Topics in Algebraic topology
Talk: Quillen’s Q-construction

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Definition 0.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \text{ob}\mathcal{D}$ a fixed object. Then we define a new category $F/d$ which consist of pairs $(c, u)$ where $u : F(c) \to d$ with $c \in \text{ob}\mathcal{C}$, in which morphisms $(c, u) \to (c', u')$ is a map $v : c \to c'$ such that the square

\[
\begin{array}{ccc}
F(c) & \xrightarrow{u} & d \\
\downarrow^{F(c)} & & \downarrow^{F(c')} \\
F(c') & \xrightarrow{u'} & d \\
\end{array}
\]

commutes.

Theorem 0.2 (Theorem A). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \text{ob}\mathcal{D}$ a fixed object. Then if the category $F/d$ is contractible for every object $d \in \text{ob}\mathcal{D}$, then the functor $F$ is a homotopy equivalence.

1 The Q-construction

1.1 Quillen’s Q-construction

Assume that $\mathcal{C}$ is an exact category. First of all we wish to define a new category $\mathcal{QC}$ called Quillen’s Q-construction. $\mathcal{QC}$ has the same objects as $\mathcal{C}$, i.e $\text{ob}\mathcal{QC} = \text{ob}\mathcal{C}$ and we define the morphisms in the following way:
Let $c_0, c_1 \in \text{ob} C$ and consider all diagrams of the form
\[
c_0 \leftarrow c_{01} \xrightarrow{p} c_0 \xrightarrow{r} c_1
\]
in $C$ with $p$ an admissible epimorphism and $r$ an admissible monomorphism. We will say that
\[
c_0 \leftarrow c_{01} \xrightarrow{p} c_0 \xrightarrow{r} c_1 \sim c_0 \leftarrow c'_{01} \xrightarrow{p'} c_0 \xrightarrow{r'} c_1
\]
if and only if there exist an isomorphism $\gamma : c_{01} \rightarrow c'_{01}$ which makes the following diagram commute:
\[
\begin{array}{ccc}
c_0 & \xrightarrow{p} & c_{01} \\
\downarrow & & \downarrow \\
c_1 & \xrightarrow{r} & c_1
\end{array}
\]
\[
\begin{array}{ccc}
c'_{01} & \xrightarrow{p'} & c_0 \\
\downarrow & & \downarrow \\
c_1 & \xrightarrow{r'} & c_1
\end{array}
\]
A morphism $f : c_0 \rightarrow c_1$ in $QC$ is all diagrams (1) up to the above equivalence. We define the composition of two morphisms $f : c_0 \rightarrow c_1$, $g : c_1 \rightarrow c_2$ in $QC$ as the pullback
\[
c_0 \xrightarrow{p_2} c_{02} \xleftarrow{p_1} c_0 \\
c_1 \xrightarrow{p_2} c_{12} \xleftarrow{p_1} c_1 \\
c_2 \xrightarrow{p_2} c_{22} \xleftarrow{p_1} c_2
\]
with $c_{02} = c_{01} \times c_1 c_2$. So $g \circ f$ is represented by
\[
c_0 \xleftarrow{c_0 \xrightarrow{c_0 \rightarrow c_2}} c_2.
\]
When the equivalence classes (1) form a set, this makes $QC$ a well-defined category. (In theory it is necessary to check that $c_{02} \in \text{ob} C$, that the morphisms $c_{02} \rightarrow c_{01}$ and $c_{02} \rightarrow c_{12}$ both are admissible, composition of admissible is admissible, the composition is well-defined on equivalence classes, is associative and has identity.)

This category can be depicted as the following:

\[
\begin{array}{ccc}
c_{01} & \xrightarrow{i} & c_{11} \\
\downarrow & & \downarrow \\
c_{01} & \xrightarrow{i} & c_{11} \\
\downarrow & & \downarrow \\
c_0 & \xrightarrow{c_0 \rightarrow c_1} & c_1
\end{array}
\]

\[
\begin{array}{ccc}
c_{02} & \xrightarrow{i} & c_{12} \\
\downarrow & & \downarrow \\
c_{02} & \xrightarrow{i} & c_{12} \\
\downarrow & & \downarrow \\
c_0 & \xrightarrow{c_0 \rightarrow c_1} & c_1
\end{array}
\]

\[
\begin{array}{ccc}
c_{03} & \xrightarrow{i} & c_{13} \\
\downarrow & & \downarrow \\
c_{03} & \xrightarrow{i} & c_{13} \\
\downarrow & & \downarrow \\
c_0 & \xrightarrow{c_0 \rightarrow c_1} & c_1
\end{array}
\]

Definition 1.1. Let $i : c_0 \rightarrow c_1$ be an admissible monomorphism in $C$ and $j : c_0 \rightarrow c_1$ and admissible epimorphism in $C$. Then we get two morphisms $i_1 : c_0 \rightarrow c_1$ and $j_1 : c_1 \rightarrow c_0$ in $QC$ respectively represented by
\[
c_0 \xleftarrow{1} c_0 \xrightarrow{i} c_1 \text{ and } c_1 \xleftarrow{j} c_0 \xrightarrow{1} c_1.
\]
1.2 An \(\infty\)-categorical \(Q\)-construction

We call \(i\) injective and \(j\) surjective. A map which is both is an isomorphism and has the form \(f = (f^{-1})^{t}\), where \(f\) is an isomorphism in \(C\).

This means that any morphism \(f : c_0 \to c_1\) in \(QC\) can be uniquely factorized, up to isomorphisms, as \(f = i_j^!\). This follows by the fact that we always can make the following pullback diagram:

\[
\begin{array}{ccc}
  c & \xleftarrow{i} & c_0 \\
  \downarrow & & \downarrow \\
  c_0 & \xrightarrow{j} & c_{01} \\
  \downarrow & & \downarrow \\
  c_0 & \xrightarrow{i} & c_{01} \\
\end{array}
\]

for some \(i\) and \(j\), when we are given a morphism \(f : c_0 \to c_1\).

1.2 An \(\infty\)-categorical \(Q\)-construction

This section uses Barwick and Rognes paper "On the \(Q\)-construction for exact \(\infty\)-categories". Throughout this let \(C\) be an exact category and recall that \(NC\) is a simplicial set. Using this we wish to make another \(Q\)-construction which can be extended to exact \(\infty\)-categories.

**Definition 1.2.** Let \(Q_n\) denote the category with objects \((i, j)\) with \(0 \leq i \leq j \leq n\) and a unique morphism \((i, j) \to (i', j')\) if \(i' \geq i\) and \(j' \leq j\), and no other morphisms. We then further define \((QC)_n\) to be subset of \(\text{Hom}(NQ_n, NC)\) such that each square of the form:

\[
\begin{array}{ccc}
  (i, j) & \xrightarrow{} & (i + 1, j) \\
  \downarrow & & \downarrow \\
  (i, j - 1) & \xrightarrow{} & (i + 1, j - 1) \\
\end{array}
\]

with \(j \geq i + 2\), is taken to a pullback square in \(C\) and such that every map \((i, j) \to (i, j - 1)\) is send to an admissible epimorphism and every map \((i, j) \to (i + 1, j)\) is send to an admissible monomorphism in \(C\).

We wish to make this into a simplicial set, so we have to choose simplicial structure maps. We will use the natural maps. We will not go into a detailed description of these, but to first understand the face maps, consider the diagram:

\[
\begin{array}{ccc}
  (i, j) & \xrightarrow{} & (i + 1, j) & \xrightarrow{} & (i + 2, j) \\
  \downarrow & & \downarrow & & \downarrow \\
  (i, j - 1) & \xrightarrow{} & (i + 1, j - 1) & \xrightarrow{} & (i, j - 2) \\
\end{array}
\]
Then the faces are

\[(i, j) \rightarrow (i + 2, j) \quad (i, j - 1) \rightarrow (i + 1, j - 1) \quad (i + 1, j) \rightarrow (i + 2, j)\]

\[(i, j - 2) \quad (i, j - 2) \quad (i + 1, j - 1)\]

To understand the degeneracy maps, consider the map

\[(i, j) \rightarrow (i + 2, j)\]

\[(i + 1, j - 1).\]

Here the degeneracies are constructed by composing the identity map in the following two ways:

\[(i, j) \rightarrow (i + 1, j) \xrightarrow{id} (i + 1, j)\]

\[(i, j) \rightarrow (i + 1, j - 1)\]

\[(i, j - 2)\]

\[(i, j) \xrightarrow{id} (i, j) \rightarrow (i + 1, j)\]

\[(i, j - 1) \xrightarrow{id} (i, j - 1)\]

\[(i, j - 1)\]

We see that the conditions for the squares and the morphisms in the definition of \(\mathbb{Q}C_n\) will be preserved under these face- and degeneracy maps, hence we have a simplicial set \(\mathbb{Q}^\infty\mathbb{C}\).

**Proposition 1.3.** If \(\mathbb{C}\) is an exact category, then \(\mathbb{Q}^\infty\mathbb{C}\) is an \(\infty\)-category.

**Proof.** To prove that this is indeed an \(\infty\)-category we have to prove that any map \(\Lambda^k_n \rightarrow \mathbb{Q}\mathbb{C}\) with \(0 < k < n\) extends to \(\Delta^n\). We will only consider the two simplest horn-filling conditions, i.e. for \(\Lambda^1_2\) and \(\Lambda^3_2\). For the first case we are given the diagram

\[
\begin{array}{ccc}
\Lambda^1_2 & \xrightarrow{c_{01}} & \Lambda^3_2 \\
\Lambda^0_2 & \xrightarrow{c_{00}} & \Lambda^1_2 \\
\Lambda^2_2 & \xrightarrow{c_{22}} & \Lambda^3_2 \\
\end{array}
\]
and we get the required extension by taking the pullback of the two central morphisms.

Considering the second case, i.e. $\Lambda^3_2$, we are given the following diagram

![Diagram](https://example.com/diagram.png)

But by construction every square in the above diagram is a pullback, hence by the universal property we get that the morphism from $c_{03}$ to $c_{01}$ factors through $c_{02}$ which results in the upper square also being a pullback. This is what was needed to fill the horn.

We are already familiar with the nerve functor $N : \text{Cat} \rightarrow s\text{Set}$. The Fundamental category functor is a left adjoint to this functor and is denoted by $\tau_{\leq 1} : s\text{Set} \rightarrow \text{Cat}$. Here we note that $\text{Cat}_\infty \subset s\text{Set}$.

Let $X$ be a simplicial set. Then $\tau_{\leq 1}(X)$ is the category freely generated by the directed graph whose vertices are $X_0$ (the 0-simplices) and whose edges are $X_1$ (1-simplices), modulo the relations

$$s_0(x) \sim \text{id}_x \text{ for } x \in X_0 \text{ and } d_1(x) \sim d_0(x) \circ d_2(x) \text{ for } x \in X_2$$

We first want to show that $QC = \tau_{\leq 1}(Q^\infty C)$.

**Theorem 1.4.** $QC = \tau_{\leq 1}(Q^\infty C)$ for an exact category $C$.

**Proof.** We start by noting that $\text{ob}(\tau_{\leq 1}(Q^\infty C)) = \text{ob}C = \text{ob}QC$, so the only non-trivial thing to check is that the morphisms agree. We have that

$$\text{Mor}(\tau_{\leq 1}(Q^\infty C)) = (Q^\infty C)[1]/ \sim = \text{Mor}(QC)$$

so we just have to check that the equivalence relation on 1-simplices in $Q^\infty C$ is the same as the one defining the morphisms in $QC$ (which is up to isomorphism). Let $f, g : c_0 \rightarrow c_1$ be isomorphic morphisms in $QC$, that means we have a commutative diagram of the form

![Diagram](https://example.com/diagram.png)
We get that the same two morphisms are equivalent as 1-simplicies of $\mathcal{Q}^\infty\mathcal{C}$ by applying the equivalence from the fundamental category functor the following two simplex

\[
\begin{array}{ccc}
c_0 & \xleftarrow{c_{01}} & c_1 \\
\downarrow & \searrow & \downarrow \\
c_0 & \xrightarrow{id} & c_1 \\
\end{array}
\]

If we are given two 1-simplices $f$ and $g$ of $\mathcal{Q}^\infty(C)$ which are equivalent using the fundamental category functor we know that there exists a 2-simplex $x \in \mathcal{Q}^\infty\mathcal{C}_2$ such that $f \simeq d_1(x) \sim d_0(x) \circ d_1(x) \simeq (d_0(x) \circ d_1(x)) \circ id \simeq g$. Using the diagrams above our results follows.

**Theorem 1.5.** Let $\mathcal{C}$ be an exact category, then $\mathcal{Q}^\infty\mathcal{C}$ is canonically equivalent to the nerve $N(\mathcal{QC})$.

We won’t discuss the proof, but note that due to the above theorem we know that the map $\mathcal{Q}^\infty\mathcal{C} \to N\mathcal{QC}$ are given by the composition $N\tau_\leq : sSet \to sSet$.

### 2 Higher algebraic K-theory

#### 2.1 Introduction

Let $\mathcal{C}$ be a small exact category and 0 a given zero-object.

**Definition 2.1.** For an exact category $\mathcal{C}$, we define the K-theory space by $K(\mathcal{C}) := \Omega B\mathcal{QC}$. The K-groups are then given by $K_i\mathcal{C} := \pi_n K\mathcal{C} = \pi_{n+1}(B\mathcal{QC}, 0)$.

Here $B\mathcal{QC}$ denotes the classifying space of $\mathcal{QC}$, i.e $|N(\mathcal{QC})|[-]$. This definition of the K-groups are independent of the choice of basepoint 0, since if we are given another zero-object 0', then there would be a unique map 0 $\to$ 0' in $\mathcal{QC}$ hence a canonical path from 0 to 0' in $B\mathcal{QC}$. Therefore we will not denote the basepoint from now on. One of the fundamental properties we wish satisfied for our K-groups is that $K_0(\mathcal{C}) = \pi_1(B\mathcal{QC})$ is canonically isomorphic to the Grothendieck group of $\mathcal{C}$, which is usually denoted by $K_0(\mathcal{C})$ by this reason. This is formulated in the following theorem:

**Theorem 2.2.** The geometric realization $B\mathcal{QC}$ is a connected CW complex with $\pi_1(B\mathcal{QC}) \cong K_0(\mathcal{C})$.

This definition of algebraic K-theory satisfies many of the same properties as the algebraic K-theory defined by the Waldhausen $S$-dot construction. Some of these basic properties are:
Proposition 2.3.

- Given an exact functor \( f : \mathcal{C} \to \mathcal{D} \) we get a functor \( \mathcal{Q}C \to \mathcal{Q}D \) which further induces a homomorphism of \( K \)-groups, which will be denoted \( f_\ast : K_i \mathcal{C} \to K_i \mathcal{D} \).

This means we have that \( K_i \) is a functor between the category of exact categories with exact functors and the category of abelian groups.

- For two exact categories \( \mathcal{C} \) and \( \mathcal{D} \) we have that \( K \) commutes with products: \( K_i (\mathcal{C} \times \mathcal{D}) \simeq K_i \mathcal{C} \times K_i \mathcal{D} \).

- The \( K \)-groups commutes with filtered colimits: Let \( j \mapsto \mathcal{C}_j \) be a functor from a small category to the category of exact categories and exact functors. Then

\[
\mathcal{Q}(\operatorname{colim} \mathcal{C}_j) \simeq \operatorname{colim} \mathcal{Q}C_j
\]

hence \( K_i (\operatorname{colim} \mathcal{C}_j) \simeq \operatorname{colim} K_i \mathcal{C}_j \).

Let \( \mathcal{C} \) be an exact category and let \( \mathcal{E} \) denote the family of short exact sequences with elements denoted by \( E, E', \ldots \) etc. We consider \( \mathcal{E} \) as an additive category. Further we let \( sE \) be the subobjects of \( E \), \( tE \) the total objects of \( E \) and \( qE \) the quotient objects of \( E \). This means we, by definition, has a short exact sequence

\[
0 \to sE \to tE \to qE \to 0
\]

in \( \mathcal{C} \) associated to each object \( E \) of \( \mathcal{E} \). We wish to turn \( \mathcal{E} \) into an exact category. We will say that a sequence \( E_0 \to E_1 \to E_2 \) is exact if it gives rise to three exact sequences in \( \mathcal{C} \) upon applying \( s, t, q \), i.e if the following is exact in \( \mathcal{C} \):

\[
0 \to sE_0 \to sE_1 \to sE_2 \to 0
\]

\[
0 \to tE_0 \to tE_1 \to tE_2 \to 0
\]

\[
0 \to qE_0 \to qE_1 \to qE_2 \to 0
\]

Due to the exactness of \( \mathcal{C} \) with \( \mathcal{E} \) as the exact sequences, it follows that the above makes \( \mathcal{E} \) an exact category.

Theorem 2.4. The functor

\[
(s, q) : \mathcal{Q}E \to \mathcal{Q}C \times \mathcal{Q}C
\]

is a homotopy equivalence.
2.2 The Devissage theorem

Proof. By Quillen's Theorem A we know it's sufficient to prove that the comma category \( \mathcal{D} := (s, q)/(M, N) \) is contractible for any pair of objects \((M, N) \in \text{ob}(C \times C)\). Note that \( \mathcal{D} \) is the filtered category consisting of triples \((E, u, v)\) where

\[
E \in \text{ob}\mathcal{E}, \quad u \in QC(sE, M), \quad v \in QC(qE, N).
\]

We then define \( \mathcal{D}' \subset \mathcal{D} \) to be the subcategory consisting of triples \((E, u, v)\) where we further assume that \(u\) is surjective and \( \mathcal{D}'' \subset \mathcal{D}' \) the subcategory where \(v\) is assumed to be injective as well. It may be proven that the two inclusion functors \( \mathcal{D}' \hookrightarrow \mathcal{D} \) and \( \mathcal{D}'' \hookrightarrow \mathcal{D}' \) has left adjoints, hence both induces homotopy equivalences. In particular \( \mathcal{D}'' \cong \mathcal{D} \), so sufficient to prove that \( \mathcal{D} \) is contractible when \(u\) is surjective and \(v\) is injective. Let \((E, j^1, i) \in \mathcal{D}''\) and let \(j_M : M \to 0\) and \(i_N : 0 \to N\) be the obvious maps. So we have two triples \((E, j^1, i), (0, j^1_M, i_N) \in \mathcal{D}''\), where the four maps can be represented by

\[
i_E : qE \xrightarrow{1} qE \xrightarrow{i} N \\
j^1 : sE \xrightarrow{j} M \xrightarrow{1} M \\
i_N : 0 \xrightarrow{1} 0 \xrightarrow{i_N} N \\
j^1_M : 0 \xrightarrow{j_M} M \xrightarrow{1} M.
\]

This gives us that a map \((0, j^1_M, i_N) \to (E, j^1, i)\) can be identified with an admissible subobject \(E'\) of \(E\) for which \(sE' = sE\) and \(qE' = 0\) due to the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{j_M} & M \\
| & \downarrow & \downarrow i \\
0 & \xrightarrow{j} & sE \\
| & \downarrow & \downarrow \\
0 & \xrightarrow{1} & sE \\
| & \downarrow & \downarrow \\
0 & \xrightarrow{0} & qE \\
| & \downarrow & \downarrow i \\
0 & \xrightarrow{0} & N
\end{array}
\]

We see that such an \(E'\) is unique up to isomorphism since by exactness \(tE' \cong sE \cong sE'\), hence \((0, j^1_M, i_N)\) is an initial object, so \( \mathcal{D}'' \) is contractible - i.e \( \mathcal{D} \) is contractible as desired. \( \square \)

2.2 The Devissage theorem

The goal for these notes is to prove the Devissage theorem, and we are now ready to describe the setting for this theorem. Let \( \mathcal{A} \) denote an abelian category with a set of
isomorphism classes of objects. Further let $\mathcal{B}$ be a non-empty full subcategory of $\mathcal{A}$ which is closed under subobjects, quotient objects and finite products in $\mathcal{A}$. We note that $\mathcal{B}$ is an abelian category and that the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact. We wish to consider both $\mathcal{A}$ and $\mathcal{B}$ as exact categories. This is done by declaring that the monomorphisms and epimorphisms are admissible. Since $\mathcal{A}$ and $\mathcal{B}$ both are abelian this will give an exact structure. Using this exact structure we get that $\mathcal{B}$ is the full subcategory of $\mathcal{A}$ consisting of those objects which are also objects of $\mathcal{B}$. We are now ready to state the theorem.

**Theorem 2.5 (Devissage).** Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Assume that every object $M \in \text{ob}\, \mathcal{A}$ has a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i \in \text{ob}\, \mathcal{A}$ such that $M_i/M_{i-1} \in \text{ob}\, \mathcal{B}$. Then the inclusion functor $\mathcal{B} \to \mathcal{A}$ is a homotopy equivalence, hence $K_i \mathcal{B} \cong K_i \mathcal{A}$.

For the proof of this theorem we will need some few definitions and a lemma:

**Definition 2.6.** Let $c \in \text{ob}\, \mathcal{C}$. An **admissible subobject** of $c$ is an isomorphism class of admissible morphisms $c' \hookrightarrow c$ in $\mathcal{C}$.

**Definition 2.7.** Let $c', c''$ be two admissible objects of $c \in \text{ob}\, \mathcal{C}$. Then if the unique morphisms $c'' \to c'$ is an admissible monomorphism we call $(c'', c')$ an **admissible layer** of $c$.

**Lemma 2.8.** A morphism $A \to B$ in $\mathcal{A}$ may be identified with an isomorphism $j : B_2/B_1 \cong A$, where $(B_1, B_2)$ is an admissible layer of $B$.

**Proof.** Let $f \in \mathcal{A}(A, B)$. Then it’s represented by

$$A \xleftarrow{h} C \xrightarrow{g} B.$$  

We note that since $\mathcal{A}$ is abelian there exists an exact sequence

$$\text{ker}(h) \hookrightarrow C \xrightarrow{h} A$$

so in particular we have an admissible layer $(\text{ker}(h), C)$ of $B$. By using that $h$ is an admissible monomorphism and the universal property of quotients, we get

$$\begin{tikzcd}
\text{ker}(h) \arrow[r] & C \arrow[r, h] & A \\
& C/\text{ker}(h) \arrow[u, \cong]
\end{tikzcd}$$

so $f : A \to B$ may be identified with the isomorphism $A \cong C/\text{ker}(h)$. \qed
2.2 The Devissage theorem

We are now ready to prove the main theorem:

Proof of Devissage. Let \( f \) denote the inclusion. By Theorem A it’s sufficient to prove that the comma category \( f/M \) is contractible for every \( M \in \text{ob}A \). We note that \( f/M \) may be considered as the fibered category over \( QB \) consisting of pairs \((N,u)\) where \( N \in \text{ob}QB \) and \( u \in QA(N,M) \). Using Lemma 2.8 we identify \( u \) with an isomorphism \( N \cong M_1/M_0 \) where \((M_0, M_1)\) is an admissible layer of \( M \) and \( M_1/M_0 \in \text{ob}B \).

Now, let \( J(M) \) denote the ordered set consisting of admissible layers \((M_0, M_1)\) of \( M \) with the ordering \((M_0, M_1) \leq (M_0', M_1')\) if and only if \( M_0' \subset M_0 \subset M_1' \subset M_1 \). We then have an equivalence between \( f/M \) and \( J(M) \).

Since we have assumed that \( M \) has a finite filtration with quotiens in \( B \) we get that it is sufficient to prove that the inclusion \( J(M') \rightarrow J(M) \) is a homotopy equivalence (with this I mean that it induces a homotopy equivalence on the classifying space) whenever \( M' \subset M \) and \( M/M' \in \text{ob}B \), since then \( J(M) \) is homotopy equivalent to \( J(0) \) which is clearly contractible.

So assume that \( M' \subset M \) and \( M/M' \in \text{ob}B \). Define the following two functors:

\[
\begin{align*}
r &: J(M) \rightarrow J(M') \\
(M_0, M_1) &\mapsto (M_0 \cap M', M_1 \cap M') \\
s &: J(M) \rightarrow J(M) \\
(M_0, M_1) &\mapsto (M_0 \cap M', M_1)
\end{align*}
\]

Here \( A \cap B \) denotes the pullback \( A \times_C B \) given by the pullback diagram

\[
\begin{array}{c}
A \times_C B \\
\downarrow \\
A \\
\end{array} \rightarrow \begin{array}{c}
B \\
\downarrow \\
C \\
\end{array}
\]

Using the universal property of pullbacks we have that \((M_0 \cap M', M_1 \cap M')\) and \((M_0 \cap M', M_1)\) are indeed admissible layers of \( M' \) and \( M \) respectively. It can also be proven that

\[
M_1 \cap M'/M_0 \cap M' \subset M_1/(M_0 \cap M') \subset M_1/M_0 \times M/M'
\]

which gives us that \( M_1 \cap M'/M_0 \cap M', M_1/M_0 \cap M' \in \text{ob}B \) since we have by assumption that \( B \) is closed under subobjects and finite projectes so \( M_1/M_0 \times M/M' \in \text{ob}B \).

Next goal is to prove that \( r \) is a homotopy inverse to \( i \). Since

\[
\begin{align*}
J(M') &\xrightarrow{i} J(M) \xrightarrow{r} J(M') \\
(M_0', M_1') &\mapsto (M_0, M_1) \mapsto (M_0 \cap M', M_1 \cap M')
\end{align*}
\]
and \((M'_0 \cap M', M_1 \cap M') = (M'_0, M'_1)\) we have \(r \circ i = id_{J(M')}\). To see that \(i \circ r \simeq id_{J(M)}\) we wish to prove that there exists natural transformations

\[i \circ r \Rightarrow s \Leftarrow id_{J(M)}.\]

We note that

\[i \circ r((M_0, M_1)) = (M_0 \cap M', M_1 \cap M') \in J(M)\]

so the first natural transformation \(i \circ r \Rightarrow s\) is represented by the ordering

\[(M_0 \cap M', M_1 \cap M') \subset (M_0 \cap M', M_1).\]

The other natural transformation \(id_{J(M)} \Rightarrow s\) is represented by

\[(M_0, M_1) \subset (M_0 \cap M', M_1).\]

It is a result that natural transformations induces homotopies, so there exist homotopies \(i \circ r \simeq s \simeq id_{J(M)}\). So we have both \(i \circ r \simeq id_{J(M)}\) and \(r \circ i \simeq id_{J(M')}\) so \(r\) is a homotopy inverse of \(i\).

We wish to consider two applications of this very strong theorem. First we note the following notation

**Definition 2.9.** Let \(R\) be a Noetherian ring and \(FMod(R)\) be the full subcategory of \(Mod(R)\) consisting of finitely generated \(R\)-modules. Then we define

\[G_i(R) := K_i(FMod(R)).\]

**Corollary 2.10.** If \(I\) is a nilpotent ideal in a Noetherian ring \(R\), then

\[G_i(R/I) \cong G_i(R).\]

**Proof.** Let \(M \in \text{ob} FMod(R)\). Then we get that

\[0 = MI^n \subseteq MI^{n-1} \subseteq \cdots \subseteq MI \subseteq M\]

is finite since \(I\) is nilpotent. We further have that \(MI^n\) is finitely generated since \(R\) is Noetherian, hence the surjection \(MI^n \twoheadrightarrow MI^n/MI^{n+1}\) gives us that all the quotients \(MI^n/MI^{n+1}\) are finitely generated \(R/I\)-modules. That means we can apply the Devissage theorem so the inclusion functor \(FMod(R/I) \hookrightarrow FMod(R)\) is a homotopy equivalence, so

\[G_i(R/I) \cong G_i(R).\]

Before we can state the next corollary we need some definitions:
Definition 2.11. An object $X \in \text{ob} \mathcal{C}$ is said to be simple if there is precisely two quotient objects of $X$—namely $X$ and $0$. If $X$ is a finite coproduct of simple objects it is said to be semi-simple.

Definition 2.12. An object $X \in \text{ob} \mathcal{C}$ is said to have finite length if there exist a finite sequence of subobject inclusions into $X$

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

such that at every state $i$, the quotient $X_i/X_{i-1}$ is a simple object of $\mathcal{C}$.

Corollary 2.13. Let $\mathcal{A}$ be an abelian category with a set of isomorphism classes, such that every object has finite length. Then

$$K_i \mathcal{A} \cong \bigoplus_{j \in J} G_i(D_j)$$

where $\{X_j, j \in J\}$ is a set of representatives for the isomorphism classes of simple objects of $\mathcal{A}$, and $D_j$ is the division ring $\text{End}(X_j)^{\text{op}}$. 