \Delta^1 \times N(\mathrm{Fin}_*) \xrightarrow{ev} \mathcal{C}^{\otimes} \times \Delta^1 \xrightarrow{\eta_{id}} \mathcal{C}^{\otimes}.$$

This map corresponds to a natural transformation $\alpha : id_{\mathrm{Fun}(N(\mathrm{Fin}_*),\mathcal{C}^{\otimes})} \to L_*$, which exhibits L_* as a localization by **Proposition 3.1.8**.

We need to show that L_* is a localization when restricted to the full subcategory spanned by the commutative algebra objects of $\mathcal{C}$, but this follows by restricting α to this subcategory. □

Corollary 3.1.10. *The localization $L_{K(n)} : Mod_E \to Mod_E$ induces a functor $L_{K(n)} : CAlg_E \to \widehat{CAlg}_E$ which is left adjoint to the inclusion $\widehat{CAlg}_E \hookrightarrow CAlg_E$.*

Proof. By **Proposition 3.1.4** we know that $L_{K(n)}$ is compatible with the symmetric monoidal structure on Mod_E, so the desired follows by **Proposition 3.1.9**. □

Remark 3.1.11. We further want to construct a $K(n)$-local version of Free_E. From [Lur17, 3.1.3.14] we know that the forgetful functor $\widehat{G} : \widehat{\mathrm{CAlg}}_E \to \widehat{\mathrm{Mod}}_E$ admits a left adjoint if $\widehat{\mathrm{Mod}}_E$ admits countable colimits, and if for each $M \in \widehat{\mathrm{Mod}}_E$ the functor $N \mapsto M \otimes_E N$ preserves countable colimits.

Since Mod_E is presentable, and localization of presentable ∞-categories are again presentable by [Lur09, 5.5.4.15 (2)], we get that $\widehat{\mathrm{Mod}}_E$ is also presentable, hence it admits countable colimits. Further we know from **Proposition 3.1.4** that $L_{K(n)}$ is compatible with the symmetric monoidal structure on Mod_E, hence the functor $N \mapsto M \otimes_E N$ in $\widehat{\mathrm{Mod}}_E$ preserves countable colimits, since this holds in Mod_E and localization preserves colimits. This gives us that the above mentioned left adjoint exists

$$\mathrm{LFree}_E : \widehat{\mathrm{Mod}}_E \to \widehat{\mathrm{CAlg}}_E.$$

Using **Corollary 3.1.10** together with **Proposition 2.0.7** we then have the following diagram, where the upper and the left arrows are the left adjoints, i and j the respective inclusions and G, $\widehat{G}$ the respective forgetful functors

$$\begin{tikzcd}
\widehat{\mathrm{CAlg}}_E \arrow[r, shift left, "\widehat{G}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, hook, "i"] & \widehat{\mathrm{Mod}}_E \arrow[l, shift left, "\mathrm{LFree}_E"'] \arrow[d, shift right, hook, "j"] \\
\mathrm{CAlg}_E \arrow[u, shift right, "L_{K(n)}"] \arrow[u, phantom, "\dashv"] \arrow[r, shift left, "G"'] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_E. \arrow[l, shift left, "\mathrm{Free}_E"'] \arrow[u, shift right, "L_{K(n)}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

The commutativity of this diagram implies that $\mathrm{LFree}_E \simeq L_{K(n)}\mathrm{Free}_E$, hence **Proposition 2.0.7** gives us

$$\mathrm{LFree}_E(M) \simeq L_{K(n)}\mathrm{Free}_E(M) \simeq L_{K(n)} \bigoplus_{m\geq 0} (M^{\otimes_E m})_{h\Sigma_m}.$$

Since E is $K(n)$-local by **Example 3.1.2** we see that

$$E \simeq L_{K(n)}E \simeq L_{K(n)} \bigoplus_{m\geq 0} (0^{\otimes_E m})_{h\Sigma_m},$$

so using the above description we get that the map $M \to 0$, for $M \in \widehat{\mathrm{Mod}}_E$, induces a natural transformation $\mathrm{LFree}_E(M) \to \mathrm{LFree}_E(0) \simeq E$. Hence we can define the $K(n)$-local analogue of the functor $\widetilde{\mathrm{Free}}_E$.

Definition 3.1.12. Let $M \in \widehat{\mathrm{Mod}}_E$. Then we define the *free non-unital $K(n)$-local commutative E-algebra* by

$$\begin{aligned}
\widetilde{\mathrm{LFree}}_E(M) &:= \mathrm{fib}(\mathrm{LFree}_E(M) \to E) \\
&\simeq \mathrm{fib}(L_{K(n)}\mathrm{Free}_E(M) \to E).
\end{aligned}$$

Remark 3.1.13. Using [Rez09, 3.4.(3)] we get that $L_{K(n)}$ commutes with finite limits, hence

$$\begin{aligned}
\widetilde{\mathrm{LFree}}_E &\simeq \mathrm{fib}(L_{K(n)}\mathrm{Free}_E \to L_{K(n)}E) \\
&\simeq L_{K(n)}\mathrm{fib}(\mathrm{Free}_E \to E) \\
&\simeq L_{K(n)}\widetilde{\mathrm{Free}}_E.
\end{aligned}$$

By further using the description of $\widetilde{\mathrm{Free}}_E$ from **Remark 2.0.13**, we get that $\widetilde{\mathrm{LFree}}_E(M) \simeq L_{K(n)}(\oplus_{m>0}(M^{\otimes_E m})_{h\Sigma_m})$ for $M \in \widehat{\mathrm{Mod}}_E$.

We wish to get a better understanding of $\widehat{\mathrm{Pow}}(E)$, in particular we wish to determine the underlying spectrum and prove a result similar to **Theorem 2.0.19**, which will describe how the operations determined by $\widehat{\mathrm{Pow}}(E)$ are compatible with the stable operations on $K(n)$-local commutative E-algebras. The following property of the forgetful functor $\widehat{F} : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$ will be necessary for both of these tasks.

Lemma 3.1.14. *The forgetful functor $\widehat{F} : \widehat{CAlg}_E \to Sp$ is represented by the cospectrum object $\{LFree_E(\Omega^m E)\} \in Sp(\widehat{CAlg}_E^{op})$.*

Proof. First we note that $\Omega^m E$ is a $K(n)$-local E-module due to **Example 3.1.2**, so we get $\{\Omega^m E\} \in \mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})$. Next, using the adjunction $\mathrm{LFree}_E \dashv \widehat{G}$ from **Remark 3.1.11**, with $\widehat{G} : \widehat{\mathrm{CAlg}}_E \to \widehat{\mathrm{Mod}}_E$ the forgetful functor, we get that for any $M \in \mathrm{CAlg}_E$

$$\mathrm{map}_{\mathrm{Sp}(\widehat{\mathrm{CAlg}}_E^{op})}(\Sigma^\infty_+(M), \{\mathrm{LFree}_E(\Omega^m E)\}) \simeq \mathrm{map}_{\mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})}(\Sigma^\infty_+(\widehat{G}(M)), \{\Omega^m E\}).$$

From **Lemma 2.0.11** we know that the cospectrum object $\{\Omega^m E\}$ represents the forgetful functor $H : \mathrm{Mod}_E \to \mathrm{Sp}$, so using that $\{\Omega^m E\} \in \mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})$ and $\mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op}) \subseteq \mathrm{Sp}(\mathrm{Mod}_E^{op})$ is a full subcategory, we get that

$$H(-) \simeq \mathrm{map}_{\mathrm{Sp}(\mathrm{Mod}_E^{op})}(\Sigma^\infty_+(-), \{\Omega^m E\}) \simeq \mathrm{map}_{\mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})}(\Sigma^\infty_+(-), \{\Omega^m E\}).$$

Since the forgetful functor $\widehat{H} : \widehat{\mathrm{Mod}}_E \to \mathrm{Sp}$ is the restriction of H to $\widehat{\mathrm{Mod}}_E$, we get that $\widehat{H}$ is represented by $\{\Omega^m E\} \in \mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})$. From this it follows that

$$\mathrm{map}_{\mathrm{Sp}(\widehat{\mathrm{Mod}}_E^{op})}(\Sigma^\infty_+(\widehat{G}(M)), \{\Omega^m E\}) \simeq \widehat{H} \circ \widehat{G}(M) \simeq \widehat{F}(M)$$

as desired. □

Remark 3.1.15. We can show that there exists a string of associative ring spectra

$$E \to \widehat{\mathrm{Pow}}(E) \to \mathrm{End}(E),$$

together with a canonical lift in the diagram

$$\begin{tikzcd}
\mathscr{S}^{op} \arrow[r, "E^{(-)}"] \arrow[dd, "L_{K(n)}E^{(-)}"'] & \mathrm{LMod}_{\mathrm{End}(E)} \arrow[d] \\
 & \mathrm{LMod}_{\widehat{\mathrm{Pow}}(E)} \arrow[d] \\
\widehat{\mathrm{CAlg}}_E \arrow[r] \arrow[ur, dashed] & \mathrm{LMod}_E.
\end{tikzcd}$$

This follows by an argument similar to the one given in the proof of **Theorem 2.0.19**, so therefore we will only go through the main points. We know from **Lemma 3.1.14** that $F : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$ is represented by the cospectrum object $\{\mathrm{LFree}_E(\Omega^m E)\}$, so by **Proposition 2.0.10** we get that

$$\widehat{\mathrm{Pow}}(E) = \mathrm{End}(\widehat{F}) \simeq \mathrm{End}_{\mathrm{Sp}(\widehat{\mathrm{CAlg}}_E^{op})}(\{\mathrm{LFree}_E(\Omega^m E)\}).$$

We further have from [GL20, 7.1.(2)] that $L_{K(n)}E^{(-)}$ is left adjoint to $\mathrm{map}_{\widehat{\mathrm{CAlg}}_E}(-, E)$, so by **Proposition 2.0.20** we get the existence of the desired string of associative ring spectra together with the desired lift in the diagram above.

This gives us a compatibility between the stable operations on the underlying spectrum of any $K(n)$-local commutative E-algebra and the actions determined by $\widehat{\mathrm{Pow}}(E)$. Next we will calculate the underlying spectrum of $\widehat{\mathrm{Pow}}(E)$, which will be an essential tool later, when we give a sketch of the calculation of $\pi_*\widehat{\mathrm{Pow}}(E(\mathbb{F}_p,\Gamma))$ for Morava E-theory at height 1.

Proposition 3.1.16. *The underlying spectrum of $\widehat{Pow}(E)$ is $lim_m \Sigma^m \widetilde{LFree}_E(\Omega^m E)$.*

Proof. We will consider [GL20, 6.1, 6.3] with $M = N = E$, $\mathcal{C} = \widehat{\mathrm{Mod}}_E$, $\mathcal{O} = N(\mathrm{Fin}_*)$ and $X = \langle 1 \rangle$. Using **Lemma 3.1.14** we know that

$$\widehat{F} : \mathrm{Alg}_{N(\mathrm{Fin}_*)}(\widehat{\mathrm{Mod}}_E) \simeq \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$$

is represented by the cospectrum object $\{\mathrm{LFree}_E(\Omega^m E)\} \in \mathrm{Sp}(\widehat{\mathrm{CAlg}}_E^{op})$, hence $\widehat{F}$ is equivalent to the functor Υ_E in [GL20, 6.1]. By further applying [GL20, 6.3] we then get that

$$\mathrm{Pow}(E) = \mathrm{End}(\widehat{F}) \simeq \mathrm{map}_{\widehat{\mathrm{Mod}}_E}(E, \lim_m \Sigma^m (\widetilde{\mathrm{LFree}}_E(\Omega^m E)),$$

which gives the desired underlying spectrum. $\square$

Note that the maps in this limit is constructed as in **Remark 2.0.16**. We can use this description of the underlying spectrum of $\widehat{\mathrm{Pow}}(E)$ to get a better understanding of the connection between $\widehat{\mathrm{Pow}}(E)$ and $\mathrm{Pow}(E)$.

Proposition 3.1.17. *There exists a canonical map $\varphi : Pow(E) \to \widehat{Pow}(E)$ of associative ring spectra.*

Proof. Consider the diagram

$$\begin{tikzcd}
\widehat{\mathrm{CAlg}}_E \arrow[d, "F'"] \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathrm{CAlg}_E \arrow[l, shift right, "L_{K(n)}"'] \arrow[d, "F"] \\
\widehat{\mathrm{Sp}} \arrow[r, hook, shift right, "j"'] \arrow[r, phantom, "\perp"] & \mathrm{Sp}, \arrow[l, shift right, "L_{K(n)}"']
\end{tikzcd}$$

where $\widehat{\mathrm{Sp}} \subseteq \mathrm{Sp}$ is the full subcategory of $K(n)$-local spectra, F' and F the respective forgetful functors and i, j the inclusions. Write $\widehat{F} : \widehat{\mathrm{CAlg}}_E \to \mathrm{Sp}$ for the forgetful functor, and note that

$$\begin{aligned}\widehat{\mathrm{Pow}}(E) &= \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E,\mathrm{Sp})}(\widehat{F},\widehat{F})\\ &\simeq \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E,\mathrm{Sp})}(jF',jF')\\ &\simeq \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E,\mathrm{Sp})}(Fi,Fi).\end{aligned}$$

Since composition is functorial we get that precomposing with i defines a functor

$$\mathrm{Fun}(\mathrm{CAlg}_E,\mathrm{Sp}) \to \mathrm{Fun}(\widehat{\mathrm{CAlg}}_E,\mathrm{Sp}),$$

which gives a map

$$\mathrm{map}_{\mathrm{Fun}(\mathrm{CAlg}_E,\mathrm{Sp})}(F,F) \to \mathrm{map}_{\mathrm{Fun}(\widehat{\mathrm{CAlg}}_E,\mathrm{Sp})}(Fi,Fi),$$

of associative ring spectra. Hence we get the desired map

$$\varphi : \mathrm{Pow}(E) \to \widehat{\mathrm{Pow}}(E).$$

□

Remark 3.1.18. By **Proposition 3.1.17** we have a canonical map $\varphi : \mathrm{Pow}(E) \to \widehat{\mathrm{Pow}}(E)$, so using that $\widehat{\mathrm{Pow}}(E)$ is $K(n)$-local by **Proposition 3.1.7**, we get a unique filler

$$\begin{tikzcd} \mathrm{Pow}(E) \arrow[r, "\varphi"] \arrow[d, "L_{K(n)}"'] & \widehat{\mathrm{Pow}}(E). \\ L_{K(n)}\mathrm{Pow}(E) \arrow[ur, "\exists!\theta"'] & \end{tikzcd}$$

Hence we have a canonical map $\theta : L_{K(n)}\mathrm{Pow}(E) \to \widehat{\mathrm{Pow}}(E)$. A natural guess would be that this is an equivalence, but this is a rather subtle point, as we know explain.

Let $U : \mathrm{Alg} \to \mathrm{Sp}$ denote the forgetful functor. Using **Proposition 3.1.16** followed by **Remark 3.1.11** and the fact that localization commutes with finite colimits, we get that

$$\begin{aligned}U(\widehat{\mathrm{Pow}}(E)) &\simeq \lim_m \Sigma^m \widetilde{\mathrm{LFree}}_E(\Omega^m E)\\ &\simeq \lim_m \Sigma^m L_{K(n)}\widetilde{\mathrm{Free}}_E(\Omega^m E)\\ &\simeq \lim_m L_{K(n)}\Sigma^m \widetilde{\mathrm{Free}}_E(\Omega^m E).\end{aligned}$$

Next, if we consider the commutative diagram

$$\begin{tikzcd} \widehat{\mathrm{Alg}} \arrow[d, "U'"'] \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, shift right, "L_{K(n)}"'] \arrow[d, "U"] \\ \widehat{\mathrm{Sp}} \arrow[r, hook, shift right, "j"'] \arrow[r, phantom, "\perp"] & \mathrm{Sp}, \arrow[l, shift right, "L_{K(n)}"'] \end{tikzcd}$$

where $\widehat{\mathrm{Alg}}$ denotes the full subcategory of $K(n)$-local associative ring spectra, U' the forgetful functor and i, j the respective inclusions, we see that $U'(L_{K(n)}\mathrm{Pow}(E)) \simeq L_{K(n)}U(\mathrm{Pow}(E))$. So using **Proposition 2.0.15** we get

$$U'(L_{K(n)}\mathrm{Pow}(E)) \simeq L_{K(n)}\mathrm{lim}_m\Sigma^m\widetilde{\mathrm{Free}}_E(\Omega^m E).$$

This means that up to commuting the localization and the infinite limit, these two spectra are very close to being equivalent. If the underlying spectra of $\widehat{\mathrm{Pow}}(E)$ and $L_{K(n)}\mathrm{Pow}(E)$ were equivalent, we would get that $U'(\theta)$ were an equivalence, hence the same would hold for θ.

3.2 p-complete K-theory

Since power operations in Morava E-theory at heights $n > 1$ are understood only partially, we will now turn our focus to height 1 where $E \simeq K_p^\wedge$ and we know a lot more, which makes this case computationally accessible. So let E denote Morava E-theory at height 1. First we will introduce an operation Q on $E_*^\wedge(X)$ following [BF15, Section 6]. These operations gives us a concrete description of $\mathbb{T}(\Omega^n E_*)$, which will be a useful result when we give a description of the underlying spectrum of $\pi_*\widehat{\mathrm{Pow}}(E)$ in the end of this thesis.

Write $K_r := K/p^r$ and note that this implies that $K_p^\wedge \simeq \mathrm{lim}_r K_r$. For X a commutative ring spectrum we further write $K_*(X;\mathbb{Z}/p^r) := \pi_*(K_r \otimes_{\mathbb{S}} X)$ for the *mod p complex K-theory.*

Using these notations we want an easier description for $E_*^\wedge(X)$ at this height, which does not use localization. From [Rez09, 3.4 (3)] we get that $L_{K(1)}M \simeq \mathrm{lim}_r(M \otimes_{\mathbb{S}} S^0/p^r)$ for $M \in \mathrm{Mod}_E$. So since $E \otimes_{\mathbb{S}} X$ is an E-module for any spectrum X, it follows that

$$L_{K(1)}(E \otimes_{\mathbb{S}} X) \simeq \mathrm{lim}_r((E \otimes_{\mathbb{S}} X) \otimes_{\mathbb{S}} S^0/p^r) \simeq \mathrm{lim}_r(E \otimes_{\mathbb{S}} X/p^r).$$

This means that for any spectrum X we have

$$\begin{aligned}E_*^\wedge(X) &= \pi_* L_{K(1)}(K_p^\wedge \otimes_{\mathbb{S}} X)\\ &\cong \pi_*(\lim_r(K_p^\wedge \otimes_{\mathbb{S}} X/p^r))\\ &\cong \pi_*((K_p^\wedge \otimes_{\mathbb{S}} X)_p^\wedge)\\ &\cong \pi_*((K \otimes_{\mathbb{S}} X)_p^\wedge)\\ &\cong \pi_*(\lim_r(K_r \otimes_{\mathbb{S}} X)).\end{aligned}$$

Using this we get that the Milnor short exact sequence for $K_r \otimes_{\mathbb{S}} X$ has the form

$$0 \to \lim_r^1 K_{*+1}(X;\mathbb{Z}/p^r) \to E_*^\wedge(X) \xrightarrow{\phi} \lim_r K_*(X;\mathbb{Z}/p^r) \to 0,$$

where the existence of the map ϕ will be necessary later. From [BMMS86, IX, 3.3] we have an operation $Q : K_*(X;\mathbb{Z}/p^r) \to K_*(X;\mathbb{Z}/p^{r-1})$ for any $X \in \mathrm{CAlg}$, which is compatible with the natural projection maps π, in the sense that

$$\begin{tikzcd} K_*(X;\mathbb{Z}/p^{r+1}) \arrow[r, "Q"] \arrow[d, "\pi"] & K_*(X;\mathbb{Z}/p^r) \arrow[d, "\pi"] \\ K_*(X;\mathbb{Z}/p^r) \arrow[r, "Q"] & K_*(X;\mathbb{Z}/p^{r-1}) \end{tikzcd}$$

commutes. This implies that Q induces natural operations on the inverse limit

$$Q : \lim_r K_*(X;\mathbb{Z}/p^r) \to \lim_r K_*(X;\mathbb{Z}/p^r).$$

We wish to argue that Q induces a natural operation on $E_*^\wedge(X)$ such that

$$\begin{tikzcd} E_*^\wedge(X) \arrow[r, "\phi"] \arrow[d, dashed, "Q"] & \lim_r K_*(X;\mathbb{Z}/p^r) \arrow[d, "Q"] \\ E_*^\wedge(X) \arrow[r, "\phi"] & \lim_r K_*(X;\mathbb{Z}/p^r) \end{tikzcd} \tag{2}$$

commutes.

We will now fix some notation for the rest of this section. For $j = 0$ or 1 we have that $E_j^\wedge(S^j) \cong \mathbb{Z}_p$, and we let z denote a generator. Then the map $S^j = \mathbb{P}_1(S^j) \to \mathrm{Free}_{\mathbb{S}}(S^j)$, which sends S^j to the first term in the direct sum, induces a map $E_j^\wedge(S^j) \to E_j^\wedge(\mathrm{Free}_{\mathbb{S}}(S^j))$, and we let x denote the image of z under this map. As mentioned above we have a natural map $\phi : E_*^\wedge(X) \to \lim_r K_*(X;\mathbb{Z}/p^r)$ so by postcomposing with the canonical map $\lim_r K_*(X;\mathbb{Z}/p^r) \to K_*(X;\mathbb{Z}/p^r)$, we get a map $E_*^\wedge(X) \to K_*(X;Z/p^r)$ for all r. The image of $x \in E_j^\wedge(\mathrm{Free}_{\mathbb{S}}(S^j))$ under this map will again be denoted by x. Note that this implies that $Q^i x$ is well-defined for $x \in K_*(\mathrm{Free}_{\mathbb{S}}(S^j);\mathbb{Z}/p^r)$, so the following description of $K_*(\mathrm{Free}_{\mathbb{S}}(S^j);\mathbb{Z}/p^r)$ makes sense.

Proposition 3.2.1 (Prop. 6.5 [BF15]). *$K_*(Free_{\mathbb{S}}(S^j);\mathbb{Z}/p^r)$ is the free strictly commutative $\mathbb{Z}/2$-graded algebra over $\mathbb{Z}/p^r$ on generators $\{Q^ix|i\geq 0\}$, i.e*

$$\begin{aligned} K_*(Free_{\mathbb{S}}(S^0);\mathbb{Z}/p^r) &\cong \mathbb{Z}/p^r[x,Qx,Q^2x,...] \\ K_*(Free_{\mathbb{S}}(S^1);\mathbb{Z}/p^r) &\cong \Lambda_{\mathbb{Z}/p^r}[x,Qx,Q^2x,...]. \end{aligned}$$

Corollary 3.2.2. *There exist isomorphisms*

$$\begin{aligned} E_*^{\wedge}(Free_{\mathbb{S}}(S^0)) &\cong L_0(\mathbb{Z}_p[x,Qx,Q^2x,...]) \\ E_*^{\wedge}(Free_{\mathbb{S}}(S^1)) &\cong L_0(\Lambda_{\mathbb{Z}_p}[x,Qx,Q^2x,...]). \end{aligned}$$

Proof. We will only prove this for even degree, since the odd case is similar. First we note that by the discussion above, we know that the Milnor short exact sequence for $K_r\otimes_{\mathbb{S}}\mathrm{Free}_{\mathbb{S}}(S^0)$ has the form

$$0\to \lim{}^1 K_{*+1}(\mathrm{Free}_{\mathbb{S}}(S^0);\mathbb{Z}/p^r)\to E_*^{\wedge}(\mathrm{Free}_{\mathbb{S}}(S^0))\to \lim{}_r K_*(\mathrm{Free}_{\mathbb{S}}(S^0);\mathbb{Z}/p^r)\to 0.$$

We will show that the $\lim^1$-term vanishes, so consider $\lim_r K_*(\mathrm{Free}_{\mathbb{S}}(S^0);\mathbb{Z}/p^r)$. Using **Proposition 3.2.1** we know that this is equivalent to the inverse system

$$\cdots \xrightarrow{\theta_{r+1}} \mathbb{Z}/p^r[x,Qx,Q^2x,...] \xrightarrow{\theta_r} \mathbb{Z}/p^{r-1}[x,Qx,Q^2x,...] \xrightarrow{\theta_{r-1}} \cdots$$

where θ_r denotes the quotient maps, which in particular are surjective. This gives us that this system satisfies the Mittage-Leffler condition, which implies that the $\lim^1$-term disappears, hence we get

$$E_*^{\wedge}(\mathrm{Free}_{\mathbb{S}}(S^0))\cong \lim{}_r K_*(\mathrm{Free}_{\mathbb{S}}(S^0);\mathbb{Z}/p^r).$$

We note that we can write $\mathbb{Z}_p[x,Qx,Q^2x,...]\cong \mathrm{colim}_n(\mathbb{Z}_p[x,Qx,Q^2x,...,Q^nx])$, so by the standard version of Lazard's theorem we get that $\mathbb{Z}_p[x,Qx,Q^2x,...]$ is a flat E_*-module. So by using **Proposition 3.0.10** together with **Proposition 3.2.1** once again, we get

$$\begin{aligned} E_*^{\wedge}(\mathrm{Free}_{\mathbb{S}}(S^0)) &\cong \lim{}_r K_*(\mathrm{Free}_{\mathbb{S}}(S^0);\mathbb{Z}/p^r) \\ &\cong \lim{}_r \mathbb{Z}/p^r[x,Qx,Q^2x,...] \\ &\cong (\mathbb{Z}_p[x,Qx,Q^2x,...])_p^{\wedge} \\ &\cong L_0(\mathbb{Z}_p[x,Qx,Q^2x,...]). \end{aligned}$$

□

As a part of this proof we showed that $E_*^{\wedge}(\mathrm{Free}_{\mathbb{S}}(S^j))\cong \lim_r K_*(\mathrm{Free}_{\mathbb{S}}(S^j);\mathbb{Z}/p^r)$, which induces the following operation on $E_*^{\wedge}(\mathrm{Free}_{\mathbb{S}}(S^j))$

$$\begin{tikzcd} E_*^\wedge(\mathrm{Free}_{\mathbb{S}}(S^j)) \arrow[r, "\cong"] \arrow[d, dashed, "Q"] & \lim_r K_*(\mathrm{Free}_{\mathbb{S}}(S^j); \mathbb{Z}/p^r) \arrow[d, "Q"] \\ E_*^\wedge(\mathrm{Free}_{\mathbb{S}}(S^j)) \arrow[r, "\cong"] & \lim_r K_*(\mathrm{Free}_{\mathbb{S}}(S^j); \mathbb{Z}/p^r). \end{tikzcd}$$

This operation is natural in the following sense.

Lemma 3.2.3 (Lem. 6.7 [BF15]). *Let $\varphi : (K \otimes_{\mathbb{S}} \mathit{Free}_{\mathbb{S}}(S^j))_p^\wedge \to (K \otimes_{\mathbb{S}} \mathit{Free}_{\mathbb{S}}(S^k))_p^\wedge$ be a map of commutative K-algebras. Then φ is compatible with Q in the sense that the following diagram commutes:*

$$\begin{tikzcd} E_i^\wedge(\mathit{Free}_{\mathbb{S}}(S^j)) \arrow[r, "\pi_i\varphi"] \arrow[d, "Q"] & E_i^\wedge(\mathit{Free}_{\mathbb{S}}(S^k)) \arrow[d, "Q"] \\ E_i^\wedge(\mathit{Free}_{\mathbb{S}}(S^j)) \arrow[r, "\pi_i\varphi"] & E_i^\wedge(\mathit{Free}_{\mathbb{S}}(S^k)). \end{tikzcd}$$

It can then be shown that this operation extends to acting on the homotopy of any p-complete commutative K-algebra.

Proposition 3.2.4 (Prop. 6.8 [BF15]). *There exists an unique operation $Q : \pi_*A \to \pi_*A$ acting on the homotopy of any p-complete commutative K-algebra A, such that Q is natural with respect to commutative K-algebra maps, and such that Q agrees with the operation $Q : E_*^\wedge(\mathit{Free}_{\mathbb{S}}(S^j)) \to E_*^\wedge(\mathit{Free}_{\mathbb{S}}(S^j))$ for $A = (K \otimes_{\mathbb{S}} \mathit{Free}_{\mathbb{S}}(S^j))_p^\wedge$.*

Remark 3.2.5. Recall that $E_*^\wedge(X) \cong \pi_*((K \otimes_{\mathbb{S}} X)_p^\wedge)$, so by applying **Proposition 3.2.4** to $A = (K \otimes_{\mathbb{S}} X)_p^\wedge$, we get that Q induces an operation

$$Q : E_*^\wedge(X) \to E_*^\wedge(X)$$

for any $X \in \mathrm{CAlg}$, which is natural with respect to commutative K-algebra maps $(K \otimes_{\mathbb{S}} X)_p^\wedge \to (K \otimes_{\mathbb{S}} X')_p^\wedge$. Using [BF15, 6.10] we get that this is the desired operation which makes diagram **(2)** commute. Noting that

$$L_{K(1)}(E \otimes_{\mathbb{S}} X) \simeq \lim_r (E \otimes_{\mathbb{S}} X/p^r) \simeq (K \otimes_{\mathbb{S}} X)_p^\wedge$$

for any spectrum X, we get that Q is also natural with respect to any commutative E-algebra map of the form $L_{K(1)}(E \otimes_{\mathbb{S}} X) \to L_{K(1)}(E \otimes_{\mathbb{S}} X')$.

These operations lets us describe $\mathbb{T}(E_*)$ in the following theorem.

Theorem 3.2.6 (Thm 6.14 [BF15]). *$\mathbb{T}(E_*)$ and $\mathbb{T}(\Omega E_*)$ are the $\mathbb{Z}/2$-graded polynomial, respectively exterior, algebras over $\mathbb{Z}_p$ on generators $\{Q^i x | i \geq 0\}$, i.e there exists isomorphisms*

$$\mathbb{T}(E_*) \cong \mathbb{Z}_p[x, Qx, Q^2x, ...][u^{\pm 1}]$$
$$\mathbb{T}(\Omega E_*) \cong \Lambda_{\mathbb{Z}_p}[x, Qx, Q^2x, ...][u^{\pm 1}]$$

with $|u| = 2$.

Warning 3.2.7. We will omit the $[u^{\pm 1}]$ in the proofs below, and simply regard $\mathbb{Z}_p[x, Qx, Q^2x, ..]$ and $\Lambda_{\mathbb{Z}_p}[x, Qx, Q^2x, ...]$ as a $\mathbb{Z}/2$-graded polynomial ring and a $\mathbb{Z}/2$-graded exterior algebra, respectively.

We are now ready to give a better description of $\lim_n \pi_*(\Sigma^n \widetilde{\mathrm{LFree}}_E(\Omega^n E))$, which we will use to sketch the calculation of $\pi_* \widehat{\mathrm{Pow}}(E)$. Write $\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ..]$ and $\widetilde{\Lambda_{\mathbb{Z}_p}}[x, Qx, Q^2x, ...]$ for $\mathbb{Z}_p[x, Qx, Q^2x, ...]$ and $\Lambda_{\mathbb{Z}_p}[x, Qx, Q^2x, ...]$ respectively, with the unit removed. We will start by proving the following two lemmas.

Lemma 3.2.8. *There exists an isomorphism*

$$\pi_* \widetilde{LFree}_E(\Omega^n E) \cong \begin{cases} (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p, & n \text{ even} \\ (\widetilde{\Lambda_{\mathbb{Z}_p}}[x, Qx, Q^2x, ...])^\wedge_p, & n \text{ odd} \end{cases},$$

with $|x| = |Q^i x| = 0$.

Proof. Using **Remark 3.1.11** together with **Theorem 3.0.17** we calculate

$$\begin{aligned} \pi_* \widetilde{\mathrm{LFree}}_E(\Omega^n E) &= \pi_* \mathrm{fib}(\mathrm{LFree}_E(\Omega^n E) \to E) \\ &\cong \pi_* \mathrm{fib}(L_{K(1)} \mathrm{Free}_E(\Omega^n E) \to E) \\ &\cong \ker(\pi_*(L_{K(1)} \mathrm{Free}_E(\Omega^n E)) \to E_*) \\ &\cong \ker(\mathbb{T}(\Omega^n E_*)^\wedge_p \to E_*). \end{aligned}$$

We know that E_* is 2-periodic, so $\Omega^{2n} E_* \cong E_*$ and $\Omega^{2n+1} E_* \cong \Omega E_*$, hence we only need to consider these two cases. Using **Theorem 3.2.6** together with the above calculation we then get the desired. □

Lemma 3.2.9. *The system* $\lim_n \pi_*(\Sigma^n \widetilde{LFree}_E(\Omega^n E))$ *is equivalent to*

$$\begin{aligned} \cdots \xrightarrow{\pi_* \theta_{2n+1}} (\mathbb{Z}_p\{x, Qx, Q^2x, ...\})^\wedge_p \xrightarrow{\pi_* \theta_{2n}} \Sigma(\Lambda_{\mathbb{Z}_p}\{x, Qx, Q^2x, ...\})^\wedge_p \\ \xrightarrow{\pi_* \theta_{2n-1}} (\mathbb{Z}_p\{x, Qx, Q^2x, ...\})^\wedge_p \xrightarrow{\pi_* \theta_{2n-2}} \cdots, \end{aligned}$$

where the maps $\pi_* \theta_n$ *are induced from those constructed in* ***Remark 2.0.16****, and where* $|x| = |Q^i x| = 0$.

Proof. We fist note that

$$\pi_*(\Sigma^n \widetilde{\mathrm{LFree}}_E(\Omega^n E)) \cong \Sigma^n \pi_*(\widetilde{\mathrm{LFree}}_E(\Omega^n E)),$$

so by applying **Lemma 3.2.8** we get that $\lim_n \pi_*(\Sigma^n \widetilde{\mathrm{LFree}}_E(\Omega^n E))$ is equivalent to the system

$$\cdots \xrightarrow{\pi_*\theta_{2n+1}} (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p \xrightarrow{\pi_*\theta_{2n}} \Sigma(\widetilde{\Lambda_{\mathbb{Z}_p}}[x, Qx, Q^2x, ...])^\wedge_p \xrightarrow{\pi_*\theta_{2n-1}} \cdots$$

Using this description of the inverse system together with **Remark 2.0.16** we get the following string of commutative squares

$$\begin{tikzcd}
\cdots \arrow[r, "\cong"] & \pi_*(\Sigma^2\Omega^2 E) \arrow[r, "\cong"] \arrow[d] & \pi_*(\Sigma\Omega E) \arrow[r, "\cong"] \arrow[d] & \pi_*(E) \arrow[d] \\
\cdots \arrow[r, "\pi_*\theta_3"] & (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p \arrow[r, "\pi_*\theta_2"] & \Sigma(\widetilde{\Lambda_{\mathbb{Z}_p}}[x, Qx, Q^2x, ...])^\wedge_p \arrow[r, "\pi_*\theta_1"] & (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p,
\end{tikzcd}$$

where the lower horizontal string is exactly the system which we are interested in. To get a better understanding of the maps $\pi_*\theta_n$ we first consider the commutative square

$$\begin{tikzcd}
\pi_*(E) \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
\pi_*(\widetilde{\mathrm{LFree}_E(E)}) \arrow[r, "\cong"] & (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p,
\end{tikzcd}$$

where we consider $\mathbb{Z}_p$ as $\mathbb{Z}/2$-graded. We need to get a better understanding of the lower isomorphism. It is induced from the isomorphism in **Proposition 3.2.1**, which is given by

$$\begin{aligned}
\mathbb{Z}/p[x, Qx, Q^2x, ...] &\xrightarrow{\cong} K_*(\mathrm{Free}_{\mathbb{S}}(S^0); \mathbb{Z}/p) \\
x &\mapsto x \\
Q^i x &\mapsto Q^i x,
\end{aligned}$$

where Q^i on the left hand side is formal and on the right hand side is the operation. This means that for the generator z of $\mathbb{Z}_p$, which by construction is mapped to x, we get the following square

$$\begin{tikzcd}
z \arrow[dd, mapsto] & \pi_*(E) \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}_p \arrow[d] \\
& \pi_*(\widetilde{\mathrm{LFree}_E(E)}) & (\widetilde{\mathbb{Z}_p}[x, Qx, Q^2x, ...])^\wedge_p. \arrow[l, "\cong"] \\
x & & x \arrow[ll, mapsto]
\end{tikzcd}$$

This gives us that the right vertical map takes z to x, so since the string of squares above commutes, we get that $\pi_*\theta_n(x) = x$. We further get from **Proposition 2.0.17** that products are annihilated. Hence $\lim_n \pi_*(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E))$ is equivalent to the desired system

$$\cdots \xrightarrow{\pi_*\theta_{2n+1}} (\mathbb{Z}_p\{x, Qx, Q^2x, ...\})_p^\wedge \xrightarrow{\pi_*\theta_{2n}} \Sigma(\Lambda_{\mathbb{Z}_p}\{x, Qx, Q^2x, ...\})_p^\wedge \xrightarrow{\pi_*\theta_{2n-1}} \cdots .$$

□

Theorem 3.2.10. *The homotopy groups $\pi_*\widehat{Pow}(E)$ of the ring of stable power operations on $K(1)$-local commutative E-algebras is equivalent to the limit of the system described in **Lemma 3.2.9**.*

Proof. We know from **Proposition 3.1.16** that the underlying spectrum of $\widehat{\mathrm{Pow}}(E)$ is $\lim_n \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)$, so to understand $\pi_*\widehat{\mathrm{Pow}}(E)$ we need to understand $\pi_* \lim_n \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)$. Consider the Milnor short exact sequence for $\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)$, which is given by

$$\begin{aligned} 0 &\to \lim^1_n \pi_{q+1}(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \\ &\to \pi_q(\lim_n \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \\ &\to \lim_n(\pi_q \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \to 0. \end{aligned}$$

We wish to argue that the $\lim^1$-term vanishes. This will be done by considering this for first q even and then for q odd. Assuming that q is even we get from **Lemma 3.2.8** that

$$\pi_{q+1}(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \cong \Sigma^n \pi_{q+1}(\widetilde{\mathrm{LFree}_E}(\Omega^n E)) \cong 0$$

for each n, hence $\lim^1_n \pi_{q+1}(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \cong 0$.

If we assume that q is odd, we again get that

$$\pi_q(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \cong 0,$$

so every term in $\lim_n(\pi_q \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E))$ is equivalent to 0. In particular this gives us that the maps in the system are surjective, hence satisfies the Mittage-Leffler condition, so the $\lim^1$-term disappears for q odd. Hence we get

$$\pi_*(\lim_n \Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)) \cong \lim_n \pi_*(\Sigma^n \widetilde{\mathrm{LFree}_E}(\Omega^n E)),$$

so the desired follows by **Lemma 3.2.9**. □

To be able to continue this calculation and give a more precise description of $\pi_*\widehat{\mathrm{Pow}}(E)$, we would need a better understanding of the maps $\pi_*\theta_n$. But these are rather complicated to determine and surprisingly it is expected that they are so complicated, that all of these $Q^i x$-terms will disappear in the limit, which means we would end up with

$$\pi_* \widehat{\mathrm{Pow}}(E) \cong \mathbb{Z}_p\{x\}[u^{\pm 1}]$$

with $|u| = 2$. This implies that there is only the trivial stable power operations on commutative $K(1)$-local E-algebras.

A way to prove this, would be to show that the 2-fold suspension in the system above maps Qx to $p \cdot Qx$, and show that this generalize to higher powers of Q, i.e. to $Q^i x$. This could probably be done by using Lawson's description of power operations [Law20].

It is described in [Law20, 4.12, 4.13] how power operations can be factored by weight, and how we can consider the power operations, both the stable and the unstable, of weight k of degree m as an abelian group, which for any commutative ring spectrum E, is denoted by $\mathrm{Pow}^E(m,k)$. Elements of this group in grading r represents a weight-k natural transformation $\pi_m \to \pi_{m+r}$ on $h\mathrm{CAlg}_E$, which induces a natural transformation $E_m \to E_{m+r}$ on $h\mathrm{CAlg}$.

It is then shown in [Law20, 4.19] that each of these $\mathrm{Pow}^E(m,k)$ are equivalent to the E-homology of a specific Thom spectrum. By [Law20, 4.25] we further have a description of the suspension maps $\sigma_r : \mathrm{Pow}^E(m,k) \to \mathrm{Pow}^E(m+r,k)$, as induced by a specific map on these Thom spectra. These maps should agree with the ones in our system, but we will not consider them in this setting, since Thom spectra is outside the scope of this thesis.

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