Topics in Algebraic topolgy
Talk: Quillens Q-construction

Julie Zangenberg Rasmusen

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Contents

Definition 0.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \mathrm{ob}\mathcal{D}$ a fixed object. Then we define a new category F/d which consist of pairs (c, u) where $u : F(c) \to d$ with $c \in \mathrm{ob}\mathcal{C}$, in which morphisms $(c, u) \to (c', u')$ is a map $v : c \to c'$ such that the square

$$\begin{tikzcd} F(c) \arrow[r, "u"] \arrow[d, "F(v)"'] & d \arrow[d, equal] \\ F(c') \arrow[r, "u'"'] & d \end{tikzcd}$$

commutes.

Theorem 0.2 (Theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in ob\mathcal{D}$ a fixed object. Then if the category F/d is contractible for every object $d \in ob\mathcal{D}$, then the functor F is a homotopy equivalence.*

1 The Q-construction

1.1 Quillens Q-construction

Assume that $\mathcal{C}$ is an exact category. First of all we wish to define a new category $\mathcal{QC}$ called Quillens Q-construction. $\mathcal{QC}$ has the same objects as $\mathcal{C}$, i.e $\mathrm{ob}\mathcal{QC} = \mathrm{ob}\mathcal{C}$ and we define the morphisms in the following way:

Let $c_0, c_1 \in \mathrm{ob}\mathcal{C}$ and consider all diagrams of the form

$$c_0 \overset{p}{\twoheadleftarrow} c_{01} \overset{r}{\rightarrowtail} c_1 \tag{1}$$

in $\mathcal{C}$ with p an admissible epimorphism and r an admissible monomorphism. We will say that

$$c_0 \overset{p}{\twoheadleftarrow} c_{01} \overset{r}{\rightarrowtail} c_1 \;\sim\; c_0 \overset{p}{\twoheadleftarrow} c'_{01} \overset{r}{\rightarrowtail} c_1$$

if and only if there exist an isomorphism $\gamma : c_{01} \to c'_{01}$ which makes the following diagram commute:

$$\begin{tikzcd}
c_0 & c_{01} \arrow[l, two heads, "p"'] \arrow[r, tail, "r"] \arrow[d, "\gamma"] & c_1 \\
& c'_{01} \arrow[ul, two heads, "p'"] \arrow[ur, tail, "r'"'] &
\end{tikzcd}$$

A morphism $f : c_0 \to c_1$ in $\mathcal{QC}$ is all diagrams (1) up to the above equivalence. We define the composition of two morphisms $f : c_0 \to c_1$, $g : c_1 \to c_2$ in $\mathcal{QC}$ as the pullback

$$\begin{tikzcd}
c_{02} \arrow[r, tail] \arrow[d, two heads] & c_{12} \arrow[r, tail] \arrow[d, two heads] & c_2 \\
c_{01} \arrow[r, tail] \arrow[d, two heads] & c_1 & \\
c_0 & &
\end{tikzcd}$$

with $c_{02} = c_{01} \times_{c_1} c_2$. So $g \circ f$ is represented by

$$c_0 \twoheadleftarrow c_{02} \rightarrowtail c_2.$$

When the equivalence classes (1) form a set, this makes $\mathcal{QC}$ a well-defined category. (In theory it is necessary to check that $c_{02} \in \mathrm{ob}\mathcal{C}$, that the morphisms $c_{02} \twoheadrightarrow c_{01}$ and $c_{02} \rightarrowtail c_{12}$ both are admissible, composition of admissible is admissible, the composition is well-defined on equivalence classes, is associative and has identity.)

This category can be depicted as the following:

$$\begin{tikzcd}
& & c_{0,3} \arrow[dl, two heads] \arrow[dr, tail] & & \\
& c_{0,2} \arrow[dl, two heads] \arrow[dr, tail] & & c_{1,3} \arrow[dl, two heads] \arrow[dr, tail] & \\
c_{0,1} \arrow[dl, two heads] \arrow[dr, tail] & & c_{1,2} \arrow[dl, two heads] \arrow[dr, tail] & & c_{2,3} \arrow[dl, two heads] \arrow[dr, tail] \\
c_{0,0} & c_{1,1} & & c_{2,2} & & c_{3,3}
\end{tikzcd}$$

Definition 1.1. Let $i : c_0 \rightarrowtail c_1$ be an admissible monomorphism in $\mathcal{C}$ and $j : c_0 \twoheadrightarrow c_1$ and admissible epimorphism in $\mathcal{C}$. Then we get two morphisms $i_! : c_0 \to c_1$ and $j^! : c_1 \to c_0$ in $\mathcal{QC}$ respectively represented by

$$c_0 \overset{1}{\twoheadleftarrow} c_0 \overset{i}{\rightarrowtail} c_1 \;\text{ and }\; c_1 \overset{j}{\twoheadleftarrow} c_0 \overset{1}{\rightarrowtail} c_1 \;.$$

We call $i_!$ *injective* and $j^!$ *surjective*. A map which is both is an *isomorphism* and has the form $f_! = (f^{-1})^!$, where f is an isomorphism in $\mathcal{C}$.

This means that any morphism $f : c_0 \to c_1$ in $\mathcal{QC}$ can be uniquely factorized, up to isomorphisms, as $f = i_! j^!$. This follows by the fact that we always can make the following pullback diagram

$$\begin{tikzcd}
c \arrow[r, tail] \arrow[d, two heads] & c_1 \arrow[r, tail, "1"] \arrow[d, two heads, "j"] & c_1 \\
c_0 \arrow[r, tail, "i"] \arrow[d, two heads, "1"] & c_{01} & \\
c_0 & &
\end{tikzcd}$$

for some i and j, when we are given a morphism $f : c_0 \to c_1$.

1.2 An ∞-categorical Q-construction

This section uses Barwick and Rognes paper "On the Q-construction for exact ∞-categories". Throughout this let $\mathcal{C}$ be an exact category and recall that $N\mathcal{C}$ is a simplicial set. Using this we wish to make another Q-construction which can be extended to exact ∞-categories.

Definition 1.2. Let $\mathcal{Q}_n$ denote the category with objects (i, j) with $0 \leq i \leq j \leq n$ and a unique morphism $(i, j) \to (i', j')$ if $i' \geq i$ and $j' \leq j$, and no other morphisms. We then further define $(\mathcal{QC})_n$ to be subset of $Hom(N\mathcal{Q}_n, N\mathcal{C})$ such that each square of the form

$$\begin{tikzcd}
(i, j) \arrow[r] \arrow[d] & (i+1, j) \arrow[d] \\
(i, j-1) \arrow[r] & (i+1, j-1)
\end{tikzcd}$$

with $j \geq i+2$, is taken to a pullback square in $\mathcal{C}$ and such that every map $(i, j) \to (i, j-1)$ is send to an admissible epimorphism and every map $(i, j) \to (i+1, j)$ is send to an admissible monomorphism in $\mathcal{C}$.

We wish to make this into a simplicial set, so we have to choose simplicial structure maps. We will use the natural maps. We will not go into a detailed description of these, but to first understand the face maps, consider the diagram

$$\begin{tikzcd}
(i, j) \arrow[r] \arrow[d] & (i+1, j) \arrow[r] \arrow[d] & (i+2, j) \\
(i, j-1) \arrow[r] \arrow[d] & (i+1, j-1) & \\
(i, j-2) & &
\end{tikzcd}$$

Then the faces are

$$\begin{tikzcd} (i,j) \arrow[r] \arrow[d] & (i+2,j) \\ (i,j-2) & \end{tikzcd} \qquad \begin{tikzcd} (i,j-1) \arrow[r] \arrow[d] & (i+1,j-1) \\ (i,j-2) & \end{tikzcd} \qquad \begin{tikzcd} (i+1,j) \arrow[r] \arrow[d] & (i+2,j) \\ (i+1,j-1) & \end{tikzcd}$$

To understand the degeneracy maps, consider the map

$$\begin{tikzcd} (i,j) \arrow[r] \arrow[d] & (i+2,j) \\ (i+1,j-1). & \end{tikzcd}$$

Here the degeneracies are constructed by composing the identity map in the following two ways:

$$\begin{tikzcd} (i,j) \arrow[r] \arrow[d, "id"] & (i+1,j) \arrow[r, "id"] \arrow[d, "id"] & (i+1,j) \\ (i,j) \arrow[r] \arrow[d] & (i+1,j-1) & \\ (i,j-2) & & \end{tikzcd}$$

$$\begin{tikzcd} (i,j) \arrow[r, "id"] \arrow[d] & (i,j) \arrow[r] \arrow[d] & (i+1,j) \\ (i,j-1) \arrow[r, "id"] \arrow[d, "id"] & (i,j-1) & \\ (i,j-1) & & \end{tikzcd}$$

We see that the conditions for the squares and the morphisms in the definition of $\mathcal{QC}_n$ will be preserved under these face - and degeneracy maps, hence we have a simplicial set $\mathcal{Q}^\infty\mathcal{C}$.

Proposition 1.3. *If $\mathcal{C}$ is an exact category, then $\mathcal{Q}^\infty\mathcal{C}$ is an ∞-category.*

Proof. To prove that this is indeed an ∞-category we have to prove that any map $\Lambda^n_k \to \mathcal{QC}$ with $0 < k < n$ extends to Δ^n. We will only consider the two simplest horn-filling conditions, i.e for Λ^2_1 and Λ^3_2. For the first case we are given the diagram

$$\begin{tikzcd} & c_{01} \arrow[dl, two heads] \arrow[dr, tail] & & c_{12} \arrow[dl, two heads] \arrow[dr, tail] & \\ c_{00} & & c_{11} & & c_{22} \end{tikzcd}$$

and we get the required extension by take the pullback of the two central morphisms.

Considering the second case, i.e Λ^3_2, we are given the following diagram

$$\begin{tikzcd}
 & & c_{03} \arrow[dl, tail] \arrow[dr, tail] \arrow[ddll, bend right] & & \\
 & c_{02} \arrow[dl, tail] \arrow[dr, tail] & & c_{13} \arrow[dl, tail] \arrow[dr, tail] & \\
c_{01} \arrow[dl, tail] \arrow[dr, tail] & & c_{12} \arrow[dl, tail] \arrow[dr, tail] & & c_{23} \arrow[dl, tail] \arrow[dr, tail] \\
c_{00} & & c_{11} & & c_{22} & & c_{33}
\end{tikzcd}$$

But by construction every square in the above diagram is a pullback, hence by the universal property we get that the morphism from c_{03} to c_{01} factors through c_{02} which results in the upper square also being a pullback. This is what was needed to fill the horn. $\square$

We are already familiar with the nerve functor $N : Cat \to sSet$. The *Fundemental category functor* is a left adjoint to this functor and is denoted by $\tau_{\leq 1} : sSet \to Cat$. Here we note that $Cat_\infty \subset sSet$. Let X be a simplicial set. Then $\tau_{\leq 1}(X)$ is the category freely generated by the directed graph whose vertices are X_0 (the 0-simplices) and whose edges are X_1 (1-simplices), modulo the relations

$$s_0(x) \sim id_x \text{ for } x \in X_0 \text{ and } d_1(x) \sim d_0(x) \circ d_2(x) \text{ for } x \in X_2$$

We first want to show that $\mathcal{QC} = \tau_{\leq 1}(\mathcal{Q}^\infty\mathcal{C})$.

Theorem 1.4. *$\mathcal{QC} = \tau_{\leq 1}(\mathcal{Q}^\infty\mathcal{C})$ for an exact category $\mathcal{C}$.*

Proof. We start by noting that $\mathrm{ob}(\tau_{\leq 1}(\mathcal{Q}^\infty\mathcal{C})) = \mathrm{ob}\mathcal{C} = \mathrm{ob}\mathcal{Q}C$, so the only non-trivial thing to check is that the morphisms agree. We have that

$$Mor(\tau_{\leq 1}(\mathcal{Q}^\infty\mathcal{C})) = (\mathcal{Q}^\infty\mathcal{C})[1]/\sim = Mor(\mathcal{QC})$$

so we just have to check that the equivalence relation on 1-simplices in $\mathcal{Q}^\infty\mathcal{C}$ is the same as the one defining the morphisms in $\mathcal{QC}$ (which is up to isomorphism). Let $f, g : c_0 \to c_1$ be isomorphic morphisms in $\mathcal{QC}$, that means we have a commutative diagram of the form

$$\begin{tikzcd}
 & c_{01} \arrow[dl, tail] \arrow[dr, tail] \arrow[dd, "\simeq"] & \\
c_0 & & c_1 \\
 & c'_{01} \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

We get that the same two morphisms are equivalent as 1-simplicies of $\mathcal{Q}^\infty\mathcal{C}$ by applying the equivalence from the fundemental category functor the following two simplex

$$\begin{tikzcd}
 & & c'_{01} \arrow[dl, tail] \arrow[dr, tail] \arrow[dd, phantom, "\simeq"] & & \\
 & c_{01} \arrow[dl, two heads] \arrow[dr, tail] & & c_1 \arrow[dr, equal] \arrow[dr, equal] & \\
c_0 & & c_1 \arrow[rr, equal, "id"] & & c_1.
\end{tikzcd}$$

If we are given two 1-simplices f and g of $\mathcal{Q}^\infty(C)$ which are equivalent using the fundemental category functor we know that there exists a 2-simplex $x \in \mathcal{Q}^\infty\mathcal{C}_2$ such that $f \simeq d_1(x) \sim d_0(x) \circ d_1(x) \simeq (d_0(x) \circ d_1(x)) \circ id \simeq g$. Using the diagrams above our results follows. $\square$

Theorem 1.5. *Let $\mathcal{C}$ be an exact category, then $\mathcal{Q}^\infty\mathcal{C}$ is canonically equivalent to the nerve $N(\mathcal{Q}\mathcal{C})$.*

We won't discuss the proof, but note that due to the above theorem we know that the map $\mathcal{Q}^\infty\mathcal{C} \to N\mathcal{Q}\mathcal{C}$ are given by the composition $N\tau_{\leq 1} : sSet \to sSet$.

2 Higher algebraic K-theory

2.1 Introduction

Let $\mathcal{C}$ be a small exact category and 0 a given zero-object.

Definition 2.1. For an exact category $\mathcal{C}$, we define the *K-theory space* by $K(\mathcal{C}) := \Omega B\mathcal{Q}\mathcal{C}$. The *K-groups* are then given by $K_i\mathcal{C} := \pi_n K\mathcal{C} = \pi_{i+1}(B(\mathcal{Q}\mathcal{C}), 0)$.

here $B(\mathcal{Q}\mathcal{C})$ denotes the classifying space of $\mathcal{Q}\mathcal{C}$, i.e $|N(\mathcal{Q}\mathcal{C})[-]|$. This definition of the K-groups are independent of the choice of basepoint 0, since if we are given another zero-object $0'$, then there would be a unique map $0 \to 0'$ in $\mathcal{Q}\mathcal{C}$ hence a canonical path from 0 to $0'$ in $B(\mathcal{Q}\mathcal{C})$. Therefore we will not denote the basepoint from now on. One of the fundemental properties we wish satisfied for our K-groups is that $K_0(\mathcal{C}) = \pi_1(B\mathcal{Q}\mathcal{C})$ is canonically isomorphic to the Groethendick group of $\mathcal{C}$, which is usually denoted by $K_0(\mathcal{C})$ by this reason. This is formulated in the following theorem:

Theorem 2.2. *The geometric realization $B\mathcal{Q}\mathcal{C}$ is a connected CW complex with $\pi_1(B\mathcal{Q}\mathcal{C}) \cong K_0(\mathcal{C})$.*

This definition of algebraic K-theory satisfies many of the same properties as the algebraic K-theory defined by the Waldhausen S-dot construction. Some of these basic properties are:

Proposition 2.3.

- *Given an exact functor $f : \mathcal{C} \to \mathcal{D}$ we get a functor $\mathcal{QC} \to \mathcal{QD}$ which further induces a homomorphism of K-groups, which will be denoted*

$$f_* : K_i\mathcal{C} \to K_i\mathcal{D}$$

  This means we have that K_i is a functor between the category of exact categories with exact functors and the category of abelian groups.

- *For two exact categories $\mathcal{C}$ and $\mathcal{D}$ we have that K commutes with products: $K_i(\mathcal{C} \times \mathcal{D}) \simeq K_i\mathcal{C} \times K_i\mathcal{D}$*

- *The K-groups commutes with filtered colimits: Let $j \mapsto \mathcal{C}_j$ be a functor from a small category to the category of exact categories and exact functors. Then*

$$\mathcal{Q}(colim_{\to}\mathcal{C}_j) \simeq colim_{\to}\mathcal{QC}_j$$

  hence $K_i(colim_{\to}\mathcal{C}_j) \simeq colim_{\to}K_i\mathcal{C}_j$.

Let $\mathcal{C}$ be an exact category and let $\mathcal{E}$ denote the family of short exact sequences with elements denoted by $E, E', ...$ etc. We consider $\mathcal{E}$ as an additive category. Further we let sE be the *subobjects* of E, tE the *total objects* of E and qE the *quotient objects* of E. This means we, by definition, has a short exact sequence

$$0 \to sE \to tE \to qE \to 0$$

in $\mathcal{C}$ associated to each object E of $\mathcal{E}$. We wish to turn $\mathcal{E}$ into an exact category. We will say that a sequence $E_0 \to E_1 \to E_2$ is *exact* if it gives rise to three exact sequneces in $\mathcal{C}$ upon applying s, t, q, i.e if the following is exact in $\mathcal{C}$:

$$0 \to sE_0 \to sE_1 \to sE_2 \to 0$$

$$0 \to tE_0 \to tE_1 \to tE_2 \to 0$$

$$0 \to qE_0 \to qE_1 \to qE_2 \to 0$$

Due to the exactness of $\mathcal{C}$ with $\mathcal{E}$ as the exact sequences, it follows that the above makes $\mathcal{E}$ an exact category.

Theorem 2.4. *The functor*

$$(s, q) : \mathcal{QE} \to \mathcal{QC} \times \mathcal{QC}$$

is a homotopy equivalence.

Proof. By Quillen's Theorem A we know it's sufficient to prove that the comma category $\mathcal{D} := (s,q)/(M,N)$ is contractible for any pair of objects $(M,N) \in \mathrm{ob}(\mathcal{C}\times\mathcal{C})$. Note that $\mathcal{D}$ is the filtered category consisting of triples (E,u,v) where

$$E \in \mathrm{ob}\mathcal{E},\ u \in \mathcal{QC}(sE,M),\ v \in \mathcal{QC}(qE,N).$$

We then define $\mathcal{D}' \subset \mathcal{D}$ to be the subcategory consisting of triples (E,u,v) where we further assume that u is surjective and $\mathcal{D}'' \subset \mathcal{D}'$ the subcategory where v is assumed to be injective as well. It may be proven that the two inclusion functors $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{D}'' \hookrightarrow \mathcal{D}'$ has left adjoints, hence both induces homotopy equivalences. In particular $\mathcal{D}'' \simeq \mathcal{D}$, so sufficient to prove that $\mathcal{D}$ is contractible when u is surjective and v is injective. Let $(E, j^!, i_!) \in \mathcal{D}''$ and let $j_M : M \twoheadrightarrow 0$ and $i_N : 0 \rightarrowtail N$ be the obvious maps. So we have two triples $(E, j^!, i_!), (0, j^!_M, i_{N_!}) \in \mathcal{D}''$, where the four maps can be represented by

$$\begin{aligned}
i_! &: qE \xleftarrow{1} qE \xrightarrow{i} N \\
j^! &: sE \xleftarrow{j} M \xrightarrow{1} M \\
i_{N_!} &: 0 \xleftarrow{1} 0 \xrightarrow{i_N} N \\
j^!_M &: 0 \xleftarrow{j_M} M \xrightarrow{1} M\ .
\end{aligned}$$

This gives us that a map $(0, j^!_M, i_{N_!}) \to (E, j^!, i_!)$ can be identified with an admissible subobject E' of E for which $sE' = sE$ and $qE' = 0$ due to the diagram

$$\begin{tikzcd}
0 \arrow[d, equal] & M \arrow[l, two heads, "j_M"'] \arrow[r, tail, "id"] \arrow[d, two heads, "j"] & M \arrow[d, two heads, "j"] \\
0 \arrow[d, equal] & sE \arrow[l, two heads] \arrow[r, tail] \arrow[d] & sE \arrow[d] \\
0 \arrow[d, equal] & tE' \arrow[l, two heads] \arrow[r, tail] \arrow[d] & tE \arrow[d] \\
0 \arrow[d, equal] & 0 \arrow[l, two heads] \arrow[r, tail] \arrow[d, equal] & qE \arrow[d, tail, "i"] \\
0 & 0 \arrow[l, two heads, "id"'] \arrow[r, tail, "i_N"] & N
\end{tikzcd}$$

We see that such an E' is unique up to isomorphism since by exactness $tE' \simeq sE \simeq sE'$, hence $(0, j^!_M, i_{N_!})$ is an initial object, so $\mathcal{D}''$ is contractible - i.e $\mathcal{D}$ is contractible as desired. $\square$

2.2 The Devissage theorem

The goal for these notes is to prove the Devissage theorem, and we are now ready to describe the settion for this theorem. Let $\mathcal{A}$ denote an abelian category with a set of

isomorphism classes of objects. Further let $\mathcal{B}$ be a non-empy full subcategory of $\mathcal{A}$ which is closed under subobjects, quotient objects and finite products in $\mathcal{A}$. We note that $\mathcal{B}$ is an abelian category and that the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ is exact. We wish to consider both $\mathcal{A}$ and $\mathcal{B}$ as exact categories. This is done by declaring that the monomorphisms and epimorhisms are admissible. Since $\mathcal{A}$ and $\mathcal{B}$ both are abelian this will give an exact structure. Using this exact structure we get that $\mathcal{QB}$ is the full subcategory of $\mathcal{QA}$ consisting of those objects which are also objects of $\mathcal{B}$. We are now ready to state the theorem.

Theorem 2.5 (Devissage)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Assume that every object $M \in ob\mathcal{A}$ has a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i \in ob\mathcal{A}$ such that $M_i/M_{i-1} \in ob\mathcal{B}$. Then the inclusion functor $\mathcal{QB} \hookrightarrow \mathcal{QA}$ is a homotopy equivalence, hence

$$K_i\mathcal{B} \cong K_i\mathcal{A}.$$

For the proof of this theorem we will need some few definitions and a lemma:

Definition 2.6. Let $c \in \mathrm{ob}\mathcal{C}$. An *admissible subobject* of c is an isomorphism class of admissible morphisms $c' \rightarrowtail c$ in $\mathcal{C}$.

Definition 2.7. Let c', c'' be two admissible objects of $c \in \mathrm{ob}\mathcal{C}$. Then if the unique morphisms $c'' \to c'$ is an admissible monomorphism we call (c'', c') an *admissible layer* of c.

Lemma 2.8. *A morphism $A \to B$ in $\mathcal{QA}$ may be identified with an isomorphism $j : B_2/B_1 \cong A$, where (B_1, B_2) is an admissible layer of B.*

Proof. Let $f \in \mathcal{QA}(A, B)$. Then it's represented by

$$A \overset{h}{\twoheadleftarrow} C \overset{g}{\rightarrowtail} B.$$

We note that since $\mathcal{A}$ is abelian there exists an exact sequence

$$ker(h) \rightarrowtail C \overset{h}{\twoheadrightarrow} A$$

so in particular we have an admissible layer $(ker(h), C)$ of B. By using that h is an admissible monomorphism and the universal property of quotients, we get

$$\begin{tikzcd} ker(h) \arrow[r, tail] & C \arrow[r, two heads, "h"] \arrow[dr] & A \arrow[d, "\cong"] \\ & & C/ker(h) \end{tikzcd}$$

so $f : A \to B$ may be identified with the isomorphism $A \cong C/ker(h)$. $\square$

We are now ready to prove the main theorem:

Proof of Devissage. Let f denote the inclusion. By Theorem A it's sufficient to prove that the comma category f/M is contractible for every $M \in \mathrm{ob}\mathcal{A}$. We note that f/M may be considered as the fibered category over $\mathcal{QB}$ consisting of pairs (N, u) where $N \in \mathrm{ob}\mathcal{Q}B$ and $u \in \mathcal{QA}(N, M)$. Using Lemma 2.8 we identify u with an isomorphism $N \cong M_1/M_0$ where (M_0, M_1) is an admissible layer of M and $M_1/M_0 \in \mathrm{ob}\mathcal{B}$.

Now, let $J(M)$ denote the ordered set consisting of admissible layers (M_0, M_1) of M with the ordering

$$(M_0, M_1) \leq (M_0', M_1') \text{ if and only if } M_0' \subset M_0 \subset M_1 \subset M_1'$$

We then have an equivalence between f/M and $J(M)$.

Since we have assumed that M has a finite filtrantion with quotiens in $\mathcal{B}$ we get that it is sufficient to prove that the inclusion $J(M') \to J(M)$ is a homotopy equivalence (with this I mean that it induces a homotopy equivalence on the classyfing space) whenever $M' \subset M$ and $M/M' \in \mathrm{ob}\mathcal{B}$, since then $J(M)$ is homotopy equivalent to $J(0)$ which is clearly contractible.

So assume that $M' \subset M$ and $M/M' \in \mathrm{ob}\mathcal{B}$. Define the following two functors:

$$\begin{aligned} r &: J(M) \to J(M') \\ (M_0, M_1) &\mapsto (M_0 \cap M', M_1 \cap M') \\ s &: J(M) \to J(M) \\ (M_0, M_1) &\mapsto (M_0 \cap M', M_1) \end{aligned}$$

Here $A \cap B$ denotes the pullback $A \times_C B$ given by the pullback diagram

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

Using the universal property of pullbacks we have that $(M_0 \cap M', M_1 \cap M')$ and $(M_0 \cap M', M_1)$ are indeed admissible layers of M' and M respectively. It can also be proven that

$$M_1 \cap M'/M_0 \cap M' \subset M_1/(M_0 \cap M') \subset M_1/M_0 \times M/M'$$

which gives us that $M_1 \cap M'/M_0 \cap M', M_1/M_0 \cap M' \in \mathrm{ob}\mathcal{B}$ since we have by assumption that $\mathcal{B}$ is closed under subobjects and finite projects so $M_1/M_0 \times M/M' \in \mathrm{ob}\mathcal{B}$.

Next goal is to prove that r is a homotopy inverse to i. Since

$$J(M') \xrightarrow{i} J(M) \xrightarrow{r} J(M')$$

$$(M_0', M_1') \longmapsto (M_0', M_1') \longmapsto (M_0' \cap M', M_1 \cap M')$$

and $(M_0' \cap M', M_1 \cap M') = (M_0', M_1')$ we have $r \circ i = id_{J(M')}$. To see that $i \circ r \simeq id_{J(M)}$ we wish to prove that there exists natural transformations

$$i \circ r \Rightarrow s \Leftarrow id_{J(M)}.$$

We note that

$$i \circ r((M_0, M_1)) = (M_0 \cap M', M_1 \cap M') \in J(M) \text{ and } s((M_0, M_1)) = (M_0 \cap M', M_1) \in J(M)$$

so the first natural transformation $i \circ r \Rightarrow s$ is represented by the ordering

$$(M_0 \cap M', M_1 \cap M') \subset (M_0 \cap M', M_1).$$

The other natural transformation $id_{J(M)} \Rightarrow s$ is represented by

$$(M_0, M_1) \subset (M_0 \cap M', M_1).$$

It is a result that natural transformations induces homotopies, so there exist homotopies $i \circ r \simeq s \simeq id_{J(M)}$. So we have both $i \circ r \simeq id_{J(M)}$ and $r \circ i \simeq id_{J(M')}$ so r is a homotopy inverse of i. □

We wish to consider two applications of this very strong theorem. First we note the following notation

Definition 2.9. Let R be a Noetherian ring and $FMod(R)$ be the fullsubcategory of $Mod(R)$ consisting of finitely generated R-modules. Then we define

$$G_i(R) := K_i(FMod(R)).$$

Corollary 2.10. *If I is a nilpotent ideal in a Noetherian ring R, then*

$$G_i(R/I) \cong G_i(R).$$

Proof. Let $M \in \text{ob}\mathcal{F}Mod(R)$. Then we get that

$$0 = MI^n \subseteq MI^{n-1} \subseteq \cdots \subseteq MI \subseteq M$$

is finite since I is nilpotent. We further have that MI^n is finitely generated since R is Noetherian, hence the surjection $MI^n \twoheadrightarrow MI^n/MI^{n+1}$ gives us that all the quotiens MI^n/MI^{n+1} are finitely generated R/I-modules. That means we can apply the Devissage theorem so the inclusion functor $FMod(R/I) \hookrightarrow FMod(R)$ is a homotopy equivalence, so

$$G_i(R/I) \cong G_i(R).$$

□

Before we can state the next corollary we need some definitions:

Definition 2.11. An object $X \in \mathrm{ob}\mathcal{C}$ is said to be *simple* if there is precisely to quotient objects of X - namely X and 0. If X is a finite coproduct of simple objects it is said to be *semi-simple*.

Definition 2.12. An object $X \in \mathrm{ob}\mathcal{C}$ is said to have *finite length* if there exist a finite sequence of subobject inclusions into X

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

such that at every state i, the quotient X_i/X_{i-1} is a simple object of $\mathcal{C}$.

Corollary 2.13. *Let $\mathcal{A}$ be an abelian category with a set of isomorphism classes, such that every object has finite length. Then*

$$K_i\mathcal{A} \cong \oplus_{j\in J} G_i(D_j)$$

where $\{X_j, j \in J\}$ is a set of representatives for the isomorphism classes of simple objects of $\mathcal{A}$, and D_j is the division ring $End(X_j)^{op}$.