§2 Waldhausen's S·dot construction

Thomas Read - 19.01.22

Let $\mathcal{C}$ be some category with a notion of exact sequences - e.g. finitely generated projective modules

Def: We define a simplicial set $sC[-]$ by saying the nth term is all functor

$$A: \text{Ar}([n]) \longrightarrow \mathcal{C}$$

s.t.

i) $A(i,i) = 0$

ii) For $0 \leq i \leq j \leq k \leq n$,

$$A(i,j) \rightarrow A(i,k) \rightarrow A(j,k)$$

is exact.

+ simplicial maps.

$sC[3]$ consists of diagrams

four interlocked exact sequences

Def: The K-theory space of $\mathcal{C}$ is

$$K(\mathcal{C}) := \Omega |sC[-]|.$$

face operators $d_0, d_1, d_2 : sC[2] \rightarrow sC[1]$

$$A_{01} \longmapsto A_{02} \longrightarrow A_{12} \overset{d_0}{\underset{d_1}{\longmapsto}} \begin{matrix} A_{12} \\ A_{02} \end{matrix}$$

$$\overset{d_2}{\searrow} A_{01}$$
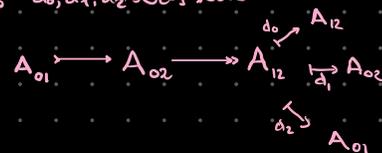
so d_k maps the sequence to the A_{ij} with $k \neq i,j$.

Claim: π_0 and π_1 of $K(\mathcal{C})$ agrees with the $K_0(\mathcal{C})$, $K_1(\mathcal{C})$ as define by Groethendieck and Whitehead.

We can 'extend' this to a simplicial category $SC[-]$ which has
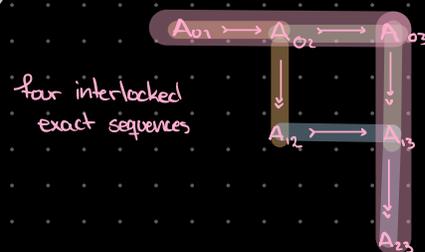
$$\text{ob} \, SC[n] = sC[n]$$

$$\text{morphisms} = \text{natural transformations.}$$

Another way to define K-theory:

Def: $k[n,m] := N(iSC[n])[m]$ bisimplicial set

isomorphism subcategory

$$k(\mathcal{C}) := \Omega |k[-,-]|$$

We want to consider this on a more general form of category, namely exact categories.

First, recall that an additive category is an Ab-enriched cat. which admits both products and coproducts.

Def: An **exact category** is a category $\mathcal{E}$ together with a set $\mathcal{E}$ of diagrams

$$A \xrightarrow{i} B \xrightarrow{P} C$$

admissible monomorphism

admissible epimorphism

in $\mathcal{E}$, called **exact sequences**, s.t.

i) $i = \ker(p)$, $p = \mathrm{coker}(i)$

ii) $\mathcal{E}$ is closed under isomorphisms

iii) Admissible monomorphism are closed under composition and pushouts along any morphism

iv) Admissible epimorphisms are closed under composition and pullbacks along any morphism

Every split exact sequence belongs to $\mathcal{E}$, i.e.

$$(A \xhookrightarrow{\text{incl.}} A \oplus B \xrightarrow{\text{proj.}} B) \in \mathcal{E}.$$

Rem: Exact categories $\longleftrightarrow$ Full subcategory of abelian categories closed under extensions.

Thm: $|S\mathcal{C}[-]|$ is homotopy equivalent to $|N(iS\mathcal{C}[-])[-]|$

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Pf: First we note that $N(i.D)[m] = \mathrm{ob} N^i(D)[m]$

$$\begin{array}{ccc} d_0 & \cong \cdots \cong & d_m \\ \downarrow \supset \downarrow & \cdots & \downarrow \\ d_0' & \cong \cdots \cong & d_m' \end{array}$$

We want to construct

$$S\mathcal{C}[-] \xrightarrow{\cong} N(iS\mathcal{C}[-])[m],$$

recalling that $sG[n] = obS G_{\bullet}[n]$.

$$N(iSG[n])[m] \stackrel{\sim}{\leftarrow} ob\ N^i\ (SG[n])[m]$$

$$\cong ob\ S(N^i\ G[m])[n]$$

← Both are full subcats of $[m] \times Ar[n] \longrightarrow G$

$$= s\ (N^i\ G[m])[n]$$

Want

$$sG[-] \stackrel{\sim}{\longrightarrow} s\ (N^i\ G[m])[n].$$

To do so we note

$$G = N^i G[0]$$
$$d_0 \uparrow \downarrow s_0$$
$$N^i G[1]$$
$$d_0 \uparrow \downarrow s_0$$
$$\vdots$$
$$d_0 \uparrow \downarrow s_0$$
$$N^i G[m],$$

← Each of these are exact categories

$$d_0 s_0 = id$$
$$id \stackrel{\sim}{\Rightarrow} s_0 d_0.$$

and it can be shown that these maps are <u>exact functors</u>, preserves exact sequences hence they induce maps on the S-dot construction

$$sG[-] = s(N^i G[0])[-] \underset{s(d_0)}{\overset{s(s_0)}{\rightleftarrows}} s(N^i G[1])[-] \overset{id \stackrel{\sim}{\Rightarrow} s_0 d_0}{\underset{}{\rightrightarrows}} \cdots$$

which will give
$$sG[-] \stackrel{\sim}{\rightarrow} s(N^i G[m])[-]$$

The result follows by applying the realization theorem.

□